Exact Computation of LTI Reach Set from Integrator Reach Set with Bounded Input

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Joint work with

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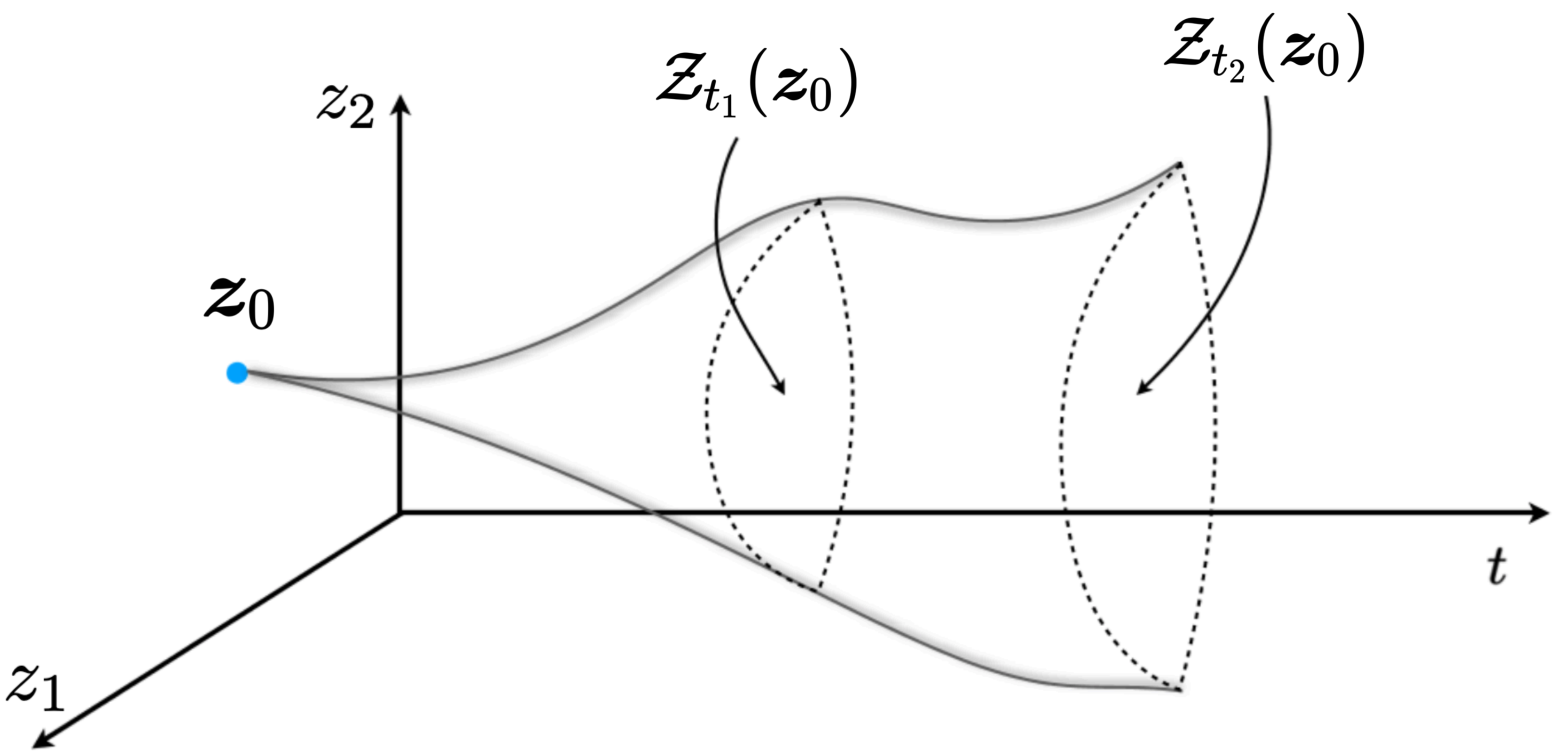
Single Input LTI Reach Set



