Prediction and Optimal Feedback Steering of Probability Density Functions for Safe Automated Driving

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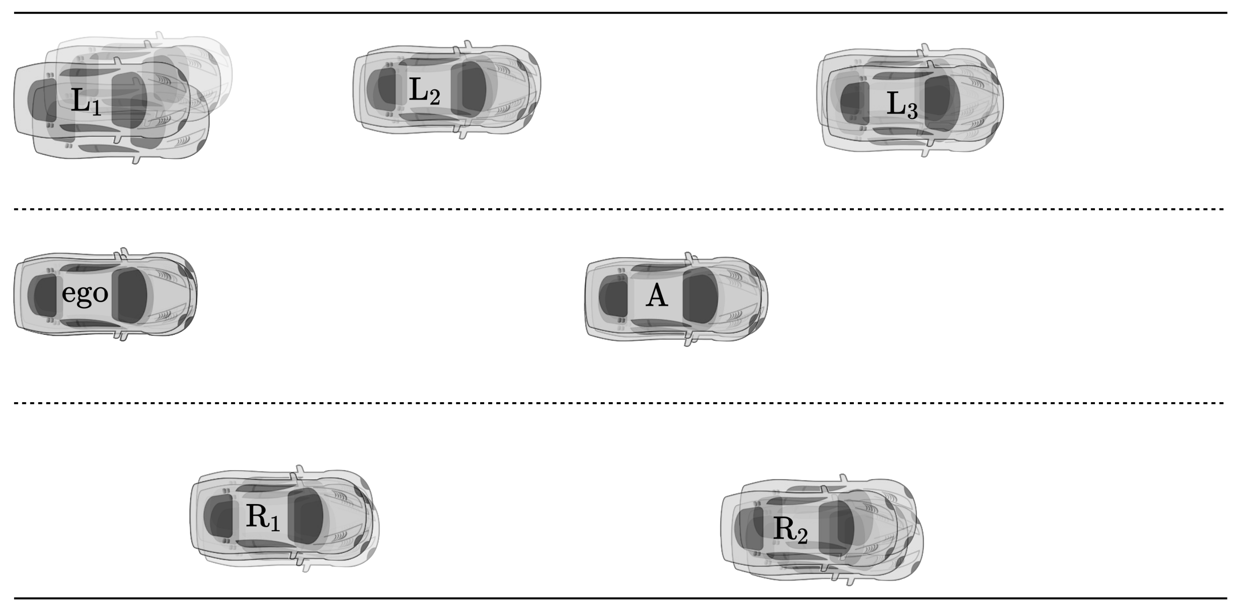
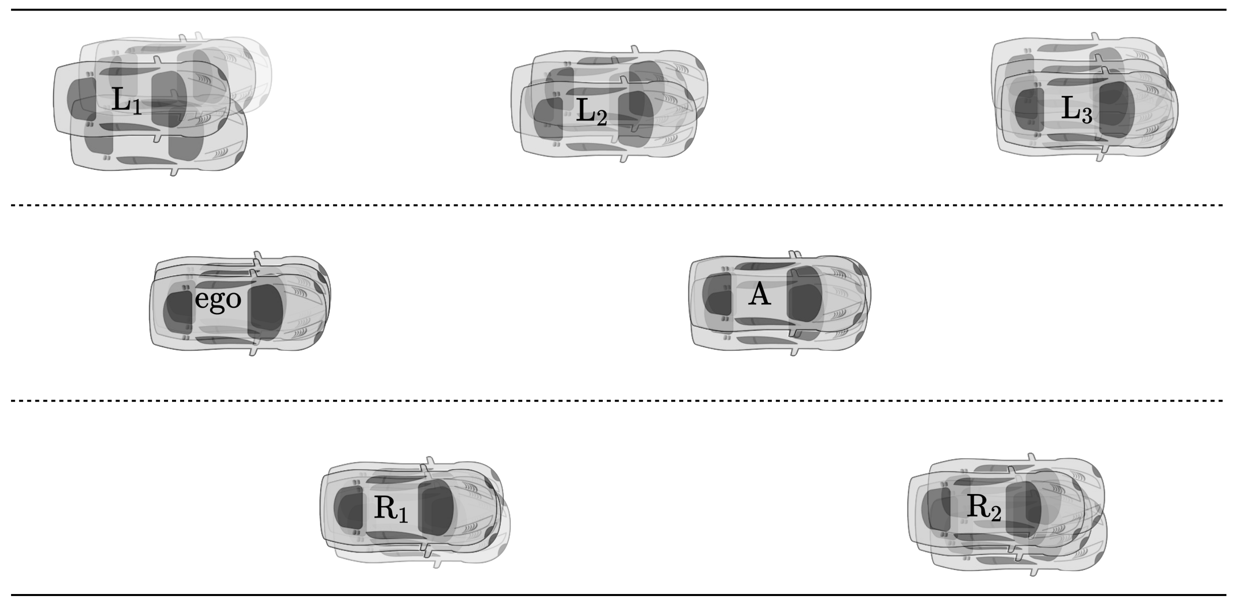
Stochastic Uncertainties in Multi-lane Highway Driving

