Boundary and Taxonomy of Integrator Reach Sets

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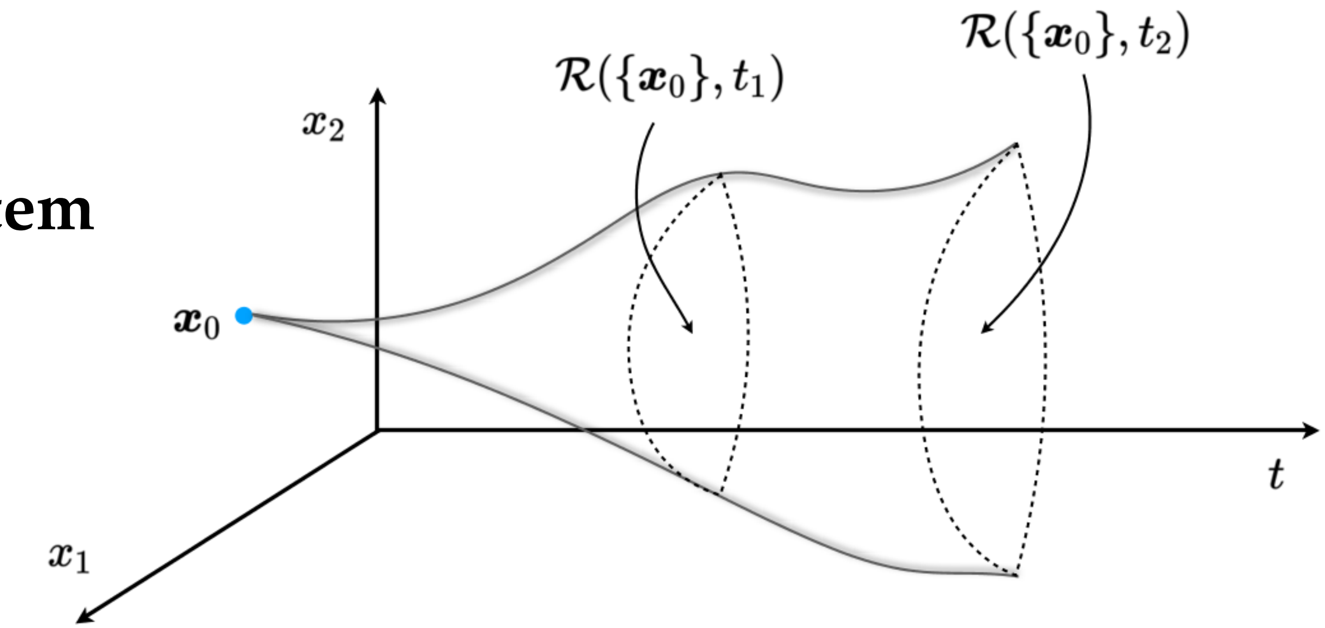
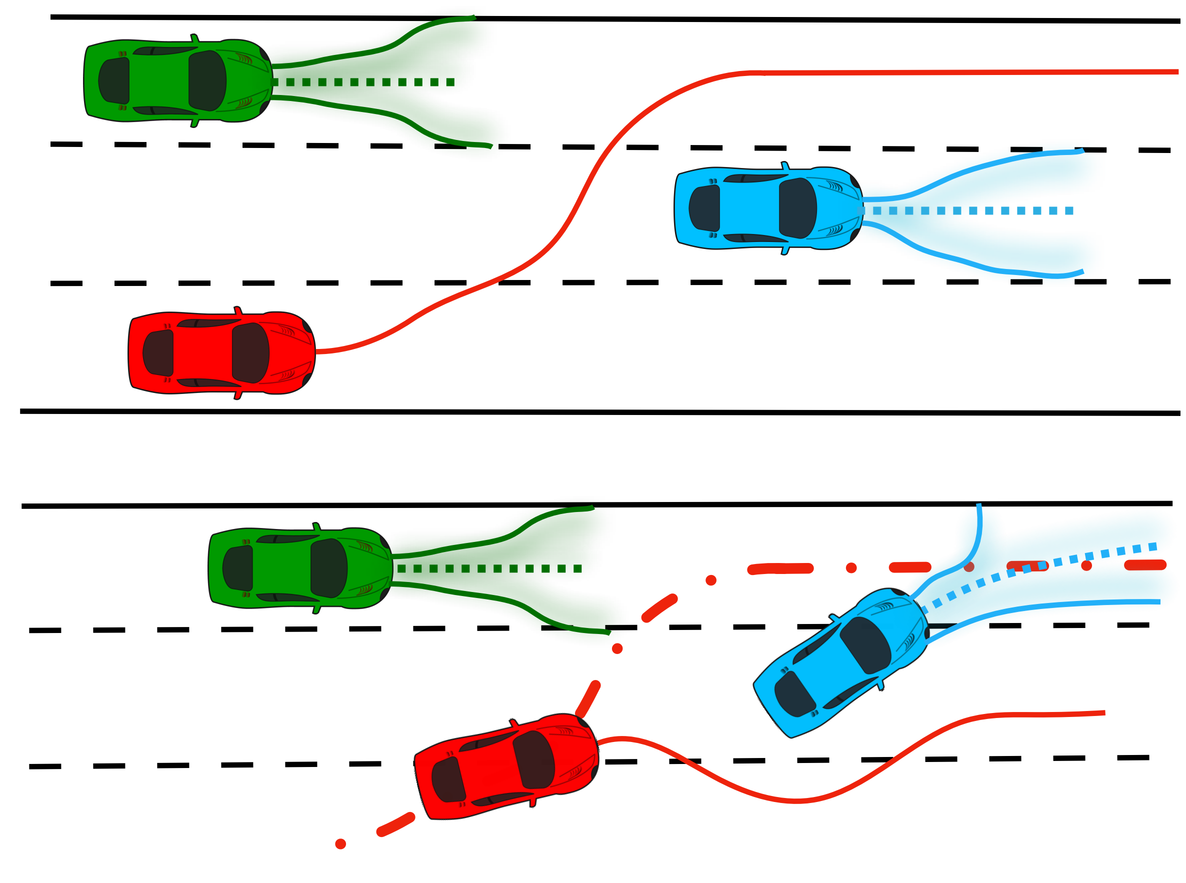
Reach set