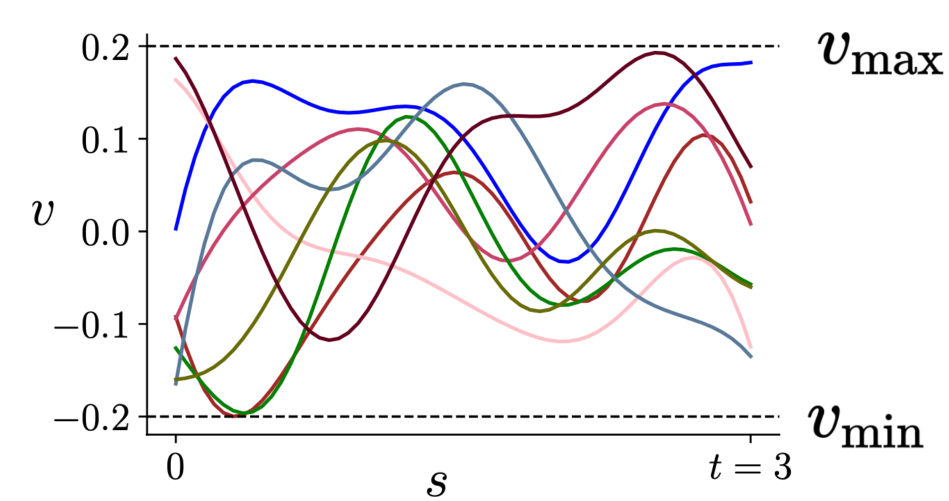
Assumption: $(\boldsymbol{A}, \boldsymbol{b})$ controllable

Reach set at time t

$$\mathcal{Z}_t(\boldsymbol{z}_0) = \bigcup_{v(\cdot)\in\mathcal{V}} \left\{ \boldsymbol{z}(t) \in \mathbb{R}^n \mid \dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{b}v,\ \boldsymbol{z}(t=0) = \boldsymbol{z}_0 \right\}$$

Set-valued input

$$\mathcal{V} = \left\{ v \in C([0,t]) \mid v(\boldsymbol{s}) \in [v_{\min}, v_{\max}] \forall \boldsymbol{s} \in [0,t] \right\}$$



Canonical Forms: Controllable and Brunovsky

Let $\boldsymbol{M} := \begin{pmatrix} \boldsymbol{q}^\top & \boldsymbol{q}^\top \boldsymbol{A} & \dots & \boldsymbol{q}^\top \boldsymbol{A}^{n-1} \end{pmatrix}^\top$

last row of the inverse of the controllability matrix

SS realization		CCF		BCF
$\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{b}v$	$\boldsymbol{z} \mapsto \boldsymbol{x} := \boldsymbol{M}\boldsymbol{z}$ ⟷	$\dot{\boldsymbol{x}} = \boldsymbol{A}_{\text{con}}\boldsymbol{x} + \boldsymbol{b}_{\text{con}}v$	$v \mapsto u := \langle \boldsymbol{c}, \boldsymbol{x} \rangle + u$ ⟷	$\dot{\boldsymbol{x}} = \boldsymbol{A}_{\text{int}}\boldsymbol{x} + \boldsymbol{b}_{\text{con}}u$

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$$\boldsymbol{A}_{\text{con}} := \boldsymbol{M}\boldsymbol{A}\boldsymbol{M}^{-1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -c_0 & -c_1 & -c_2 & \dots & -c_{n-1} \end{pmatrix} = \begin{bmatrix} \boldsymbol{0}_{(n-1)\times 1} & \boldsymbol{I}_{n-1} \\ \multicolumn{2}{c}{-\boldsymbol{c}^\top} \end{bmatrix}$$

coeff vector for charpoly of A

$$\boldsymbol{b}_{\text{con}} := \boldsymbol{M}\boldsymbol{b} = (0 \quad 0 \quad \dots \quad 1)^\top$$

$$\boldsymbol{A}_{\text{int}} := \begin{bmatrix} \boldsymbol{0}_{(n-1)\times 1} & \boldsymbol{I}_{n-1} \\ \multicolumn{2}{c}{\boldsymbol{0}_{1\times n}} \end{bmatrix}$$