The ego vehicle's estimate at time $t = t_0$



The ego vehicle's estimate at time $t = t_0 + T$



The Present Paper

Stochastic uncertainties: joint state PDFs

Nonparametric prediction of PDFs: characteristic ODEs

Feedback synthesis for the ego vehicle's stochastic states

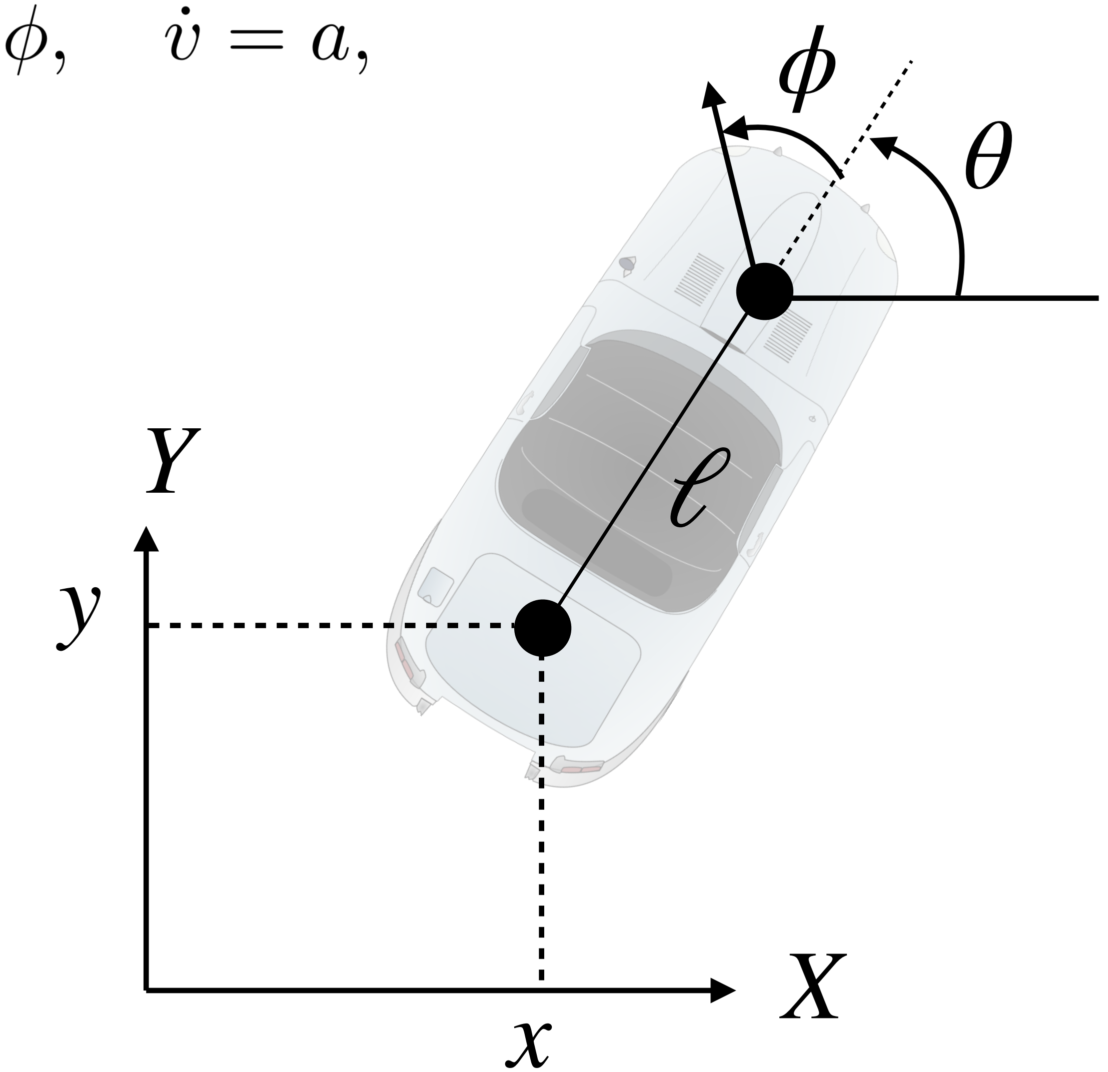
Prediction Problem

Kinematic bicycle model

$$\dot{x} = v\cos\theta, \quad \dot{y} = v\sin\theta, \quad \dot{\theta} = \frac{v}{\ell}\tan\phi, \quad \dot{v} = a,$$