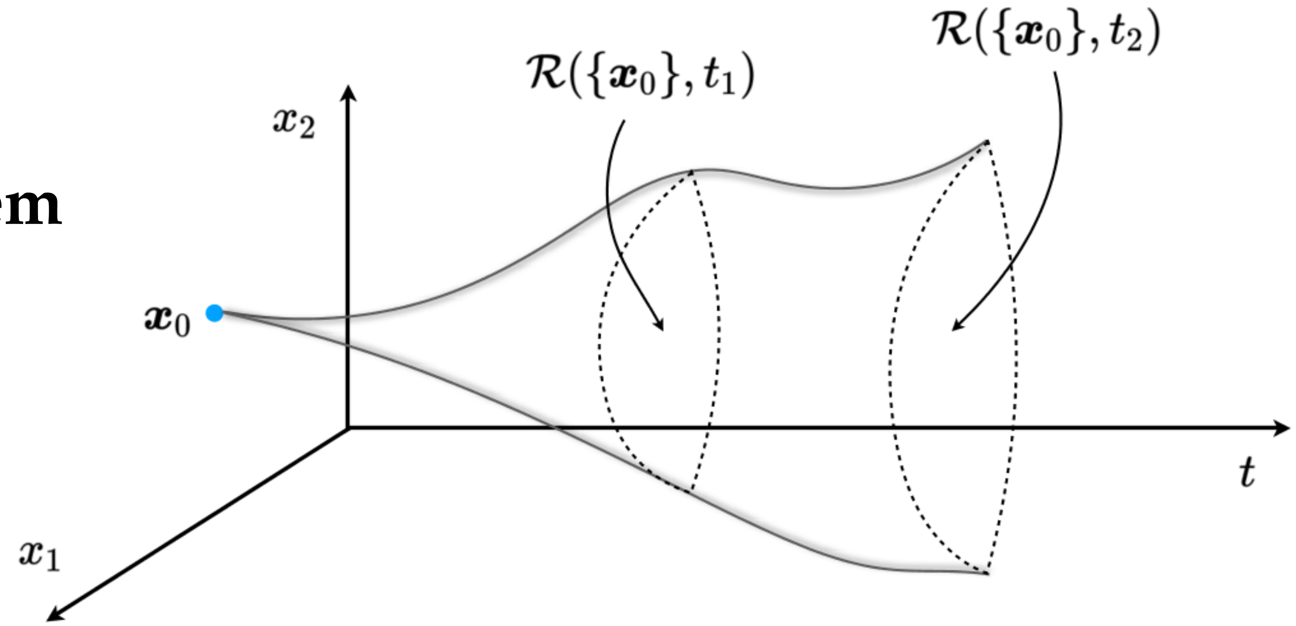


Predicting the states of an uncertain system



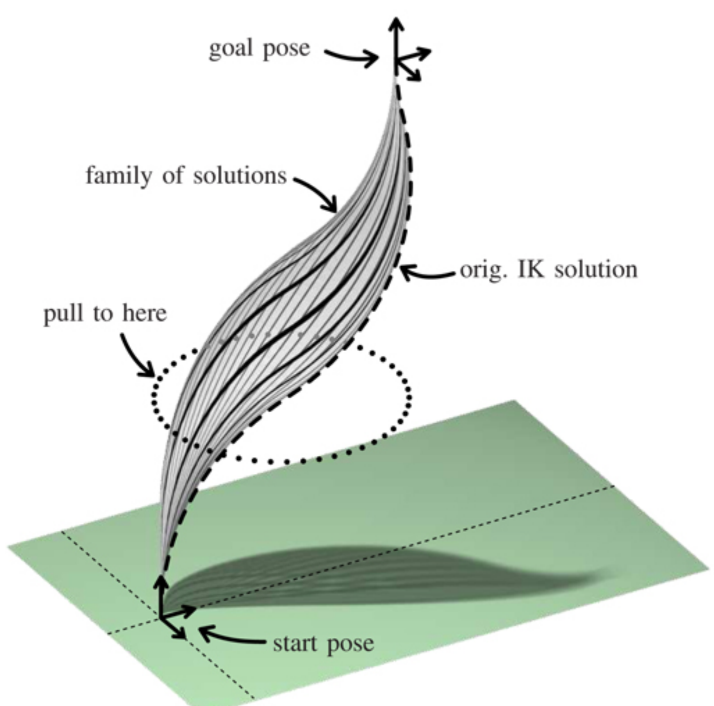
Safety critical applications such as motion planning & collision warning systems