Idea for Reach Set Computation

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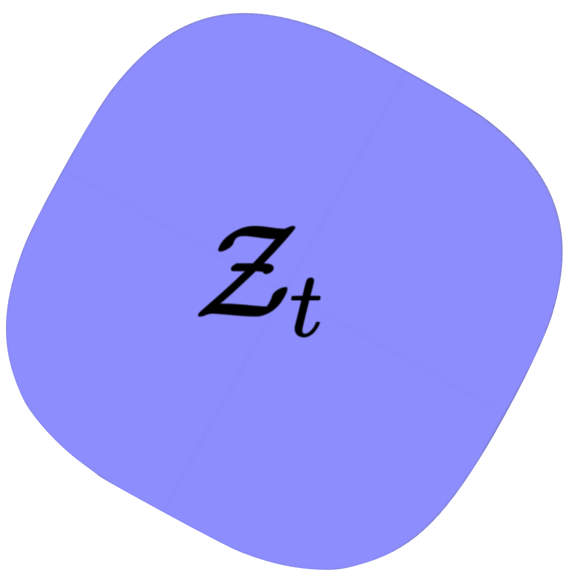

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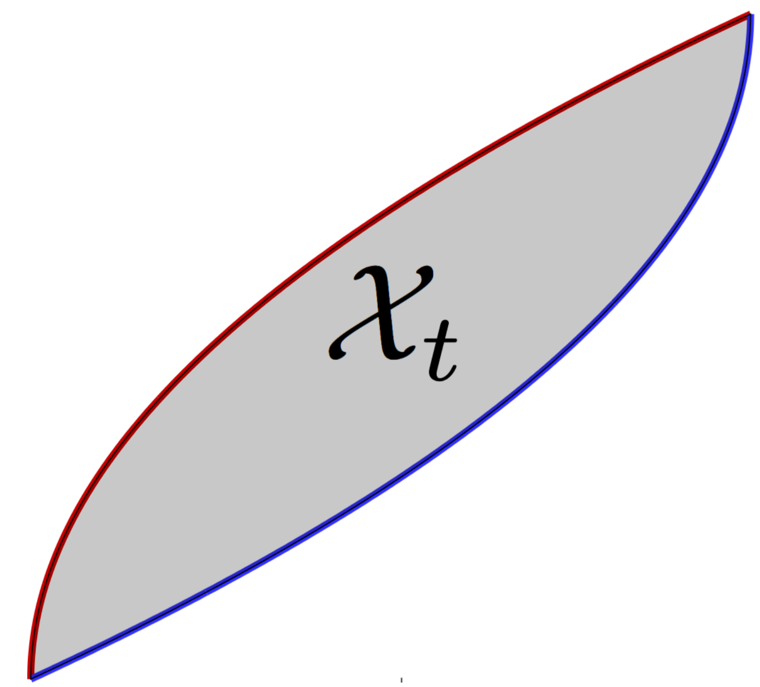
topologically equivalent



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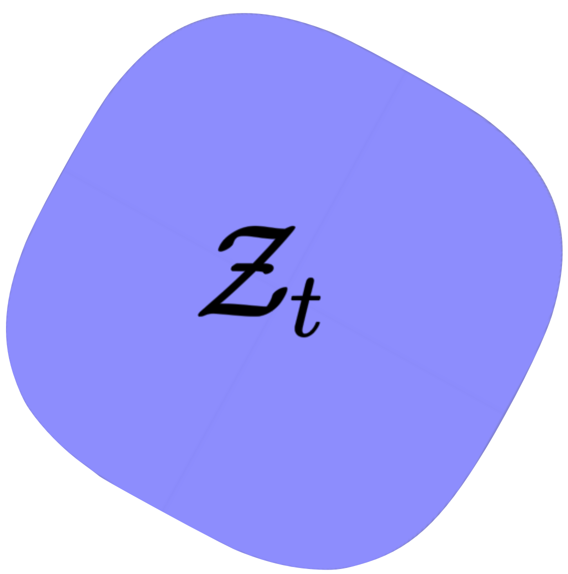

