**AERO 320: Numerical Methods**
Fall 2013

Mid Term 1

**Academic Integrity**. I will enforce the Aggie Code of Honor: "An Aggie does not lie, cheat or steal, or tolerate those who do." There is a zero tolerance policy for academic dishonesty.

Name:________________________________________ Date:_________________

For this exam you only need a pen/pencil and a calculator. Write your answers in the spaces provided. If you need more space, work on the other side of the page.
Unless explicitly mentioned, use your calculator's precision for all your calculations.

1. The function $\sin(x)$ has a Taylor expansion of the form

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots$$

(a) For $x = 1$, approximate $\sin(x)$ using the right-hand-side with (i) just the first term, (ii) the first and second terms, and (iii) the three terms.

(i) 1, (ii) $1 - \frac{1}{6} = \frac{5}{6}$, (iii) $1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120}$.

(b) Compute the exact value of $\sin(x)$ at $x = 1$ (radians). Estimate *absolute* and *relative errors* as well as *significant digits* for the three approximations above.

Exact value: $\sin(1) \approx 0.8415$, keeping four significant digits (s.d.).

| Approximation | Absolute error | Relative error |
|---|---|---|
| 1 term | $\lvert\sin(1) - 1\rvert \approx 0.1585$ (s.d. $= 4$) | $\frac{\lvert\sin(1) - 1\rvert}{\lvert\sin(1)\rvert} \approx 0.1884$ (s.d. $= 4$) |
| 2 term | $\lvert\sin(1) - \frac{5}{6}\rvert \approx 0.0081$ (s.d. $= 4$) | $\frac{\lvert\sin(1) - \frac{5}{6}\rvert}{\lvert\sin(1)\rvert} \approx 0.0097$ (s.d. $= 4$) |
| 3 term | $\lvert\sin(1) - \frac{101}{120}\rvert \approx 1.9568 \times 10^{-4}$ (s.d. $= 5$) | $\frac{\lvert\sin(1) - \frac{101}{120}\rvert}{\lvert\sin(1)\rvert} \approx 2.3255 \times 10^{-4}$ (s.d. $= 5$) |

(c) What kind of error is this?

Truncation error.

(d) How can you reduce it?

By keeping more terms in the Taylor expansion.

(e) Just as we did for other sequences in class, we can determine the *order of convergence* $\alpha$ for this sequence. What is it?

$$\alpha \approx \frac{\log(e_3) - \log(e_2)}{\log(e_2) - \log(e_1)} = \frac{\log\left(1.9568 \times 10^{-4}\right) - \log(0.0081)}{\log(0.0081) - \log(0.1585)} \approx 1.2519.$$

This implies faster than linear convergence.

2. Consider the fixed point iteration

$$x_{k+1} = (\alpha + 1)\, x_k - x_k^2, \quad k = 0, 1, 2, \ldots$$

where $\alpha$ satisfying $1 \geq \alpha \geq 1/2$ is given.

(a) Show that the iteration converges for any initial guess $x_0$ satisfying

$$\alpha/2 \leq x_0 \leq \alpha/2 + 1.$$

$g(x) = (\alpha + 1)\, x - x^2, \Rightarrow g'(x) = \alpha + 1 - 2x$. For convergence, we must have $|g'(x)| < 1 \Leftrightarrow -1 < \alpha + 1 - 2x < +1 \Leftrightarrow \alpha/2 < x < \alpha/2 + 1$. For any initial guess $x = x_0$ satisfying this condition, convergence is guaranteed.

(b) What would happen if I started from $x_0 = \alpha/3$?

$|g'\left(x = \frac{\alpha}{3}\right)| = |\alpha + 1 - 2\frac{\alpha}{3}| = |\frac{\alpha}{3} + 1|$. For this choice of $x_0$, convergence is possible if $|g'\left(x = \frac{\alpha}{3}\right)| < 1 \Leftrightarrow -1 < \frac{\alpha}{3} + 1 < +1 \Leftrightarrow -6 < \alpha < 0$. This is impossible since the question tells that $\alpha$ satisfies $1 \geq \alpha \geq 1/2$. Thus, the fixed point iteration with initial guess $x_0 = \alpha/3$ will diverge.

3. For a quadratically convergent sequence, we know that

$$e_n = \lambda e_{n-1}^2,$$

where $e_n = |x_n - x_{\text{exact}}|$. Show that

$$\lambda e_n = (\lambda e_0)^{2^n}.$$

$e_n = \lambda e_{n-1}^2 = \lambda\left(\lambda e_{n-2}^2\right)^2 = \lambda\left(\lambda\left(\lambda e_{n-3}^2\right)^2\right)^2 = \ldots = \lambda^{2^n-1} e_0^{2^n}$. Multiplying both sides by $\lambda$, we get $\lambda e_n = (\lambda e_0)^{2^n}$.