$$\boldsymbol{x} := (x, y, \theta, v)^{\top}$$

$$\boldsymbol{u} := (a, \phi)^{\top}$$



Nominal MPC policy for each vehicle

$$\boldsymbol{u} = \boldsymbol{\pi}_{\text{MPC}}(\boldsymbol{x}, t)$$

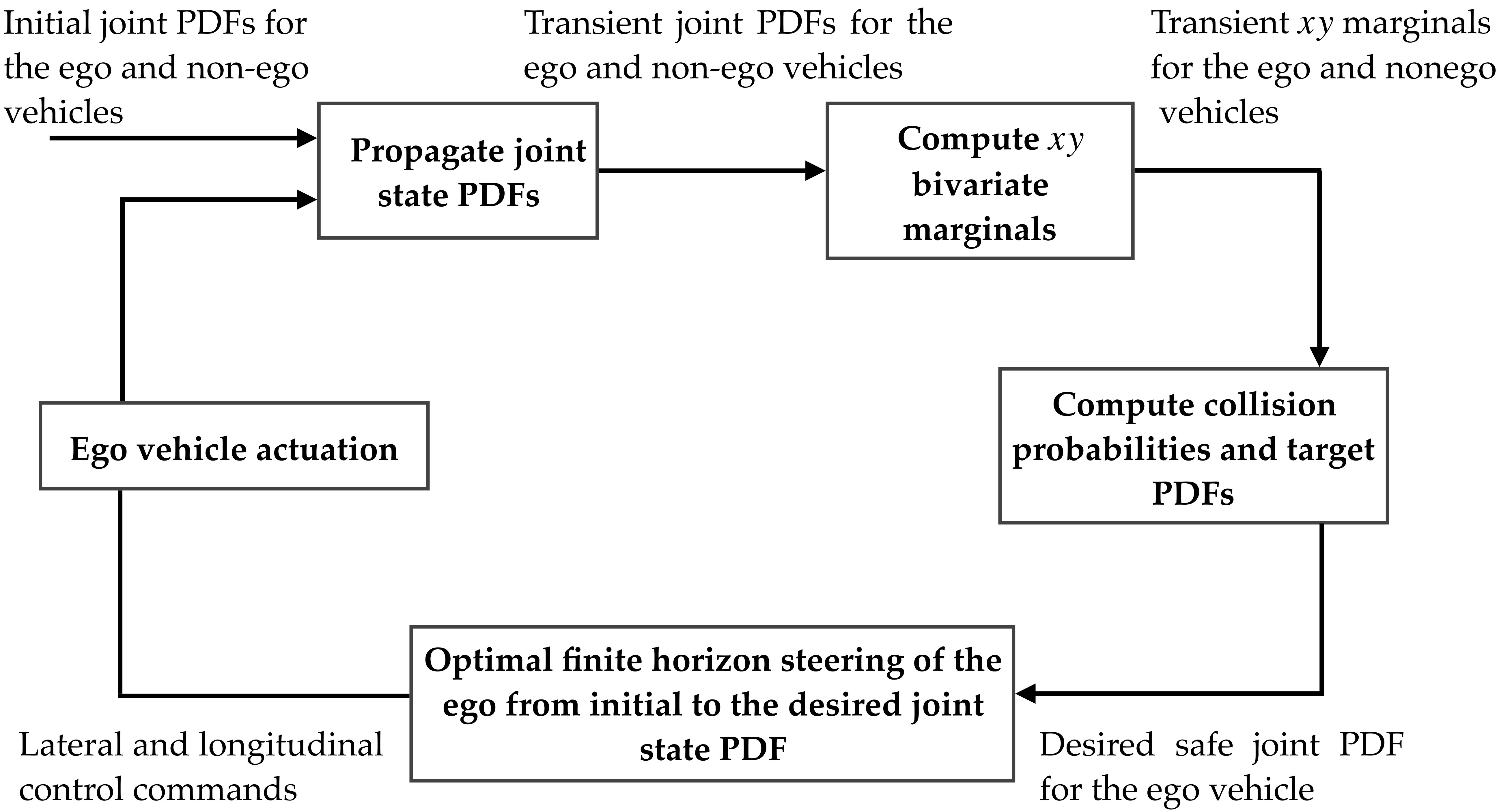
Known initial joint PDFs at $t = t_0$

$$\rho_0^{\text{ego}}, \rho_0^{\text{A}}, \rho_0^{\text{L}_i}, \rho_0^{\text{R}_j}$$

Want to predict joint PDFs at $t = t_0 + T$

$$\rho^{\text{ego}}(\boldsymbol{x}, t), \rho^{\text{A}}(\boldsymbol{x}, t), \rho^{\text{L}_i}(\boldsymbol{x}, t), \rho^{\text{R}_j}(\boldsymbol{x}, t)$$

Framework


Initial joint PDFs for the ego and non-ego vehicles
Propagate joint state PDFs
Transient joint PDFs for the ego and non-ego vehicles
Compute xy bivariate marginals
Transient xy marginals for the ego and nonego vehicles
Compute collision probabilities and target PDFs
Desired safe joint PDF for the ego vehicle
Optimal finite horizon steering of the ego from initial to the desired joint state PDF
Lateral and longitudinal control commands
Ego vehicle actuation