Reach set

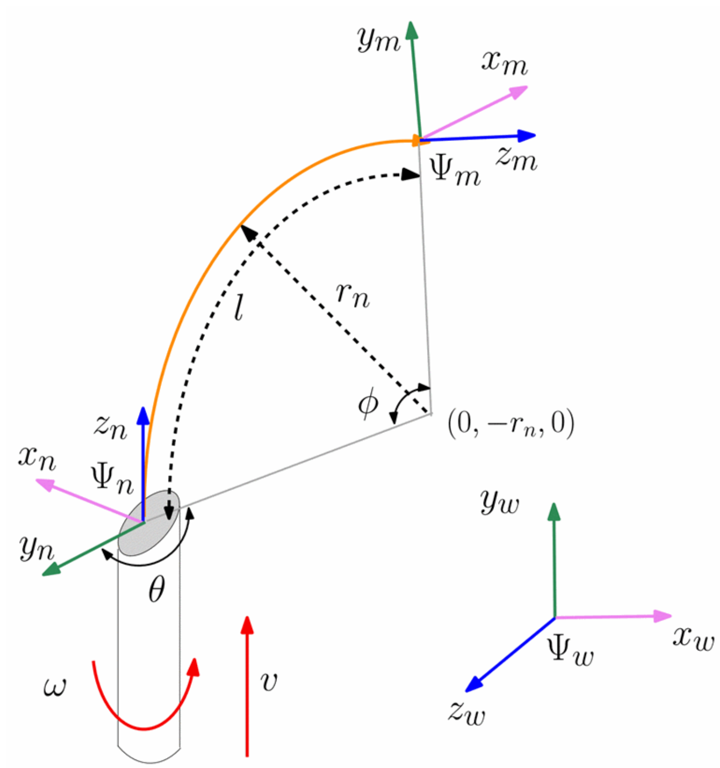


Predicting the states of an uncertain system

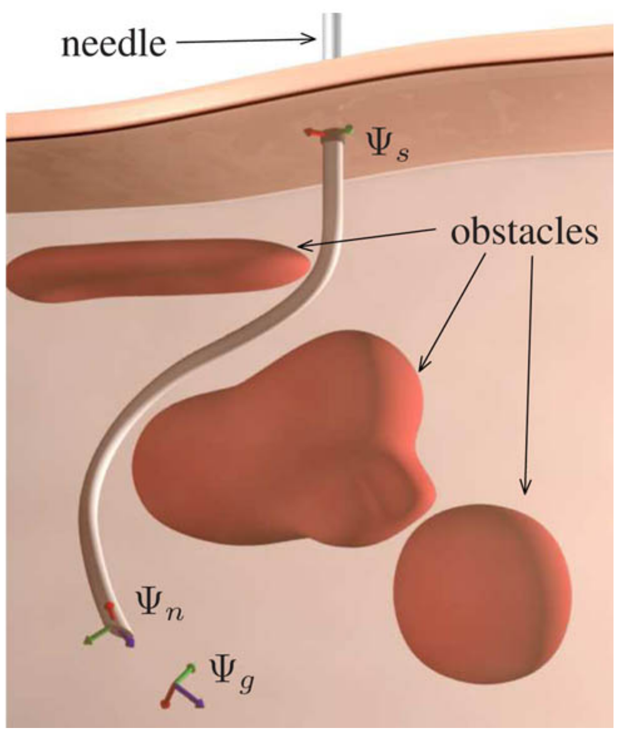
Needle steering w. input uncertainties



Credit: Duindam *et al., 2009*



Credit: Patil and Alterovitz, *2010*

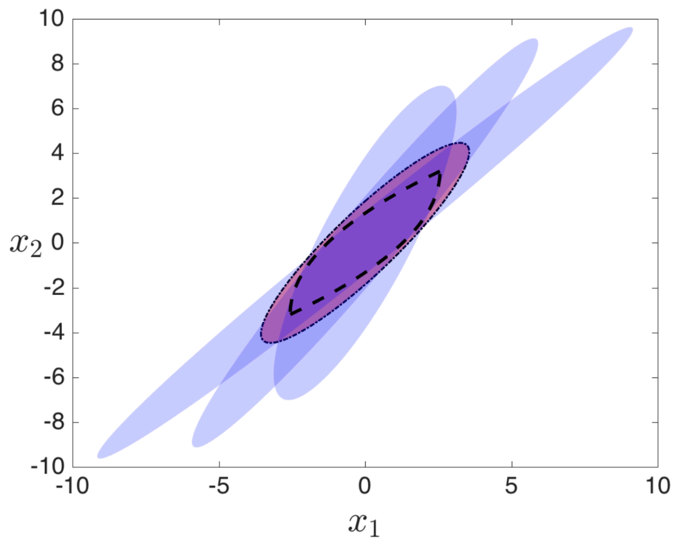


Credit: Duindam *et al., 2009*

Existing algorithms for reach set computation

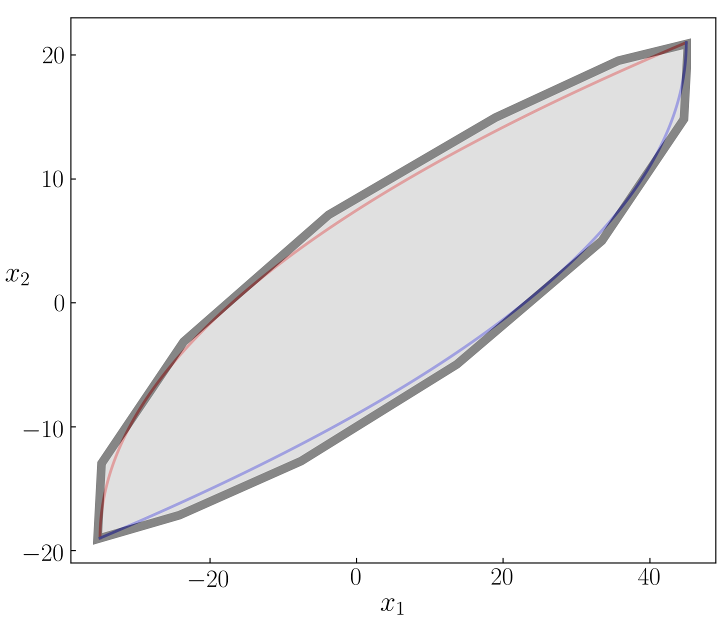
Parametric

Ellipsoidal over-approximation



Elipsoidal toolbox
[Kurzhansky et al., 2006]

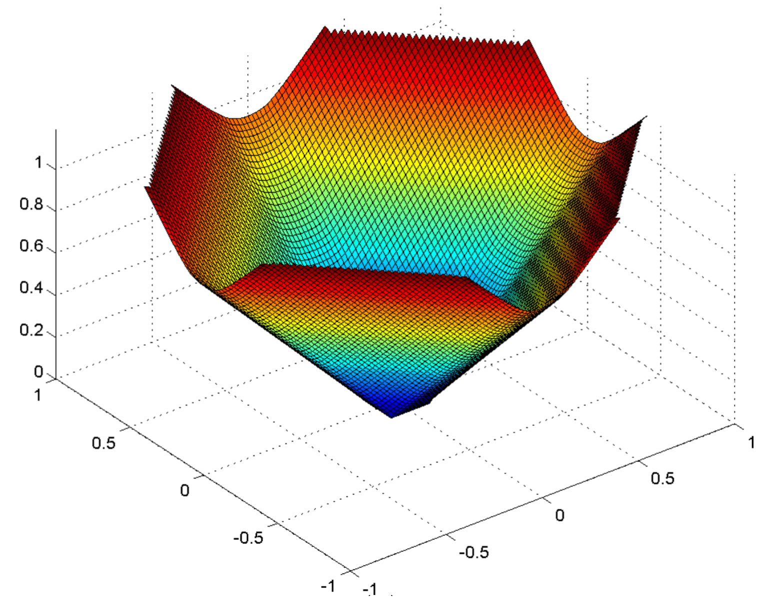
Zonotopic over-approximation

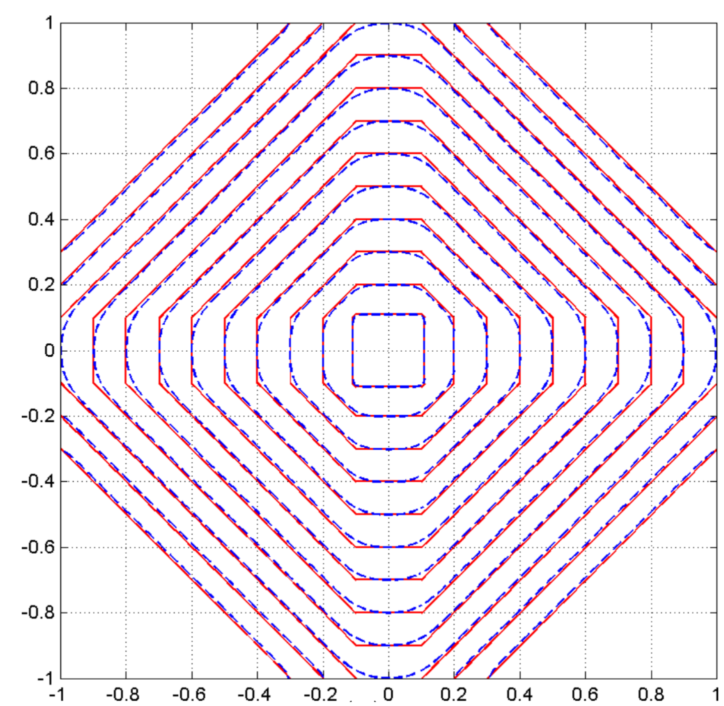


CORA toolbox
[Althoff et al., 2015]

Nonparametric

Zero sub-level set of the viscosity solution of HJB PDE

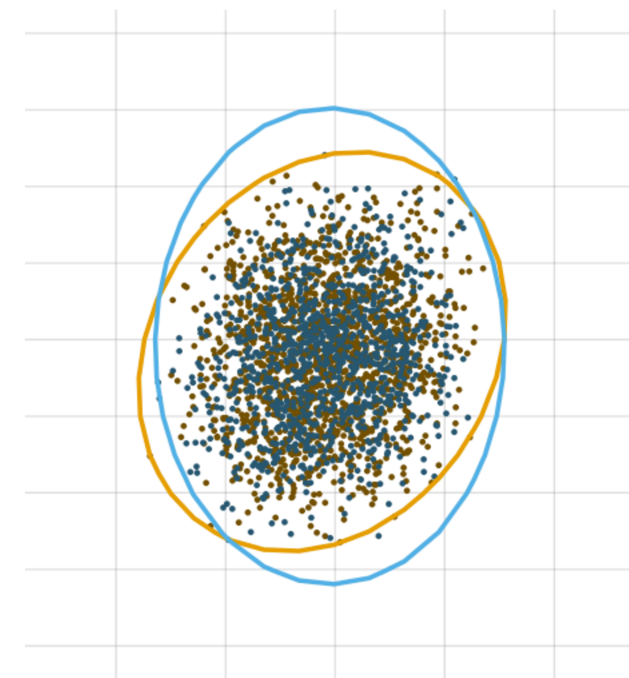


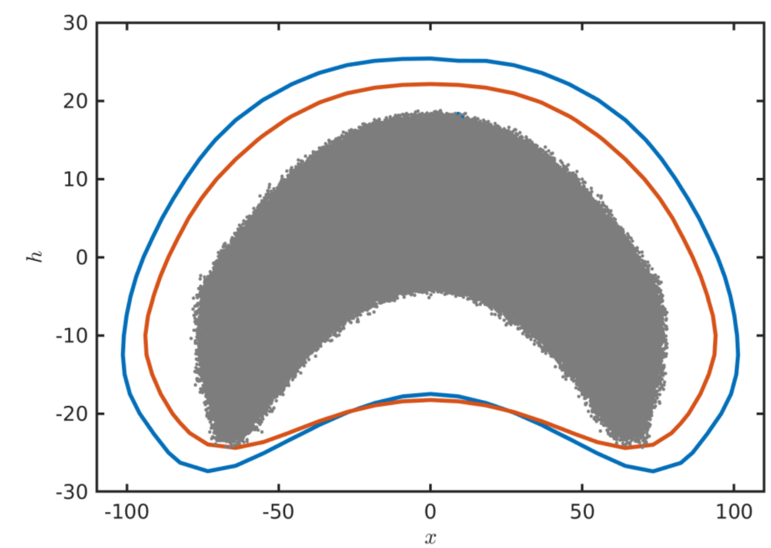


Level set toolbox
[Mitchell et al., 2008]