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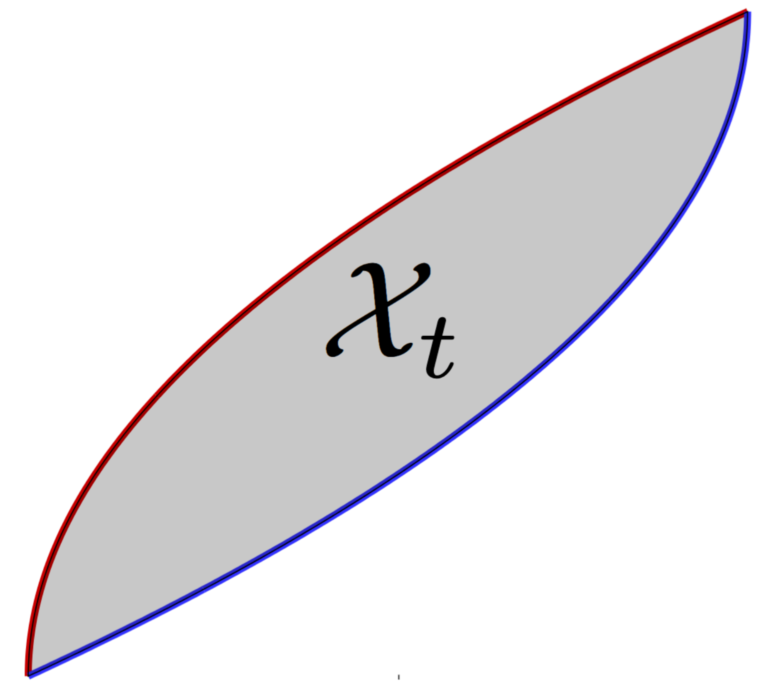


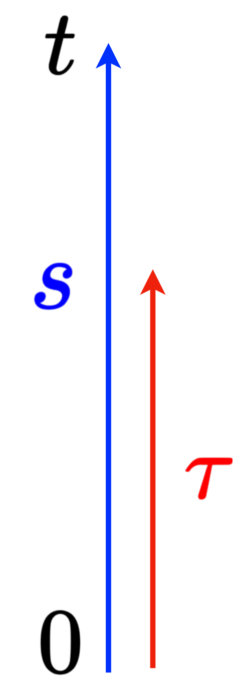
Algorithm

Step 1: Analytically compute (the boundary of) compact set $\mathcal{X}_t(\boldsymbol{M}\boldsymbol{z}_0)$ subject to TBD input range $[u_{\min}(s), u_{\max}(s)] \forall s \in [0, t]$

Step 2: Compute $\partial\mathcal{Z}_t(\boldsymbol{z}_0) = \boldsymbol{M}^{-1}\partial\mathcal{X}_t(\boldsymbol{M}\boldsymbol{z}_0)$

Step 1.1: Determining $[u_{\min}(s), u_{\max}(s)]\ \forall s \in [0, t]$

$$u_{\min}(s) = -\langle \boldsymbol{c}, e^{s\boldsymbol{A}_{\text{con}}}\boldsymbol{M}\boldsymbol{z}_0 \rangle + I_{\min}(s)$$
$$u_{\max}(s) = -\langle \boldsymbol{c}, e^{s\boldsymbol{A}_{\text{con}}}\boldsymbol{M}\boldsymbol{z}_0 \rangle + I_{\max}(s)$$



Variational problems:

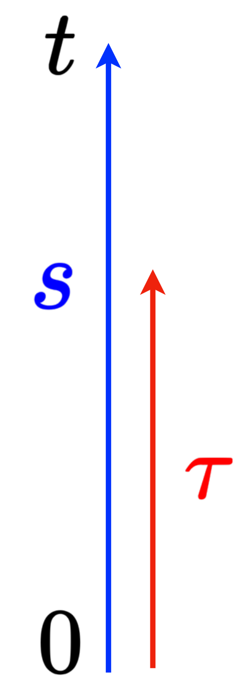
$$I_{\min}(s) := \inf_{v(\cdot)\in C([0,s])} I(v) \quad \text{subject to } v_{\min} \leq v(\cdot) \leq v_{\max}$$
$$I_{\max}(s) := \sup_{v(\cdot)\in C([0,s])} I(v) \quad \text{subject to } v_{\min} \leq v(\cdot) \leq v_{\max}$$

and

$$I(v) := v(s) - \int_0^s f(\tau)v(\tau)\mathrm{d}\tau, \quad f(\tau) := \left\langle \boldsymbol{c}, e^{(s-\tau)\boldsymbol{A}_{\text{con}}}\boldsymbol{b}_{\text{con}} \right\rangle$$