PDF Prediction Layer

Joint state PDF propagation using Liouville PDE

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\pi}_{\text{MPC}}(\boldsymbol{x}, t))\rho) = 0,$$

vehicle dynamics

Solve characteristic ODE over $t \in [t_0, t_0 + T]$

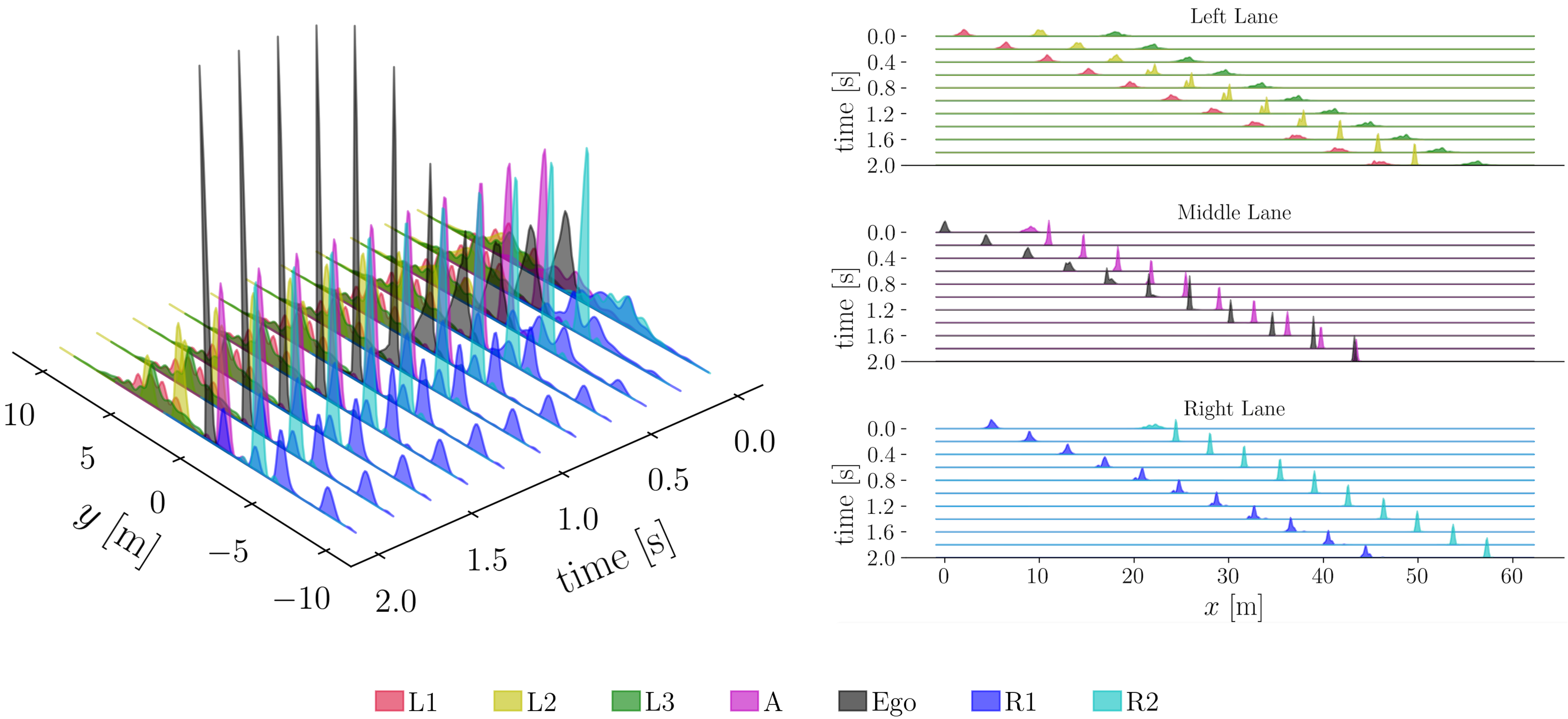
$$\{\boldsymbol{x}_0^i, \rho_0^i\}_{i=1}^N \longrightarrow \{\boldsymbol{x}^i(t), \rho^i(t)\}_{i=1}^N$$

$$\dot{\rho}^i = -\nabla_{\boldsymbol{x}^i} \cdot \boldsymbol{f}(\boldsymbol{x}^i, \boldsymbol{\pi}_{\text{MPC}}(\boldsymbol{x}^i, t)), \quad i = 1, \ldots, N$$

- **Probability weighted scattered point cloud evolution: method of characteristics**
- **No approximation of the statistics**
- **No approximation of the dynamics**

PDF Prediction Layer

y marginals (left figure), *x* marginals (right figure)