4. It is possible for Newton's method to iterate forever. To see this notice that the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

cycles back and forth around a point $a$ if

$$x_{n+1} - a = -(x_n - a).$$

(a) Show that for this to happen, the function $f(x)$ must satisfy

$$\frac{f'(x)}{f(x)} = \frac{1}{2(x-a)}.$$

Eliminating $x_{n+1}$ from the two given formulas, we get $x_n - \frac{f(x_n)}{f'(x_n)} = -x_n + 2a \Rightarrow \frac{f'(x_n)}{f(x_n)} = \frac{1}{2(x_n - a)}$. Since this must happen for all $x = x_n$, we get the result by replacing $x_n$ with $x$.

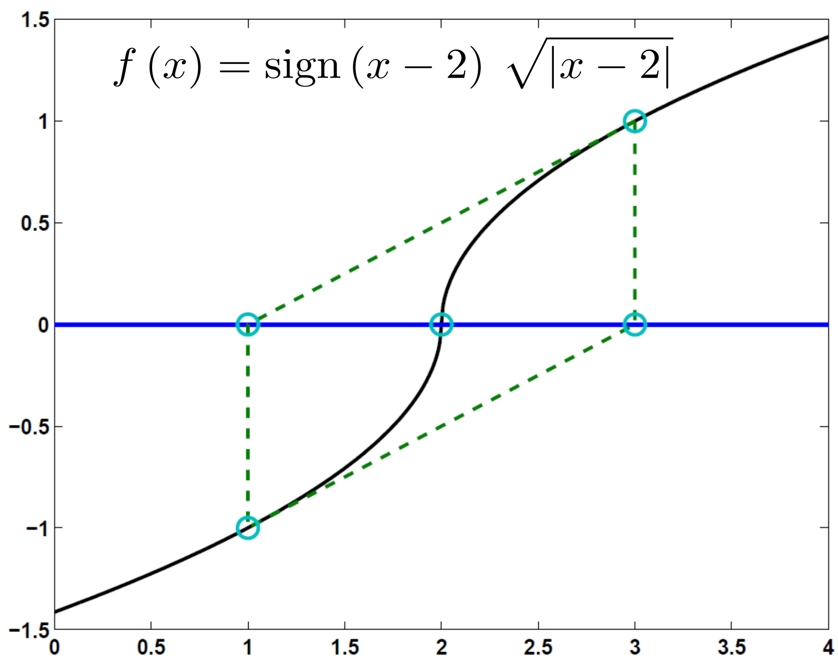
(b) From your answer to part (a), prove that the function must be

$$f(x) = \operatorname{sign}(x-a)\ \sqrt{|x-a|}.$$

We can solve the ordinary differential equation as $\frac{df}{f} = \frac{1}{2}\frac{dx}{(x-a)} \Rightarrow \ln\left(\frac{f(x)}{\operatorname{sign}(x-a)\ \sqrt{|x-a|}}\right) = \ln(1) \Rightarrow f(x) = \operatorname{sign}(x-a)\ \sqrt{|x-a|}$.

**Alternatively**, we can verify that $f(x) = \operatorname{sign}(x-a)\ \sqrt{|x-a|}$ indeed satisfies the differential equation in part (a).

(c) By hand, draw a plot of the function $f(x)$ found in part (b), for $a = 2$ and $x \in [0, 4]$. Using your plot, explain why Newton's method will fail to converge to the root $x = a$.




Newton's method fails to converge to the root $x = a$, since $f'(x) \to \infty$ as $x \to a$. For example, in the above plot, the slope at $a = 2$ is 90°.

5. Consider the following:

```
float u = 1.;
for (int i=1; 1 + u > 1; i++){
u = u/2.;
cout << u << i << endl;
}
```

   (a) Is this an infinite loop? If not, what can you say about the final value of $u$? Explain.

   No, this is *not* an infinite loop. The value of $u$ will keep decreasing until it becomes the smallest number that the `float` variable-type can represent. Then it will exit the loop. This termination of loop is due to the round-off error.

   (b) Will your answer change if I define $u$ as double precision? Explain.

   No, it will still be a *finite* loop. However, the number of iterations will increase since `double` variable-type has smaller (but still non-zero) round-off error than `float`.

6. (a) Determine the *minimum number of iterations* $n$, required by the *bisection method* to converge to within an absolute error tolerance of $\varepsilon$, starting from the initial interval $(a, b)$.