Step 1.1: Determining $\left[u_{\min}(s), u_{\max}(s)\right] \; \forall s \in [0, t]$

Lemma: (Spectral representation)



For A with distinct eigenvalues $\{\lambda_i\}_{i=1}^{n} \in \mathbb{C}^n$,

$$f(\tau) = -\sum_{i=1}^{n} \frac{\lambda_i^n}{\prod_{j \neq i}(\lambda_i - \lambda_j)} e^{\lambda_i (s-\tau)}, \quad 0 \leq \tau \leq s$$

Example: $n = 3, \lambda_1 = 1, \lambda_{2,3} = \pm \mathrm{i}$

$$f(\tau) = -\frac{1}{2}\left(e^{s-\tau} + \cos(s-\tau) - \sin(s-\tau)\right), 0 \leq \tau \leq s$$

Step 1.1: Determining $[u_{\min}(s), u_{\max}(s)]\ \forall s \in [0, t]$

Theorem:

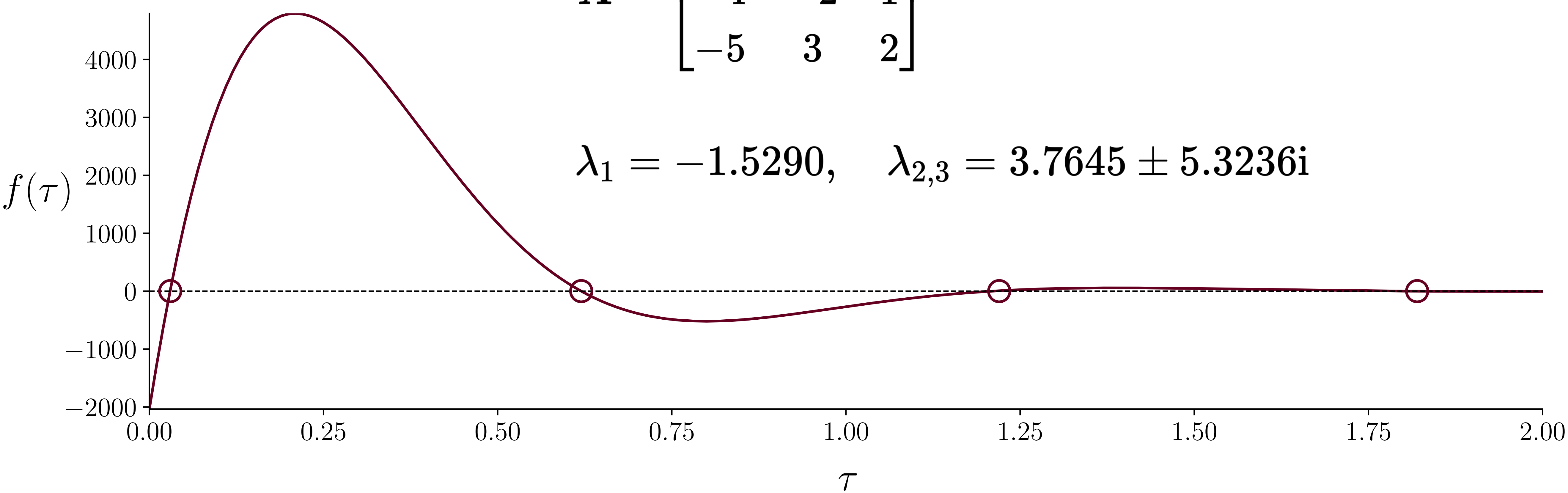
$$I_{\min}(s) = v_{\min} - v_{\min} \int_{\tau \in \text{zero sublevel set of} f} f(\tau)\mathrm{d}\tau - v_{\max} \int_{\tau \in \text{strict zero superlevel set of} f} f(\tau)$$

$$I_{\max}(s) = v_{\max} - v_{\max} \int_{\tau \in \text{zero sublevel set of} f} f(\tau)\mathrm{d}\tau - v_{\min} \int_{\tau \in \text{strict zero superlevel set of} f} f(\tau)$$