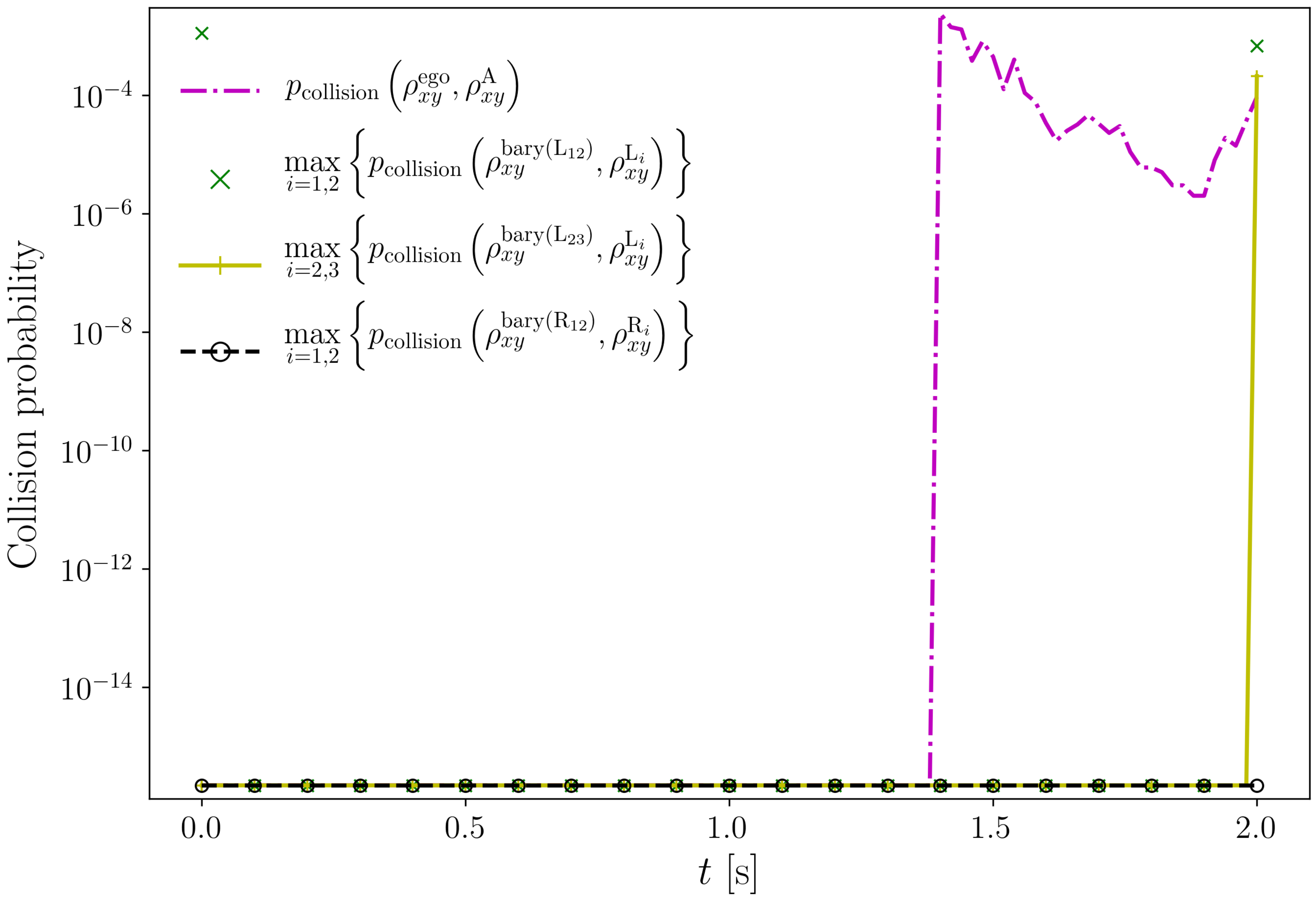


Compute Desired State PDF for the Ego at $t = t_0 + T$

Wasserstein barycenter

$$\rho^{\text{bary}} := \arg\inf_{\rho}\{\lambda_1 W^2(\rho,\rho_1) + \lambda_2 W^2(\rho,\rho_2)\}$$

Compute collision probabilities using xy bivariate marginals



PDF Control Layer

State feedback linearization

$$\boldsymbol{x} \mapsto \boldsymbol{z} := \boldsymbol{\tau}(\boldsymbol{x}) \qquad \dot{\boldsymbol{z}} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{=:\boldsymbol{A}} \boldsymbol{z} + \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}_{=:\boldsymbol{B}} \widetilde{\boldsymbol{u}}.$$

