Finite Horizon Density Steering for Multi-input State Feedback Linearizable Systems

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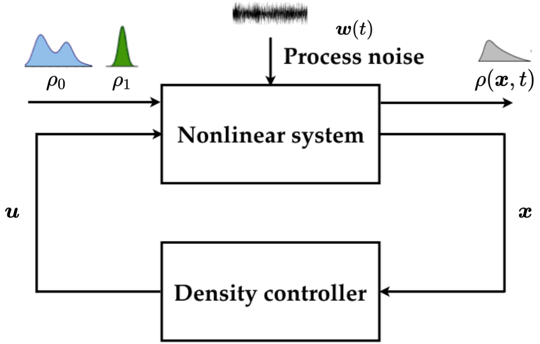
Problem: Finite Horizon Feedback Density Control



$$\inf_{u \in \mathcal{U}} \quad \mathbb{E}\left\{\int_0^1 \mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t)\,\mathrm{d}t\right\}$$

subject to $\mathrm{d}\boldsymbol{x} = \boldsymbol{f}(x,t)\,\mathrm{d}t + \boldsymbol{G}(x,t)u(\boldsymbol{x},t)\,\mathrm{d}t + \sqrt{2\epsilon}\boldsymbol{C}(x,t)\,\mathrm{d}\boldsymbol{w}$

$$\boldsymbol{x}(t=0) \sim \rho_0(\boldsymbol{x}), \quad \boldsymbol{x}(t=1) \sim \rho_1(\boldsymbol{x})$$

Density Control: Schrödinger Bridge Problem

- $\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, t) \equiv \|\boldsymbol{u}(\boldsymbol{x}, t)\|_2^2, \boldsymbol{f} \equiv \boldsymbol{0}, \boldsymbol{G} \equiv \boldsymbol{I}, \boldsymbol{C} \equiv \boldsymbol{I},$
- Compute $\rho^{\rm opt}(\boldsymbol{x}, t), \boldsymbol{u}^{\rm opt}(\boldsymbol{x}, t) = \nabla\psi(\boldsymbol{x}, t)$ from

$$\frac{\partial \rho^{\rm opt}}{\partial t} + \nabla \cdot \left(\rho^{\rm opt} \nabla \psi\right) = \epsilon \Delta \rho^{\rm opt}$$

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\|\nabla \psi\|_2^2 = -\epsilon \Delta \psi$$

$$\rho^{\rm opt}(\boldsymbol{x}, 0) = \rho_0(\boldsymbol{x}), \quad \rho^{\rm opt}(\boldsymbol{x}, 1) = \rho_1(\boldsymbol{x})$$

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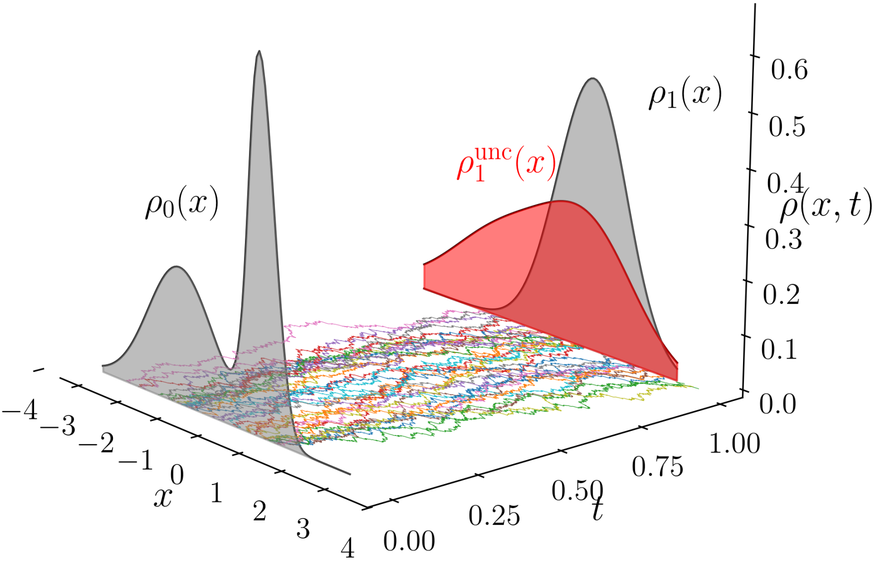
$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\|\nabla \psi\|_2^2 = -\epsilon \Delta \psi$$
$$\rho^{\rm opt}(\boldsymbol{x}, 0) = \rho_0(\boldsymbol{x}), \quad \rho^{\rm opt}(\boldsymbol{x}, 1) = \rho_1(\boldsymbol{x})$$