Semiparametric

Sample-based statistical learning





[Devonport and Arcak, 2020]

Existing algorithms for reach set computation

No specific algebraic or topological results about the ground truth

Difficult to quantitatively compare performance between two given algorithms

One-size-fits-all algorithms ignore the specific geometry induced by different class of systems

Our approach

Generic $\longrightarrow$ specific algorithm exploiting geometry of the true set

Contribution

Computing the exact reach set of integrator dynamics

$$\mathcal{R}\left(\{\boldsymbol{x}_0\}, t\right) := \left\{\boldsymbol{x}(t) \in \mathbb{R}^d \;\; | \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),\; \boldsymbol{u}(t) \in \mathcal{U}\right\}$$

$$\mathcal{U} := [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times [\alpha_m, \beta_m] \subset \mathbb{R}^m$$ **Boxed-valued input set**

$$\boldsymbol{A} := \text{blkdiag}(\boldsymbol{A}_1, ..., \boldsymbol{A}_m)\,, \quad \boldsymbol{B} := \text{blkdiag}(\boldsymbol{b}_1, ..., \boldsymbol{b}_m)\,,$$

$$\boldsymbol{A}_j = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \ldots & \boldsymbol{e}_{r_j - 1} \end{bmatrix}, \quad \boldsymbol{b}_j = \boldsymbol{e}_{r_j}$$ **Brunovsky normal form**

$$\boldsymbol{r} = (r_1, r_2, \ldots, r_m)^\top \in \mathbb{Z}_+^m$$ **Relative degree vector**

These reach sets are in general, compact and convex

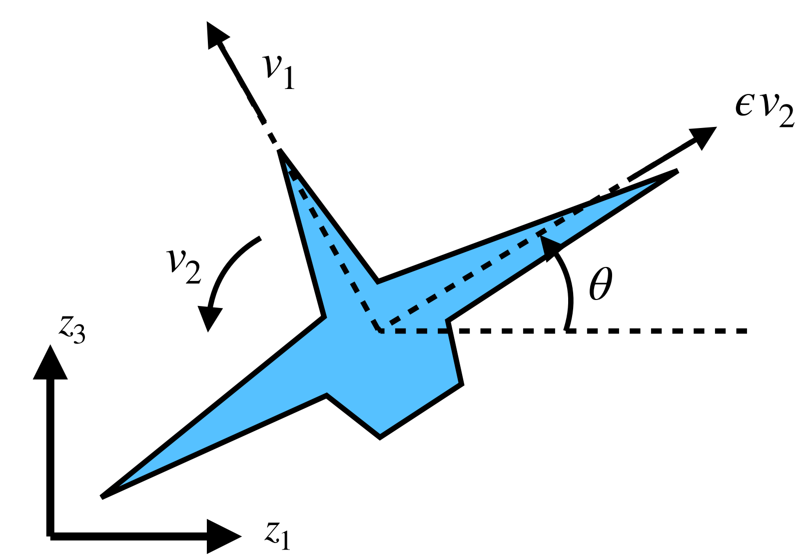
Motivation

Benchmarking the performance of over-approximation algorithms

Estimating the reach set of differentially flat nonlinear systems

Dynamics of VTOL aircraft

$$\dot{z}_1 = z_2 \qquad \dot{z}_4 = v_1 \cos(z_5) + \epsilon v_2 \sin(z_5) - g$$
$$\dot{z}_2 = -v_1 \sin(z_5) + \epsilon v_2 \cos(z_5) \qquad \dot{z}_5 = z_6$$
$$\dot{z}_3 = z_4 \qquad \dot{z}_6 = v_2$$



The VTOL aircraft

Normal form

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_2 \end{pmatrix}$$

Compute the reach set and its functionals in normal coordinate x → **Map them back to original coordinate z via known diffeomorphism**

Support function

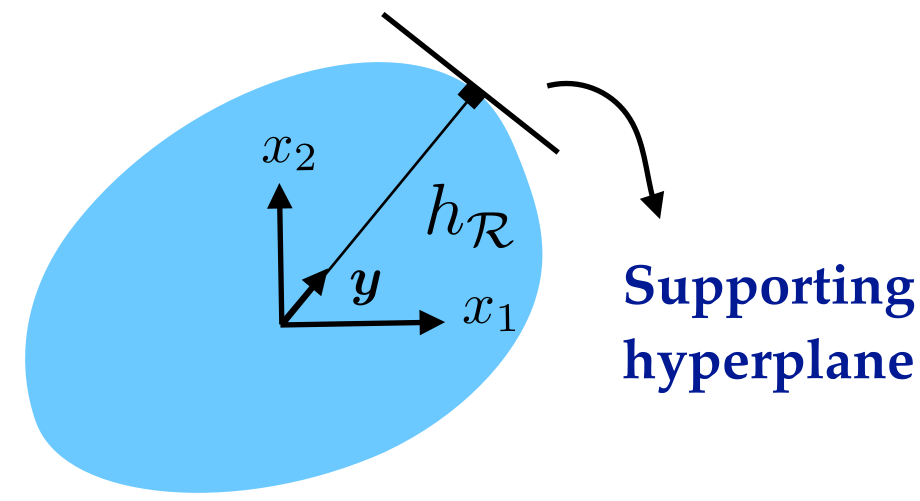
Theorem: The support function of $\mathcal{R}(\{\boldsymbol{x}_0\}, t)$ is

$$h_{\mathcal{R}(\{\boldsymbol{x}_0\},t)}(\boldsymbol{y}) = \sum_{j=1}^{m} \left\{ \langle \boldsymbol{y}_j, \exp(t\boldsymbol{A})\boldsymbol{x}_{j0} \rangle + \nu_j \langle \boldsymbol{y}_j, \boldsymbol{\zeta}_j(t) \rangle + \mu_j \int_0^t |\langle \boldsymbol{y}_j, \boldsymbol{\xi}_j(s) \rangle| \, \mathrm{d}s \right\}$$

where $\mu_j := \dfrac{\beta_j - \alpha_j}{2}, \quad \nu_j := \dfrac{\beta_j + \alpha_j}{2}, \; j = 1, \cdots, m,$

$$\boldsymbol{\zeta}_j(t_0, t) := \int_{t_0}^{t} \boldsymbol{\xi}_j(s) \, \mathrm{d}s \in \mathbb{R}^{r_j}$$

$$\boldsymbol{\xi}(s) := \begin{pmatrix} \mu_1 \boldsymbol{\xi}_1(s) \\ \vdots \\ \mu_m \boldsymbol{\xi}_m(s) \end{pmatrix}, \boldsymbol{\xi}_j(s) := \begin{pmatrix} s^{r_j-1}/(r_j-1)! & s^{r_j-2}/(r_j-2)! & \cdots & s & 1 \end{pmatrix}^\top$$