Feedback steering of the ego toward the desired joint state PDF

$$\inf_{(\sigma^{\widetilde{\boldsymbol{u}}}, \widetilde{\boldsymbol{u}})} \quad \mathbb{E}_{\sigma^{\widetilde{\boldsymbol{u}}}} \left[\int_{t_0}^{t_0+T} \frac{1}{2} \|\widetilde{\boldsymbol{u}}\|_2^2 \, \mathrm{d}t \right]$$

$$\text{subject to} \quad \frac{\partial \sigma^{\widetilde{\boldsymbol{u}}}}{\partial t} + \nabla_{\boldsymbol{z}} \cdot \left(\left(\boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\widetilde{\boldsymbol{u}} \right) \sigma^{\widetilde{\boldsymbol{u}}} \right) = 0$$

$$\sigma^{\widetilde{\boldsymbol{u}}}(\boldsymbol{z}, t = 0) = \boldsymbol{\tau}_\sharp \rho_0^{\text{ego}}$$

$$\sigma^{\widetilde{\boldsymbol{u}}}(\boldsymbol{z}, t = T) = \boldsymbol{\tau}_\sharp \rho_T^{\text{desired}}$$

$\sigma^{\widetilde{\boldsymbol{u}}}(\boldsymbol{z}, t)$	Controlled joint PDF
$\widetilde{\boldsymbol{u}}(\boldsymbol{z}, t)$	State feedback
$\sharp$	Pushforward
$\mathbb{E}_\rho[\cdot]$	Expectation operator

PDF Control Layer

Stochastic regularization: actuation noise

$$\mathrm{d}\boldsymbol{z} = \left(\boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\widetilde{\boldsymbol{u}}\right)\,\mathrm{d}t + \sqrt{2\varepsilon}\boldsymbol{B}\,\mathrm{d}\boldsymbol{w}$$

Schrodinger Bridge Problem

$$\inf_{\left(\sigma^{\widetilde{\boldsymbol{u}}},\widetilde{\boldsymbol{u}}\right)} \quad \mathbb{E}_{\sigma^{\widetilde{\boldsymbol{u}}}}\left[\int_{t_0}^{t_0+T} \frac{1}{2}\|\widetilde{\boldsymbol{u}}\|_2^2\,\mathrm{d}t\right]$$

$$\text{subject to} \quad \frac{\partial\sigma^{\widetilde{\boldsymbol{u}}}}{\partial t} + \nabla_{\boldsymbol{z}}\cdot\left(\left(\boldsymbol{A}\boldsymbol{z}+\boldsymbol{B}\widetilde{\boldsymbol{u}}\right)\sigma^{\widetilde{\boldsymbol{u}}}\right) = \varepsilon\left\langle\boldsymbol{B}\boldsymbol{B}^{\top},\mathrm{Hess}(\sigma^{\widetilde{\boldsymbol{u}}})\right\rangle$$

$$\sigma^{\widetilde{\boldsymbol{u}}}\left(\boldsymbol{z},t=0\right) = \boldsymbol{\tau}_{\sharp}\rho_0^{\text{ego}}$$

$$\sigma^{\widetilde{\boldsymbol{u}}}\left(\boldsymbol{z},t=T\right) = \boldsymbol{\tau}_{\sharp}\rho_T^{\text{desired}}$$

$\sigma^{\widetilde{\boldsymbol{u}}}(\boldsymbol{z},t)$	Controlled joint PDF
$\widetilde{\boldsymbol{u}}(\boldsymbol{z},t)$	State feedback
$\sharp$	Pushforward
$\mathbb{E}_{\rho}\left[\cdot\right]$	Expectation operator

PDF Control Layer

Solution of Stochastic Optimal Control

$$\left(\sigma_\varepsilon^{\widetilde{\boldsymbol{u}}}, \widetilde{\boldsymbol{u}}_\varepsilon\right)_{\text{opt}} = \left(\widehat{\varphi}(\boldsymbol{z},t)\varphi(\boldsymbol{z},t),\ 2\varepsilon \boldsymbol{B}^\top \nabla_{\boldsymbol{z}}\varphi(\boldsymbol{z},t)\right)$$

Where

$$\widehat{\varphi}(\boldsymbol{z},t) = \int_{\mathcal{Z}_0} \kappa(t_0, \widetilde{\boldsymbol{z}}, t, \boldsymbol{z})\, \widehat{\varphi}_0\left(\boldsymbol{z}_0\right) \mathrm{d}\boldsymbol{z}_0,$$

$$\varphi(\boldsymbol{z},t) = \int_{\mathcal{Z}_T} \kappa(t, \boldsymbol{z}, t_0+T, \boldsymbol{z}_T)\, \varphi_T\left(\boldsymbol{z}_T\right) \mathrm{d}\boldsymbol{z}_T$$

$$\kappa(s, \boldsymbol{z}, t, \widetilde{\boldsymbol{z}}) := \frac{\det\left(\boldsymbol{M}_{ts}\right)^{-1/2}}{(4\pi\varepsilon)^{n/2}} \exp\left(-\frac{1}{4\varepsilon} \times \left(\boldsymbol{z} - \boldsymbol{\Phi}_{ts}\widetilde{\boldsymbol{z}}\right)^\top \boldsymbol{M}_{ts}^{-1}\left(\boldsymbol{z} - \boldsymbol{\Phi}_{ts}\widetilde{\boldsymbol{z}}\right)\right)$$