- **Schrödinger System**: $(\rho^{\rm opt}, \psi) \mapsto (\varphi, \hat{\varphi})$

$$\frac{\partial \varphi}{\partial t} = -\epsilon \Delta \varphi$$
$$\frac{\partial \hat{\varphi}}{\partial t} = \epsilon \Delta \hat{\varphi}$$
$$\varphi_0(\boldsymbol{x})\hat{\varphi}_0(\boldsymbol{x}) = \rho_0^{\rm opt}(\boldsymbol{x}), \varphi_1(\boldsymbol{x})\hat{\varphi}_1(\boldsymbol{x}) = \rho_1^{\rm opt}(\boldsymbol{x})$$

- Recover $\rho^{\rm opt}(\boldsymbol{x}, t) = \varphi(\boldsymbol{x}, t)\hat{\varphi}(\boldsymbol{x}, t), \boldsymbol{u}^{\rm opt}(\boldsymbol{x}, t) = 2\epsilon \nabla \log \varphi(\boldsymbol{x}, t)$

Density Control: Schrödinger Bridge Problem



Density Control

- **Challenge:** How do we solve this problem for nonlinear f?

- **Applications:** dynamic shaping of swarms, stochastic motion planning

- **Idea:** in many applications, (f, G) is feedback linearizable

Density Control for Feedback Linearizable Systems

$$\inf_{u \in \mathcal{U}} \quad \mathbb{E}\left\{\int_0^1 \frac{1}{2}\|u(\boldsymbol{x},t)\|_2^2 \,\mathrm{d}t\right\}$$

$$\text{subject to} \quad \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}$$

$$\boldsymbol{x}(t=0) \sim \rho_0(\boldsymbol{x}) \quad \boldsymbol{x}(t=1) \sim \rho_1(\boldsymbol{x})$$

- $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$ $m \leq n$
- $\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{G}(\boldsymbol{x})$ feedback linearizable $\Leftrightarrow$ there exist $(\boldsymbol{\delta}(\boldsymbol{x}), \Gamma(\boldsymbol{x}), \boldsymbol{\tau}(\boldsymbol{x}))$ s.t.

$$\left(\nabla\boldsymbol{\tau}(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\delta(\boldsymbol{x}))\right)_{\boldsymbol{x}=\boldsymbol{\tau}^{-1}(\boldsymbol{z})} = \boldsymbol{A}\boldsymbol{z}$$
$$\left(\nabla\boldsymbol{\tau}(\boldsymbol{G}(\boldsymbol{x})\Gamma(\boldsymbol{x}))\right)_{\boldsymbol{x}=\boldsymbol{\tau}^{-1}(\boldsymbol{z})} = \boldsymbol{B}$$

  and $(\boldsymbol{A}, \boldsymbol{B})$ controllable.
- change of coordinates from $(\boldsymbol{x}, \boldsymbol{u}) \mapsto (\boldsymbol{z}, \boldsymbol{v})$ where $\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{v}$ and recover $\boldsymbol{u} = \delta(\boldsymbol{x}) + \Gamma(\boldsymbol{x})\boldsymbol{v}$.

Density Control for Feedback Linearizable Systems

Main Idea: Use the diffeomorphism $\tau : \mathcal{X} \mapsto \mathcal{Z}$

$$\sigma_i(\boldsymbol{z}) := \boldsymbol{\tau}_{\sharp}\rho_i = \frac{\rho_i(\tau^{-1}(\boldsymbol{z}))}{|\det(\nabla_{\boldsymbol{x}}\boldsymbol{\tau}_{\boldsymbol{x}=\boldsymbol{\tau}^{-1}(\boldsymbol{z})})|}, \quad i \in \{0, 1\}.$$

Define $\boldsymbol{\delta}_{\boldsymbol{\tau}} := \boldsymbol{\delta} \circ \boldsymbol{\tau}^{-1}, \boldsymbol{\Gamma}_{\boldsymbol{\tau}} := \boldsymbol{\Gamma} \circ \boldsymbol{\tau}^{-1}$ to reformulate the problem as:

$$\inf_{\boldsymbol{v} \in \mathcal{V}} \quad \mathbb{E}\left\{\int_0^1 \frac{1}{2}\mathcal{J}(\boldsymbol{z}, \boldsymbol{v}, t)\,\mathrm{d}t\right\}$$

subject to

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{v}$$

$$\boldsymbol{z}(t=0) \sim \sigma_0(\boldsymbol{z}) \quad \boldsymbol{z}(t=1) \sim \sigma_1(\boldsymbol{z})$$

where $\mathcal{J}(\boldsymbol{z}, \boldsymbol{v}, t) = \|\boldsymbol{\delta}_{\boldsymbol{\tau}}(\boldsymbol{z}) + \boldsymbol{\Gamma}_{\boldsymbol{\tau}}(\boldsymbol{z})\boldsymbol{v}\|_2^2$.