$\mathcal{R}(\{\boldsymbol{x}_0\}, t)$ over-approximates the integrator reach set with any compact $\mathcal{U}$

$$\alpha_j := \min_{\boldsymbol{u} \in \mathcal{U}} u_j, \quad \beta_j := \max_{\boldsymbol{u} \in \mathcal{U}} u_j, \quad j = 1, \cdots, m,$$

Parametric formula of boundary

Theorem. Parametric boundary of $\mathcal{R}\left(\{\boldsymbol{x}_0\}, t\right)$

Components of the boundary

$$\boldsymbol{x}_j^{\text{bdy}}(k) = \sum_{\ell=1}^{r_j} \mathbf{1}_{k\leq\ell} \frac{t^{\ell-k}}{(\ell-k)!} \boldsymbol{x}_{j0}(\ell) + \frac{\nu_j\, t^{r_j-k+1}}{(r_j-k+1)!}$$
$$\pm \frac{\mu_j}{(r_j-k+1)!}\left\{(-1)^{r_j-1}\, t^{r_j-k+1} + 2\sum_{q=1}^{r_j-1}(-1)^{q+1}\, s_q^{r_j-k+1}\right\},$$

Parameters: $0 \leq s_1 \leq s_2 \leq \ldots \leq s_{r_j-1} \leq t, \quad j = 1, \ldots, m$

Each single input integrator reach set has two bounding surfaces:

$$\mathcal{R}_j(\{\boldsymbol{x}_{0j}\}, t) = \{\boldsymbol{x} \in \mathbb{R}^{r_j} \mid p_j^{\text{upper}}(\boldsymbol{x}) \leq 0,\ p_j^{\text{lower}}(\boldsymbol{x}) \leq 0\},$$

with boundary:

$$\partial\mathcal{R}_j\left(\{\boldsymbol{x}_{j0}\}, t\right) = \{\boldsymbol{x} \in \mathbb{R}^{r_j} \mid p_j^{\text{upper}}(\boldsymbol{x}) = 0\} \cup \{\boldsymbol{x} \in \mathbb{R}^{r_j} \mid p_j^{\text{lower}}(\boldsymbol{x}) = 0\}.$$

Implicit formula of boundary

Generating function of the parametric form:

$$F(\tau) = \sum_{k \geq 0} A_k \tau^k = \frac{(1 - s_1\tau)(1 - s_3\tau)\cdots}{(1 - s_2\tau)(1 - s_4\tau)\cdots}, \qquad \textbf{(1)}$$

Taking the logarithmic derivative for $q = 1, \cdots, n_x - 1$

$$\frac{F'(\tau)}{F(\tau)} = -s_1 \sum_{k \geq 0} (s_1\tau)^k + s_2 \sum_{k \geq 0} (s_2\tau)^k - s_3 \sum_{k \geq 0} (s_3\tau)^k + \ldots,$$

Integrating with respect to τ:

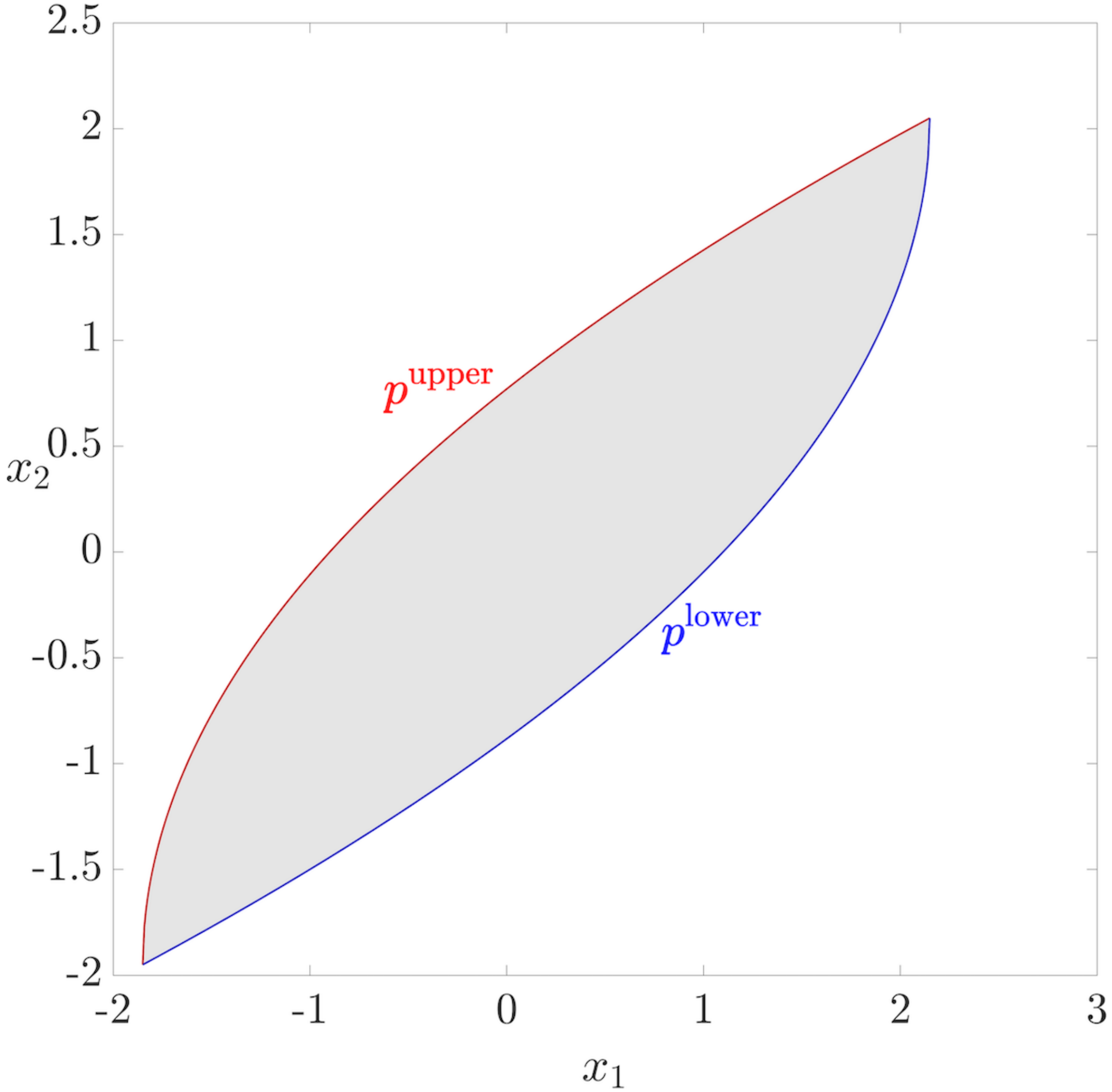
$$F(\tau) = \exp\left(-\sum_{k=1}^{n_x} \frac{\lambda_k}{k} \tau^k\right), \qquad \textbf{(2)}$$