Example:

$$\boldsymbol{A} = \begin{bmatrix} 6 & 7 & 2 \\ -4 & -2 & 1 \\ -5 & 3 & 2 \end{bmatrix}$$

$$\lambda_1 = -1.5290, \quad \lambda_{2,3} = 3.7645 \pm 5.3236\mathrm{i}$$



Step 1.2: Parametric Boundary $\partial\mathcal{X}_t(\boldsymbol{x}_0)$

Theorem: [Generalizes Haddad and Halder, *TAC 68(11), 2023, 6680-6695*]

Define parameter vector

$$\boldsymbol{\sigma} \in \mathcal{W}_t := \{\boldsymbol{\sigma} \in \mathbb{R}^{n-1} \mid 0 \leq \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_{n-1} \leq t\}$$

Weyl chamber

and for $0 \leq s \leq t$,

$$\mu(s) := (u_{\max}(s) - u_{\min}(s))/2, \quad \nu(s) := (u_{\max}(s) + u_{\min}(s))/2,$$

$$\boldsymbol{\xi}(s) := \left(\frac{s^{n-1}}{(n-1)!}, \frac{s^{n-2}}{(n-2)!}, \ldots, s, 1\right)^\top.$$

Then $\boldsymbol{x}^{\text{bdy}} \in \partial\mathcal{X}_t(\boldsymbol{x}_0)$ has $\boldsymbol{\sigma}$ parameterization

$$\boldsymbol{x}^{\text{bdy}}(\boldsymbol{\sigma}) = \boldsymbol{\chi}(t, \boldsymbol{x}_0) + \int_0^t \nu(s)\boldsymbol{\xi}(t-s)\mathrm{d}s \pm \int_0^{\sigma_1} \mu(s)\boldsymbol{\xi}(t-s)\mathrm{d}s \mp \int_{\sigma_1}^{\sigma_2} \mu(s)\boldsymbol{\xi}(t-s)\mathrm{d}s \pm \ldots \pm (-1)^n \int_{\sigma_{n-1}}^{t} \mu(s)\boldsymbol{\xi}(t-s)\mathrm{d}s$$

$$\chi_k := \sum_{\ell=1}^{n} \mathbf{1}_{k\leq\ell} \frac{t^{\ell-k}}{(\ell-k)!} \boldsymbol{x}_{\ell 0} \; \forall k \in [n]$$

Consequences

Corollary: $\partial \mathcal{X}_t(\boldsymbol{x}_0) = \partial \mathcal{X}_t^{\text{upper}}(\boldsymbol{x}_0) \cup \partial \mathcal{X}_t^{\text{lower}}(\boldsymbol{x}_0)$

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Theorem: (Volume)

$$\mathrm{vol}_n(\mathcal{Z}_t(\boldsymbol{z}_0)) = \frac{1}{\det(\boldsymbol{M})} \int_0^1 \int_{\mathcal{W}_t} |\det(\mathrm{D}\pi)| \mathrm{d}\boldsymbol{\sigma} \mathrm{d}\lambda$$