Optimal Solution in Feedback Linearized Coordinates

Solve for optimal PDF σ^{opt} and optimal control given by

$$\boldsymbol{v}^{\mathrm{opt}}(\boldsymbol{z},t) = (\boldsymbol{\Gamma}_{\boldsymbol{\tau}}^{\top}\boldsymbol{\Gamma}_{\boldsymbol{\tau}}(\boldsymbol{z}))^{-1}\boldsymbol{B}^{\top}\nabla_{\boldsymbol{z}}\psi - \boldsymbol{\Gamma}_{\boldsymbol{\tau}}^{-1}(\boldsymbol{z})\boldsymbol{\delta}_{\boldsymbol{\tau}}(\boldsymbol{z})$$

$$\frac{\partial\psi}{\partial t} + \langle\nabla_{\boldsymbol{z}}\psi, \boldsymbol{A}\boldsymbol{z}\rangle - \langle\nabla_{\boldsymbol{z}}\psi, \boldsymbol{B}\boldsymbol{\Gamma}_{\boldsymbol{\tau}}^{-1}(\boldsymbol{z})\boldsymbol{\delta}_{\boldsymbol{\tau}}(\boldsymbol{z})\rangle$$

$$+ \frac{1}{2}\langle\nabla_{\boldsymbol{z}}\psi, \boldsymbol{B}\left(\boldsymbol{\Gamma}_{\boldsymbol{\tau}}^{\top}(\boldsymbol{z})\boldsymbol{\Gamma}_{\boldsymbol{\tau}}(\boldsymbol{z})\right)^{-1}\boldsymbol{B}^{\top}\nabla_{\boldsymbol{z}}\psi\rangle = 0$$

$$\frac{\partial\sigma^{\mathrm{opt}}}{\partial t} + \nabla_{\boldsymbol{z}}\cdot\left(\left(\boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{v}^{\mathrm{opt}}\right)\sigma^{\mathrm{opt}}\right) = 0$$

$$\sigma^{\mathrm{opt}}(\boldsymbol{z}, t=0) = \sigma_0(\boldsymbol{z}), \quad \sigma^{\mathrm{opt}}(\boldsymbol{z}, t=1) = \sigma_1(\boldsymbol{z})$$

Challenge: How to solve coupled nonlinear system of PDE's?

Schrödinger System for Feedback Linearizable Systems

Idea: Stochastic regularization $\sqrt{2\epsilon}\boldsymbol{B}\boldsymbol{\Gamma}_{\boldsymbol{\tau}}^{-1}(\boldsymbol{z})\mathrm{d}\boldsymbol{w}$

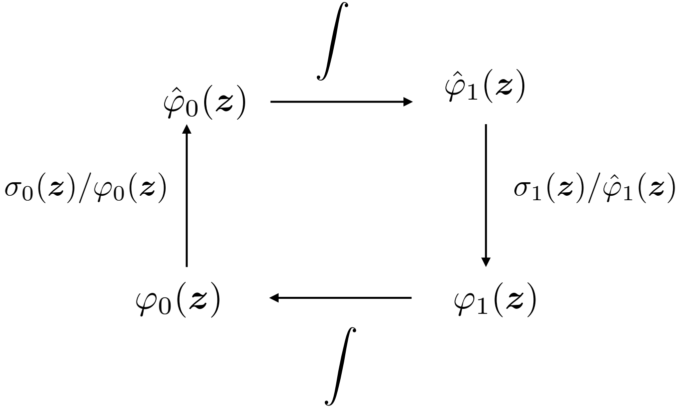
Theorem: Let $\boldsymbol{D}(\boldsymbol{z}) := \boldsymbol{B}\boldsymbol{\Gamma}_{\boldsymbol{\tau}}^{-1}(\boldsymbol{z})(\boldsymbol{B}\boldsymbol{\Gamma}_{\boldsymbol{\tau}}^{-1}(\boldsymbol{z}))^{\top}$