Equating (1) and (2), the following Hankel determinant gives implicit formula

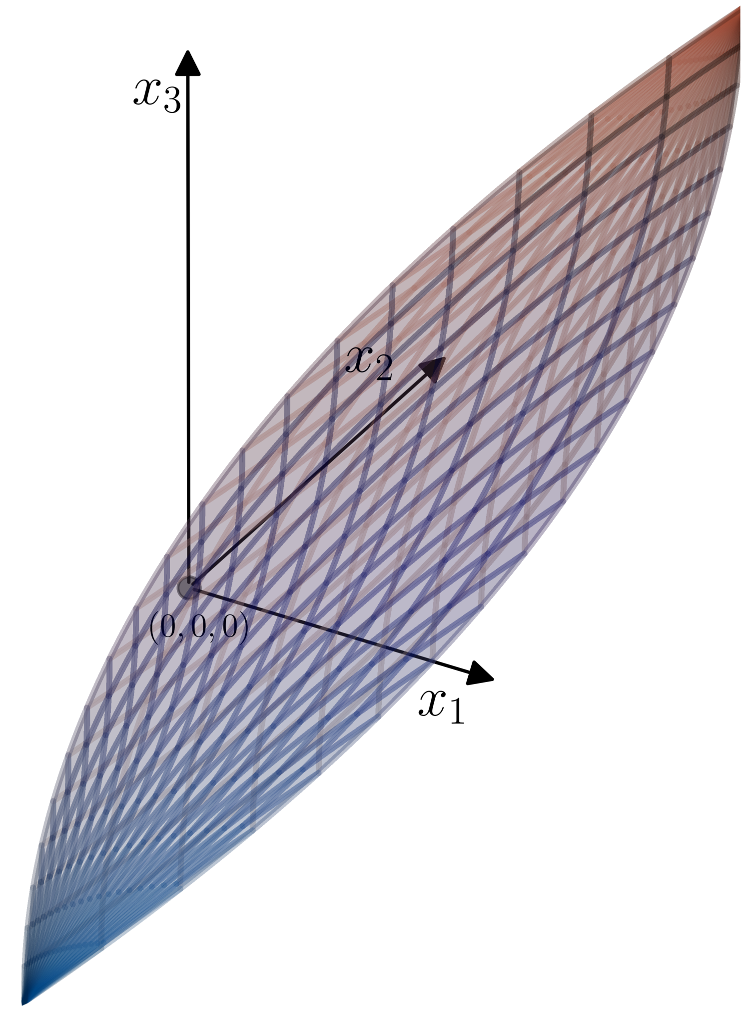
$$\det[A_{n_x - 2\delta + i + j}]_{i,j=0}^{\delta} = 0.$$

Taxonomy

Theorem. The set $\mathcal{R}\left(\{\boldsymbol{x}_0\}, t\right)$ is semialgebraic



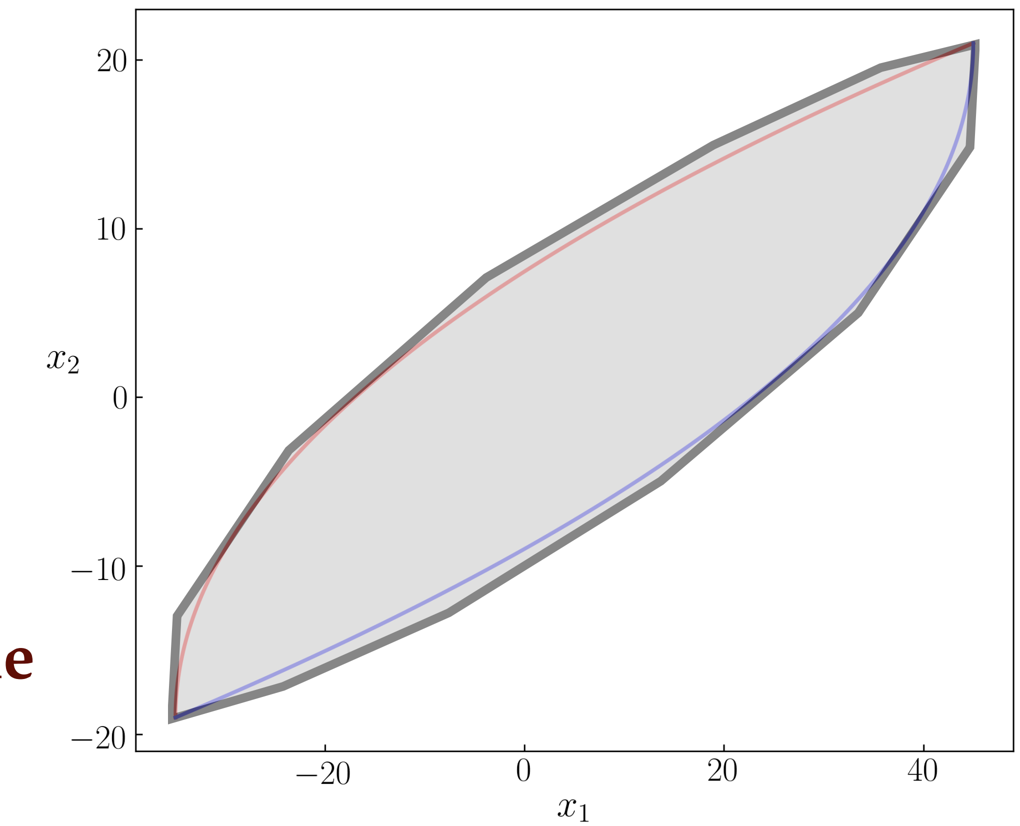
The single input double integrator reach set



The single input triple integrator reach set

Taxonomy

Theorem. The set $\mathcal{R}\left(\{\boldsymbol{x}_0\}, t\right)$ is a zonoid



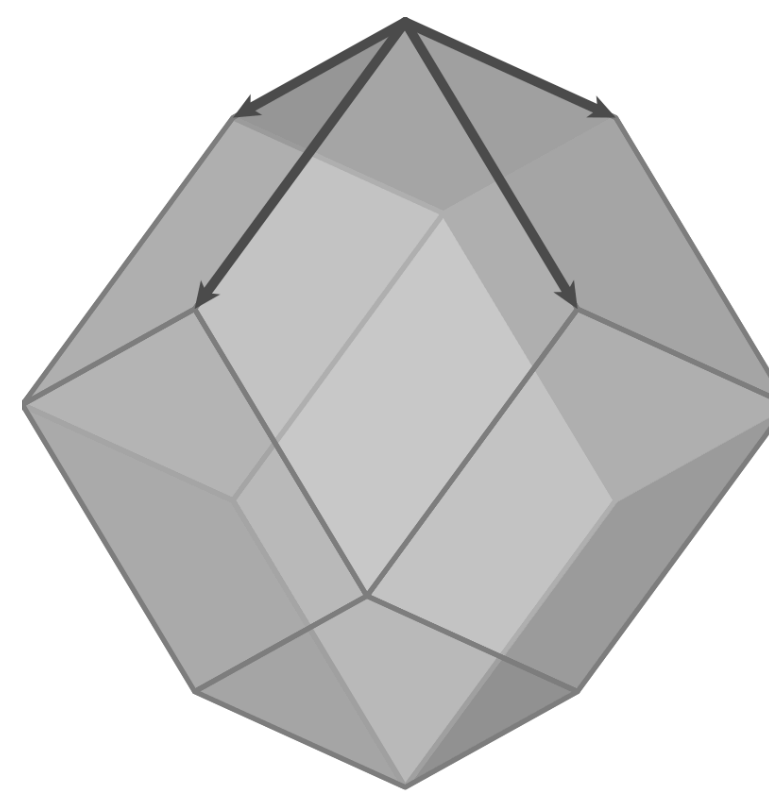
Zonoid: Limiting set of the Minkowski sum of line segments

Zonotope of dimension d

Generators

$$\mathcal{Z}_n := \left\{ \sum_{j=1}^{n} \gamma_j \boldsymbol{v}_j \,|\, \gamma_j \in [-1, 1], \boldsymbol{v}_j \in \mathbb{R}^d, j = 1, \ldots, n \right\}.$$

$$h_{\mathcal{Z}_n}(\boldsymbol{y}) = \sum_{j=1}^{n} |\langle \boldsymbol{y}, \boldsymbol{v}_j \rangle|, \quad \boldsymbol{y} \in \mathbb{R}^d$$