$\left(\sigma_\varepsilon^{\widetilde{\boldsymbol{u}}}, \widetilde{\boldsymbol{u}}_\varepsilon\right)_{\text{opt}}$	Optimal pair based on the choice $\varepsilon > 0$
$\kappa(s, \boldsymbol{z}, t, \widetilde{\boldsymbol{z}})$	Markov Kernel for $t_0 \leq s < t \leq t_0 + T$
$\boldsymbol{M}_{ts}$	Controllability Gramian
$\boldsymbol{\Phi}_{ts}$	State transition matrix

PDF Control Layer

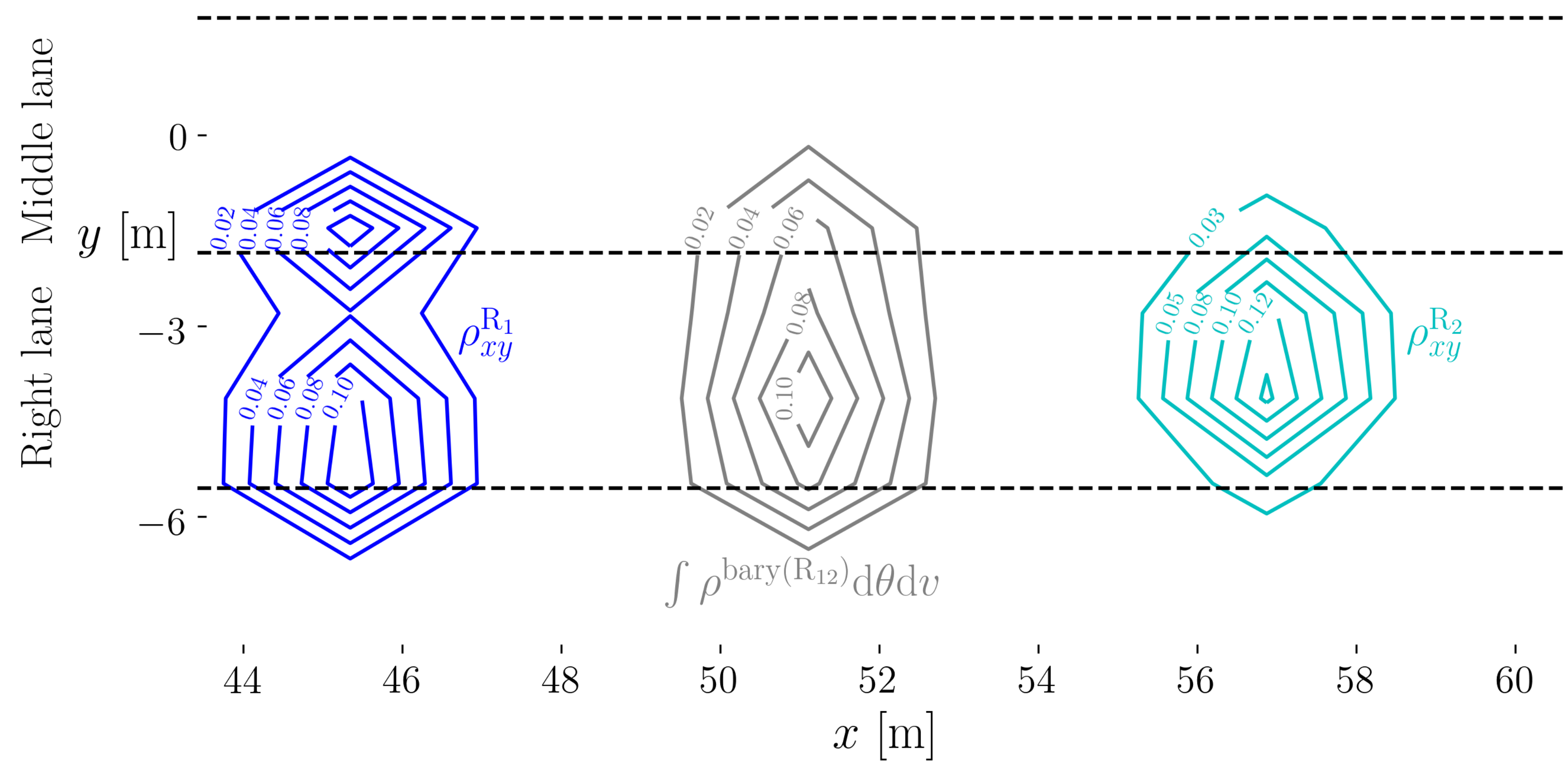
Contractive fixed point recursion

$$\widehat{\varphi}_0 \varphi_0 = \boldsymbol{\tau}_{\sharp}\rho_0^{\text{ego}}, \quad \widehat{\varphi}_T \varphi_T = \boldsymbol{\tau}_{\sharp}\rho_T^{\text{desired}}$$