$$\varphi(\boldsymbol{z},t) := \exp\left(\frac{\psi(\boldsymbol{z},t)}{2\epsilon}\right), \quad \hat{\varphi}(\boldsymbol{z},t) := \sigma^{\rm opt}(\boldsymbol{z},t)\exp\left(-\frac{\psi(\boldsymbol{z},t)}{2\epsilon}\right),$$

then we have the boundary coupled linear system

$$\begin{aligned}
&\frac{\partial\varphi}{\partial t} + \langle\nabla_{\boldsymbol{z}}\varphi, \boldsymbol{A}\boldsymbol{z} - \boldsymbol{B}\boldsymbol{\Gamma}_{\boldsymbol{\tau}}^{-1}\boldsymbol{\delta}_{\boldsymbol{\tau}}(\boldsymbol{z})\rangle + \epsilon\langle\boldsymbol{D}, \mathrm{Hess}(\varphi)\rangle = 0,\\
&\frac{\partial\hat{\varphi}}{\partial t} + \nabla_{\boldsymbol{z}}\cdot\left(\left(\boldsymbol{A}\boldsymbol{z} - \boldsymbol{B}\boldsymbol{\Gamma}_{\boldsymbol{\tau}}^{-1}\boldsymbol{\delta}_{\boldsymbol{\tau}}(\boldsymbol{z})\right)\hat{\varphi}\right)\\
&\qquad - \epsilon\mathbf{1}\left(\boldsymbol{D}(\boldsymbol{z})\odot\mathrm{Hess}(\hat{\varphi})\right)\mathbf{1} = 0,\\
&\varphi_0(\boldsymbol{z})\hat{\varphi}_0(\boldsymbol{z}) = \sigma_0(\boldsymbol{z}), \quad \varphi_1(\boldsymbol{z})\hat{\varphi}_1(\boldsymbol{z}) = \sigma_1(\boldsymbol{z})
\end{aligned}$$

Algorithm: Fixed Point Recursion on $(\varphi_1, \hat{\varphi}_0)$



This recursion is contractive in the Hilbert Metric.

- Y. Chen, T. T. Georgiou, and M. Pavon, "Entropic and Displacement Interpolation: A Computational Approach Using the Hilbert metric,"SIAM Journal on Applied Mathematics"

Optimal Solution in Original Coordinates

- Fixed point recursion gives $\varphi_1(\boldsymbol{z}), \hat{\varphi}_0(\boldsymbol{z}) \to \varphi(\boldsymbol{z}, t), \hat{\varphi}(\boldsymbol{z}, t)$
- Obtain
  - optimal controlled PDF:
  $$\sigma^{\mathrm{opt}}(\boldsymbol{z}, t) = \varphi(\boldsymbol{z}, t)\hat{\varphi}(\boldsymbol{z}, t)$$
  - optimal control:
  $$\boldsymbol{v}^{\mathrm{opt}}(\boldsymbol{z}, t) = (\boldsymbol{\Gamma}_{\boldsymbol{\tau}}^{\top}\boldsymbol{\Gamma}_{\boldsymbol{\tau}}(\boldsymbol{z}))^{-1}\boldsymbol{B}^{\top}2\epsilon\nabla_{\boldsymbol{z}}\log\varphi - \boldsymbol{\Gamma}_{\boldsymbol{\tau}}^{-1}(\boldsymbol{z})\boldsymbol{\delta}_{\boldsymbol{\tau}}(\boldsymbol{z})$$
- Recover $\rho^{\mathrm{opt}}(\boldsymbol{x}, t)$ and $\boldsymbol{u}^{\mathrm{opt}}(\boldsymbol{x}, t)$ as
$$\rho^{\mathrm{opt}}(\boldsymbol{x}, t) = \sigma^{\mathrm{opt}}(\boldsymbol{\tau}(\boldsymbol{x}), t)|\det(\nabla_{\boldsymbol{x}}\boldsymbol{\tau})|$$
$$\boldsymbol{u}^{\mathrm{opt}}(\boldsymbol{x}, t) = \boldsymbol{\delta}(x) + \boldsymbol{\Gamma}(\boldsymbol{x})\boldsymbol{v}^{\mathrm{opt}}(\boldsymbol{\tau}^{-1}(\boldsymbol{x}), t)$$

Summary

- Density Control subject to Multi input state feedback linearizable dynamics
- Reduced the system of coupled nonlinear PDEs to a system of boundary-coupled linear PDE's
- Algorithm via Fixed Point Recursion on initial-terminal condition pair

Thank You!

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