   Starting from the initial interval $(a, b)$, the bisection method generates a sequence $\{x_n\}$ that converges to the root $x$, with $|x_n - x| \leq \dfrac{b-a}{2^n}$. To converge to within an absolute error tolerance of $\varepsilon$, means we need to have $|x_n - x| \leq \varepsilon$. This will happen if $\dfrac{b-a}{2^n} \leq \varepsilon$. Solving for $n$, we get $n \geq \frac{\log\left(\frac{b-a}{\varepsilon}\right)}{\log(2)}$. Hence, the *minimum* number of iterations is the right-hand-side of the previous inequality, rounded above to its nearest integer, i.e.

$$n_{\text{minimum}} = \left\lceil \frac{\log\left(\dfrac{b-a}{\varepsilon}\right)}{\log(2)} \right\rceil .$$

   (b) Starting with the initial guess $x_0 = 0.3$, perform two iterations (i.e. find $x_1, x_2$) of *Newton's method*, to solve the equation $f(x) = \dfrac{1}{x} - 3 = 0$. What is the exact root for this problem?

   We have $f(x) = x^{-1} - 3$, $f'(x) = -x^{-2}$. Hence, Newton's iteration becomes $x_{n+1} = x_n + \dfrac{x_n^{-1} - 3}{x_n^{-2}} = 2x_n - 3x_n^2$. So we get $x_0 = 0.3$, $x_1 = 2 \times 0.3 - 3 \times (0.3)^2 = 0.33$, $x_2 = 2 \times 0.33 - 3 \times (0.33)^2 = 0.3333$.

   Exact root: $x = \dfrac{1}{3} = 0.33333\ldots$ (recurring).

   (c) Using the last two iterations in part (b), and the convergence properties you know about Newton's method, estimate the *asymptotic error constant* $\lambda$.

Since $x = \dfrac{1}{3}$ is the *unique* root, hence $\alpha = 2$ (given in last page). From part (b),
$e_1 = |x_1 - x| = |0.33 - \dfrac{1}{3}| \approx 0.003333$ (6 significant digits),
$e_2 = |x_2 - x| = |0.3333 - \dfrac{1}{3}| \approx 0.000033$ (6 significant digits). Thus, we have
$\lambda = \dfrac{e_2}{e_1^\alpha} = \dfrac{0.000033}{(0.003333)^2} \approx 2.97059$ (6 significant digits).

**Alternatively**, one can estimate $\alpha$ from part (b):
$\alpha \approx \dfrac{\log\left(\frac{e_2}{e_1}\right)}{\log\left(\frac{e_1}{e_0}\right)} = \dfrac{\log\left(\frac{0.000033}{0.003333}\right)}{\log\left(\frac{0.003333}{0.033333}\right)} \approx 2.00424$ (6 significant digits), which leads to
$\lambda \approx \dfrac{0.000033}{(0.003333)^{2.00424}} \approx 3.04331$ (6 significant digits).

**The Grading Rubric for Midterm 1 (Total 60 points)**

Problem 1: **(1 + 1 + 1) + (1 + 1 + 1) + 1 + 1 + 2 = 10 points**

Problem 2: **5 + 5 = 10 points**

Problem 3: **10 points**

Problem 4: **3 + 3 + (2 + 2) = 10 points**

Problem 5: **5 + 5 = 10 points**

Problem 6: **4 + (1 + 1 + 1) + 3 = 10 points**

# Some useful information

Theorems and definitions

- Definition of rate of convergence for a sequence of approximations $y_n$ to the real value $y$:

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^{\alpha}} = \lambda,$$

  where $e_n = |y_n - y|$. For large $n$:

$$\alpha \approx \frac{\log(e_{n+1}) - \log(e_n)}{\log(e_n) - \log(e_{n-1})}.$$

- Theorem for bisection: The error at the $n^{\text{th}}$ iteration is bounded according to

$$|x_n - x| \leq \frac{b-a}{2^n},$$

  where $(a, b)$ is the initial interval.

- The function $\operatorname{sign}(\cdot)$ is defined as follows:

$$\operatorname{sign}(y) = \begin{cases} +1 & \text{if } y > 0, \\ 0 & \text{if } y = 0, \\ -1 & \text{if } y < 0. \end{cases}$$

Some iterative methods

- $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ (order of convergence: $\alpha = 2$ for single root, $\alpha = 1$ for multiple roots)

- $x_{n+1} = x_n - f(x_n)\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$ (order of convergence: $1 < \alpha < 2$)

C++

- A non-zero number divided by zero returns "Inf".

- Zero divided by zero returns "NaN". So does the logarithm of a negative number.