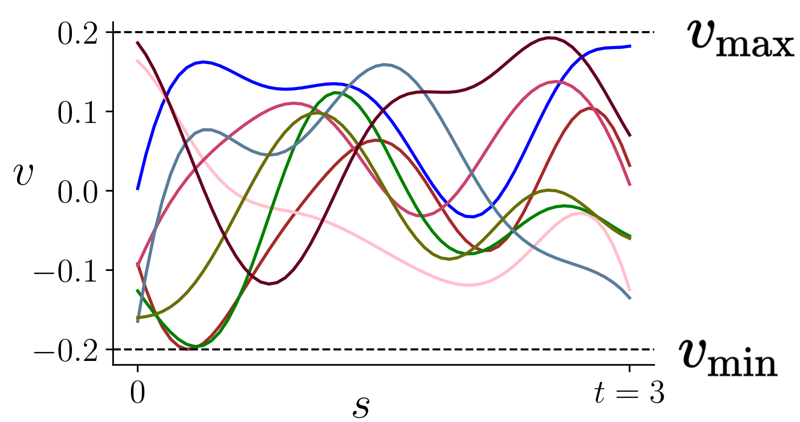
where

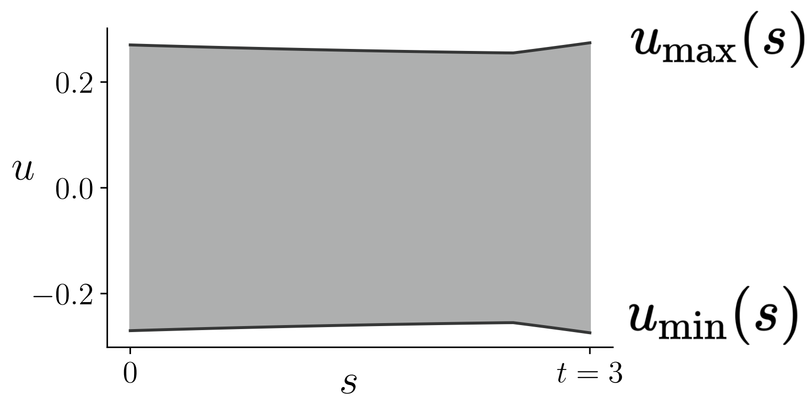
$$|\det(\mathrm{D}\pi)| = \mu(\sigma_1)\ldots\mu(\sigma_{n-1})|(4\lambda-2)^{n-1}| \left| \det \begin{pmatrix} \frac{(t-\sigma_1)^{n-1}}{(n-1)!} & \cdots & \frac{(t-\sigma_{n-1})^{n-1}}{(n-1)!} & \zeta_1(\boldsymbol{\sigma}) \\ \vdots & \vdots & \vdots & \vdots \\ (t-\sigma_1) & \cdots & (t-\sigma_{n-1}) & \zeta_{n-1}(\boldsymbol{\sigma}) \\ 1 & \cdots & 1 & \zeta_n(\boldsymbol{\sigma}) \end{pmatrix} \right|$$

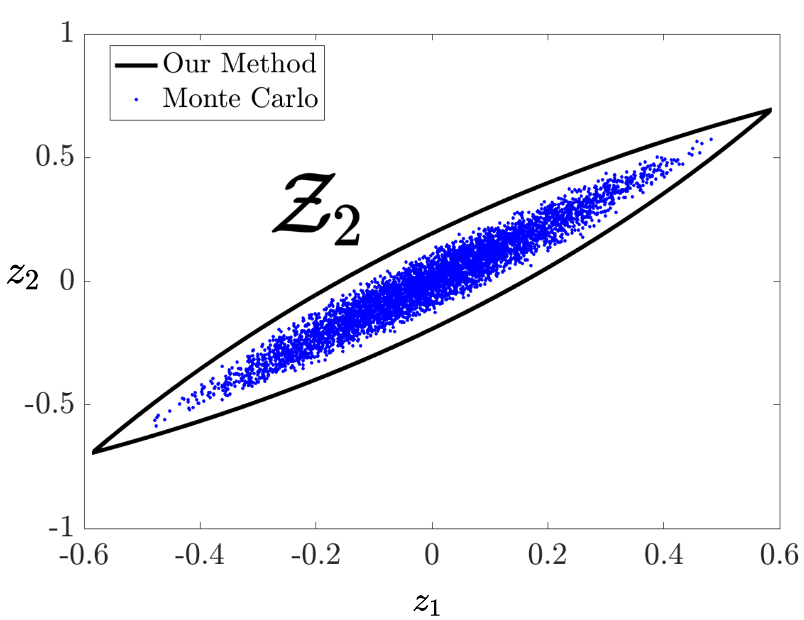
and $\boldsymbol{\zeta}(\boldsymbol{\sigma}) := \boldsymbol{x}^{\text{upper}}(\boldsymbol{\sigma}) - \boldsymbol{x}^{\text{lower}}(\boldsymbol{\sigma})$