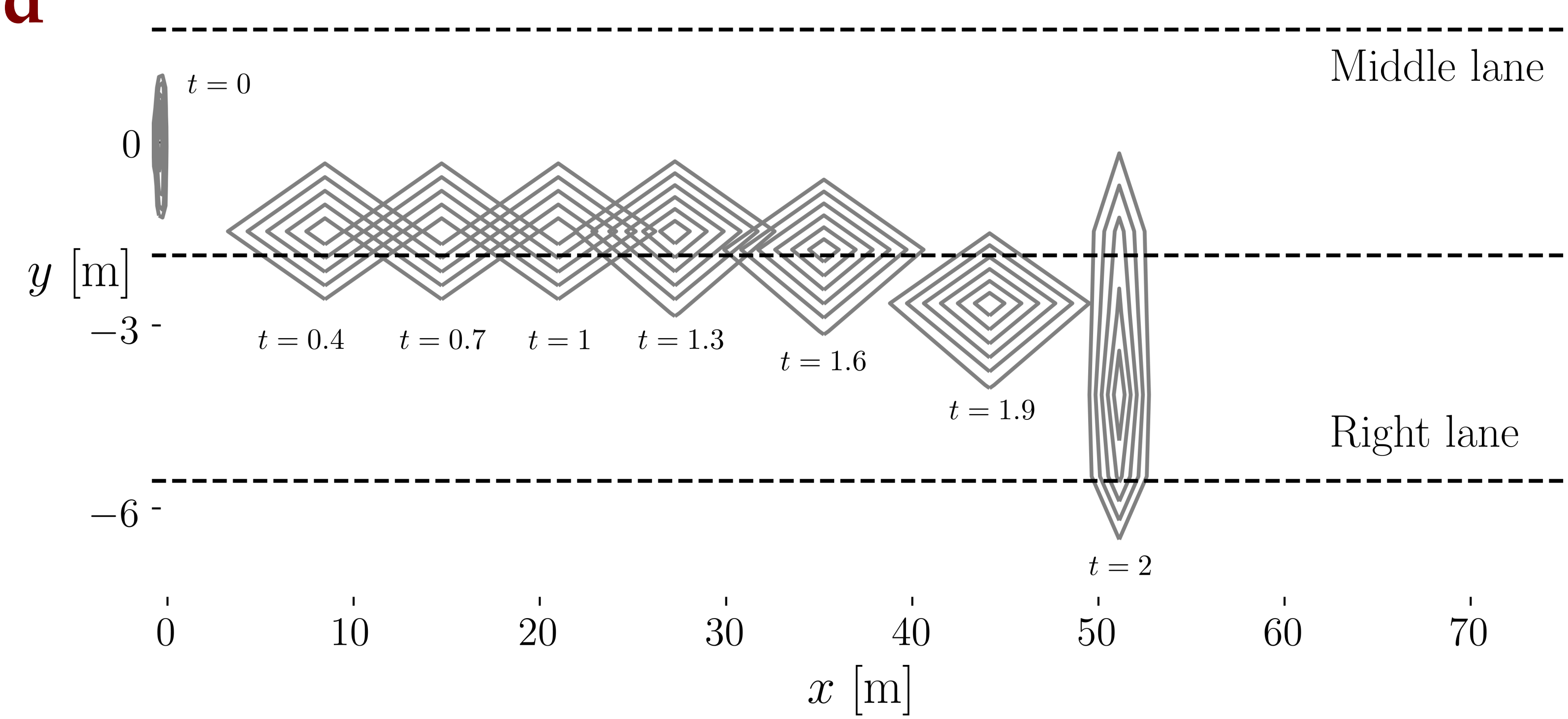
$$\begin{tikzcd}
\widehat{\varphi}_0 \arrow[rr, "\int_{\mathcal{Z}_0} \kappa(t_0, \boldsymbol{z}_0, t_0+T, \boldsymbol{z}_T)\, \widehat{\varphi}_0(\boldsymbol{z}_0) \mathrm{d}\boldsymbol{z}_0"] & & \widehat{\varphi}_T \arrow[d, "\left(\boldsymbol{\tau}_{\sharp}\rho_T^{\text{desired}}\right) \oslash \widehat{\varphi}_T"] \\
\varphi_0 \arrow[u, "\left(\boldsymbol{\tau}_{\sharp}\rho_0^{\text{ego}}\right) \oslash \varphi_0"] & & \varphi_T \arrow[ll, "\int_{\mathcal{Z}_T} \kappa(t_0, \boldsymbol{z}_0, t_0+T, \boldsymbol{z}_T)\, \varphi_T(\boldsymbol{z}_T) \mathrm{d}\boldsymbol{z}_T"]
\end{tikzcd}$$

Numerical Simulation

Target PDF



Optimal controlled xy marginals



Summary

Moving horizon nonparametric prediction of joint state PDFs

Compute safest terminal PDF for the ego vehicle

Feedback synthesis for joint PDF steering for the ego vehicle

Thank You

Support: Ford University Research Project