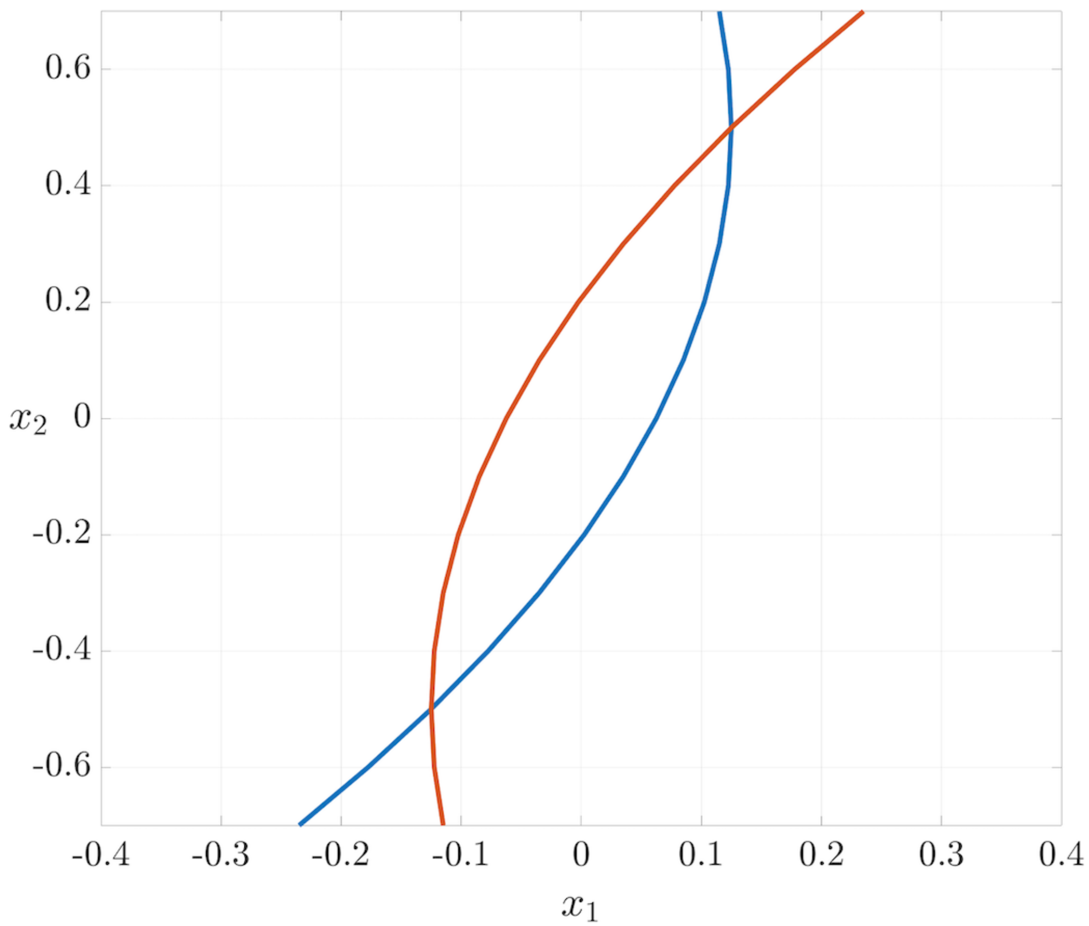


Taxonomy

Integrator Reach Set Is Not Spectrahedron

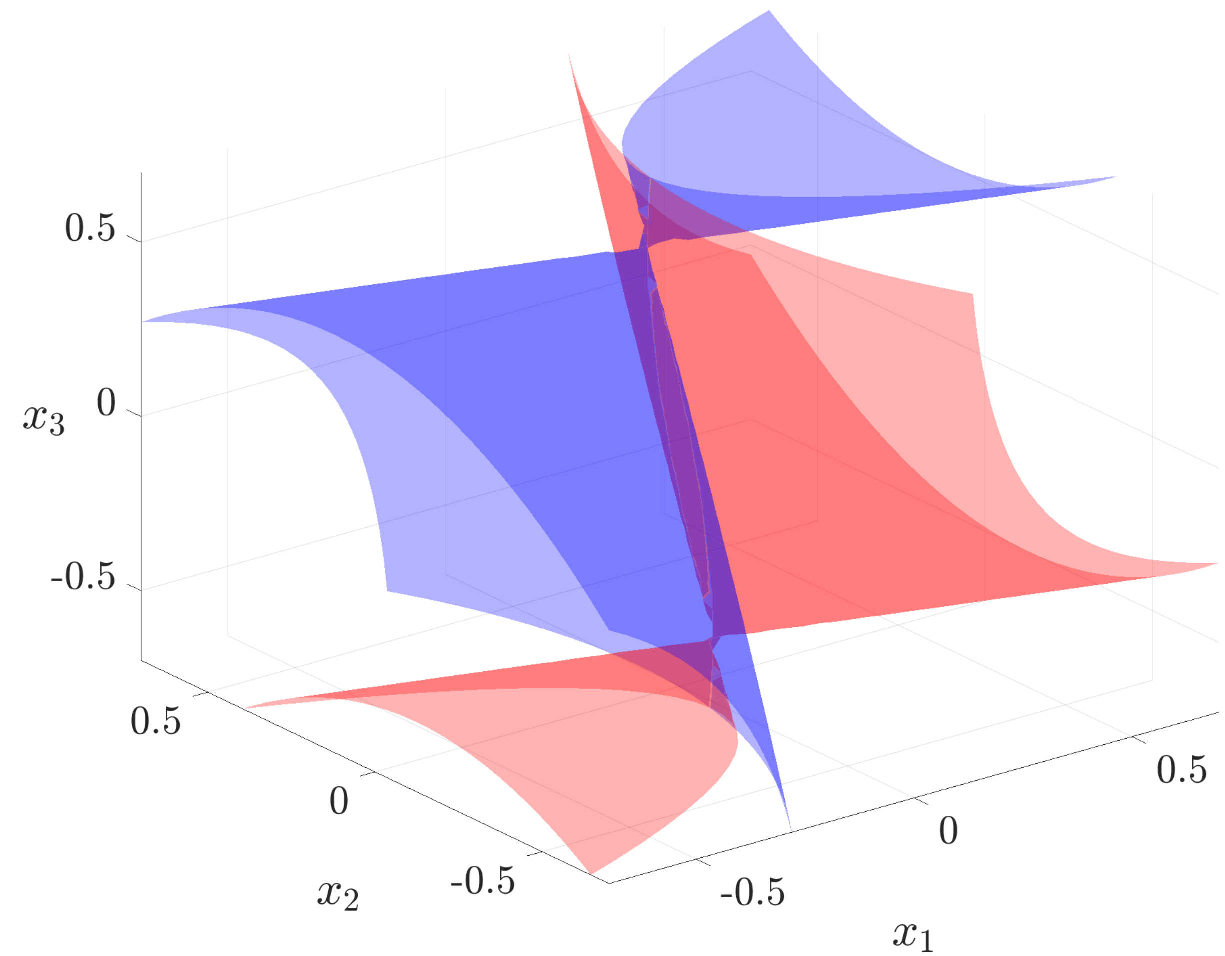
Polynomial degree of d dimensional integrator reach set surface:

$$\left(\left\lfloor \frac{d-1}{2} \right\rfloor + 1\right)\left(d - \left\lfloor \frac{d-1}{2} \right\rfloor\right)$$