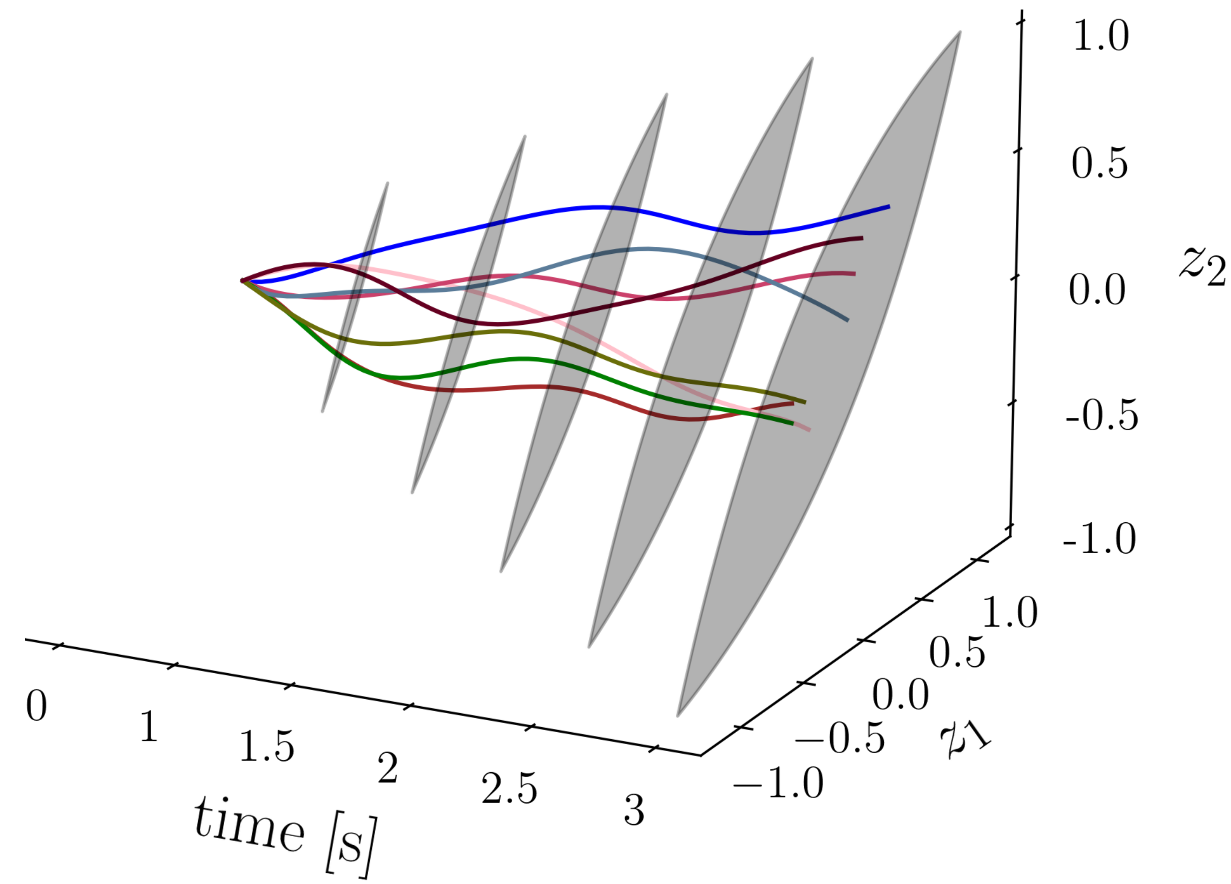
Example

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\end{pmatrix} = \underbrace{\begin{pmatrix}0.1 & 0.2\\ -0.3 & 0.1\end{pmatrix}}_{\boldsymbol{A}}\begin{pmatrix}z_1\\ z_2\end{pmatrix} + \underbrace{\begin{pmatrix}1\\ 2\end{pmatrix}}_{\boldsymbol{b}} v, \quad v_{\min} = -0.2, \quad v_{\max} = 0.2, \quad \boldsymbol{z}_0 = \boldsymbol{0}$$









Our vol Thm: $\text{vol}_2(\mathcal{Z}_2) \approx 0.3043$

5000 sample MC estimate using MATLAB polyarea: ≈ 0.2301

Thank You