Degree of $\partial\mathcal{R}$ is 2

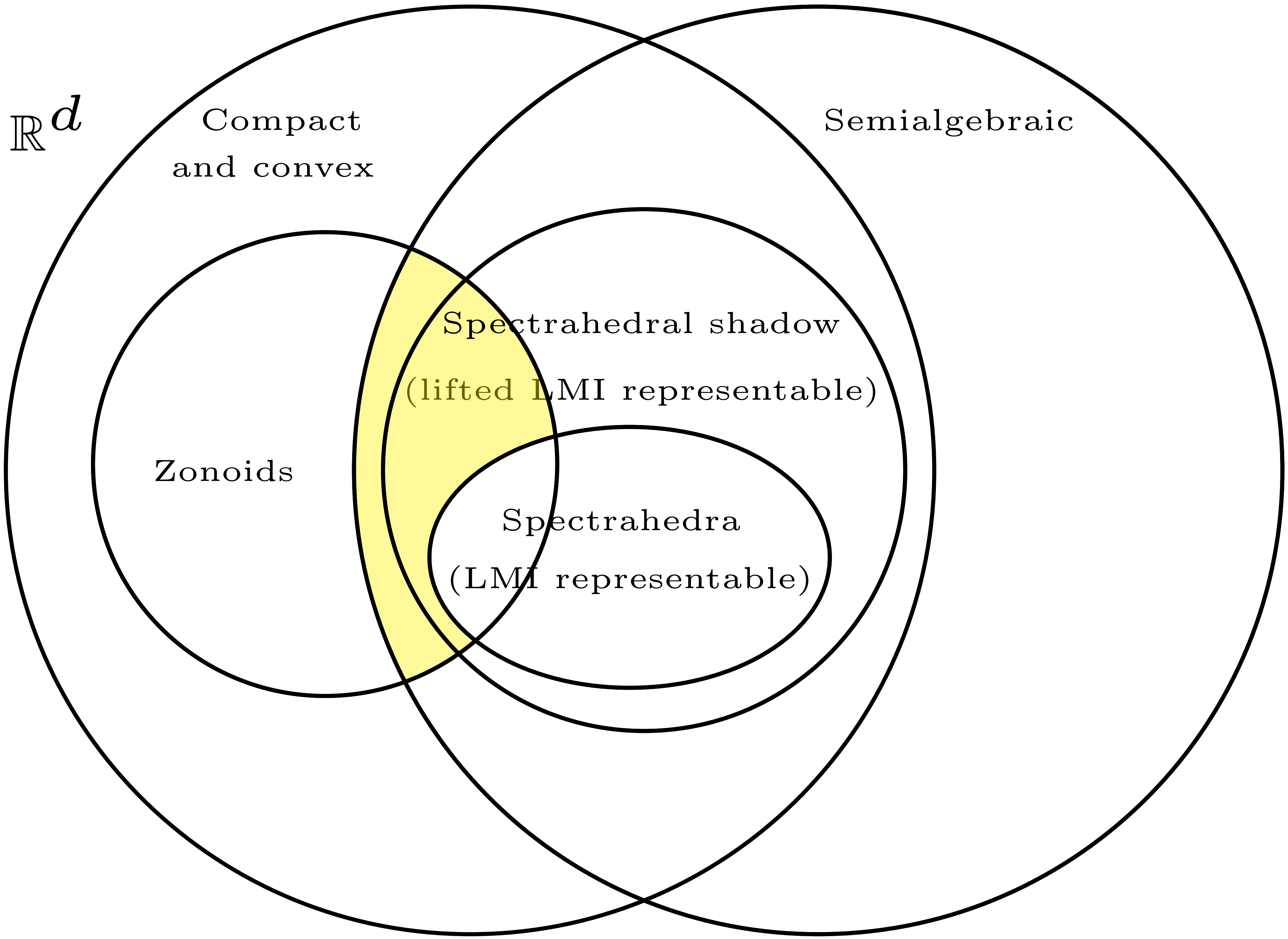
Number of intersections by generic line is 4



Degree of $\partial\mathcal{R}$ is 4

Number of intersections by generic line is 6

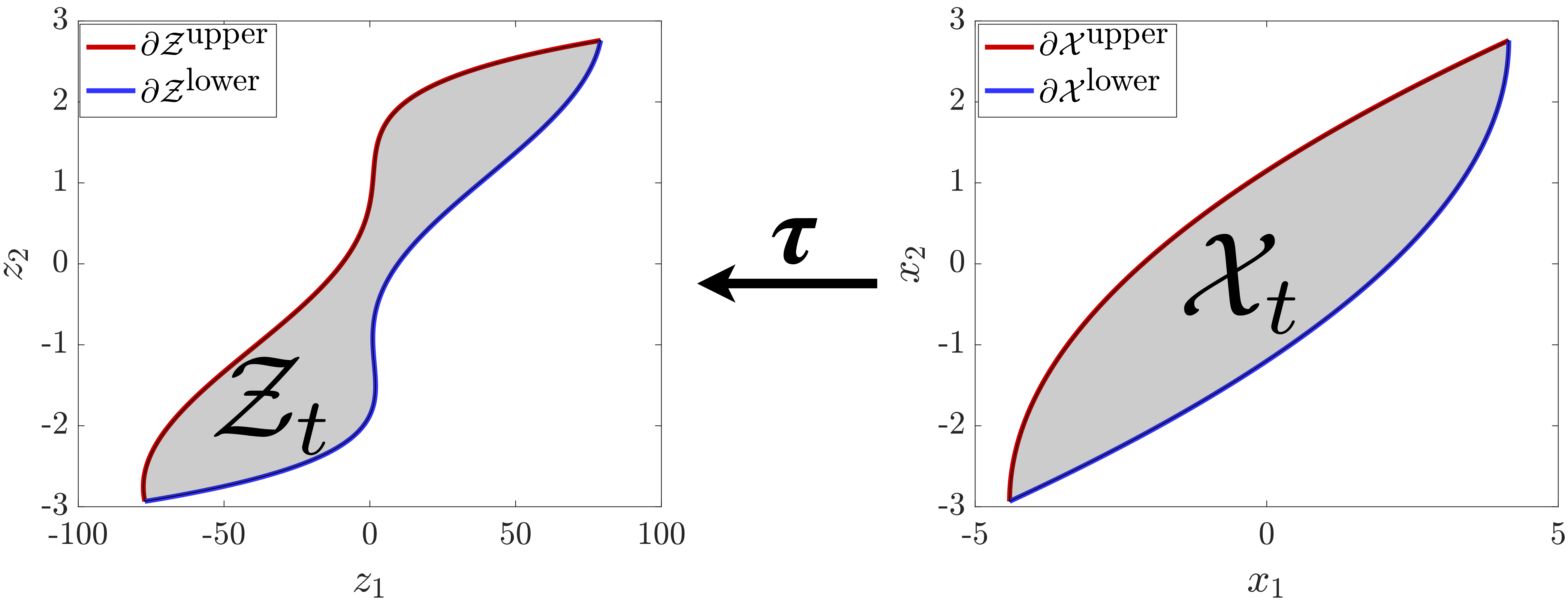
Summary of Taxonomy



Ongoing work for differentially flat systems

Map them back to original coordinate z via known diffeomorphism

Compute the reach set and its functionals in normal coordinate x



$\text{vol}\left(\mathcal{Z}_t\right) = \mathbf{206.7362}$

$\text{vol}\left(\mathcal{X}_t\right) = \mathbf{15.4292}$

Thank You