## Weyl Calculus and Exactly Solvable Schrödinger Bridges with Quadratic State Cost

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#### **The Problem**

Solve linear reaction-diffusion PDE IVP:

$$rac{\partial}{\partial t}\widehat{arphi} = -\mathcal{L}\widehat{arphi}, \quad \widehat{arphi}(t=t_0,oldsymbol{x}) = \widehat{arphi}_0(oldsymbol{x}) ext{ (given)}$$

where 
$$\mathcal{L} \equiv -\Delta_{x} + q(x)$$
  
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given bounded continuous reaction rate

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where 
$$\mathcal{L} \equiv -\Delta_{m{x}} + q(m{x})$$

#### given bounded continuous reaction rate

Our contribution: closed-form Green's function for convex quadratic  $q(\boldsymbol{x})$  $\widehat{\varphi}(t, \boldsymbol{x}) = \int_{\mathcal{X}} \kappa(t_0, \boldsymbol{x}, t, \boldsymbol{y}) \widehat{\varphi}_0(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}$ 

#### State-of-the-art

Randomized numerical solution via Feynman-Kac path integral

Estimates  $\widehat{\varphi}(t, \boldsymbol{x})$  as conditional expectation

**Good:** meshless/particle-based

**Bad:** still needs function approximation

#### **Our Motivation**

Schrödinger bridge problem with state cost

$$egin{aligned} &\inf_{(\mu^{m{u}},m{u})} \int_{\mathbb{R}^n} \int_{t_0}^{t_1} \left\{ rac{1}{2} |m{u}|^2 + m{q}(m{x}^{m{u}}_t) 
ight\} \mathrm{d}t \, \mathrm{d}\mu^{m{u}}(m{x}^{m{u}}_t) \ & ext{subject to} \quad \mathrm{d}m{x}^{m{u}}_t = m{u}_t(t,m{x}^{m{u}}_t) \, \mathrm{d}t + \sqrt{2} \, \mathrm{d}m{w}_t, \ &m{x}^{m{u}}_t(t=t_0) \sim \mu_0, \quad m{x}^{m{u}}_t(t=t_1) \sim \mu_1, \end{aligned}$$



#### Can be reduced to solving

Nonlinearly boundary coupled system of linear PDEs

$$egin{aligned} &rac{\partial \widehat{arphi}}{\partial t} = (\Delta_{oldsymbol{x}} - q(oldsymbol{x})) \widehat{arphi}, \ &rac{\partial arphi}{\partial t} = (-\Delta_{oldsymbol{x}} + q(oldsymbol{x})) arphi, \ &\widehat{arphi}(t = t_0, \cdot) arphi(t = t_0, \cdot) = \mu_0, \quad \widehat{arphi}(t = t_0, \cdot) arphi(t = t_1, \cdot) = \mu_1 \end{aligned}$$

Equivalently, nonlinear integral equations (Schrödinger system)

$$egin{aligned} \mu_0(oldsymbol{x}) &= \widehat{arphi}_0(oldsymbol{x}) \int \kappa(t_0,oldsymbol{x},t_1,oldsymbol{y}) arphi_1(oldsymbol{y}) \mathrm{d}oldsymbol{y} \ \mu_1(oldsymbol{x}) &= \widehat{arphi}_1(oldsymbol{x}) \int \kappa(t_0,oldsymbol{y},t_1,oldsymbol{x}) \widehat{arphi}_0(oldsymbol{y}) \mathrm{d}oldsymbol{y} \end{aligned}$$

Optimal solution:  $\mu_{\text{opt}}^{\boldsymbol{u}}(t,\cdot) = \widehat{\varphi}(t,\cdot)\varphi(t,\cdot), \quad \boldsymbol{u}_{\text{opt}}(t,\cdot) = \nabla_{(\cdot)}\log\varphi(t,\cdot)$ 

The Case 
$$q(\mathbf{x}) = \frac{1}{2} \mathbf{x}^{\mathsf{T}} \mathbf{Q} \mathbf{x}, \quad \mathbf{Q} \ge \mathbf{0}$$

Gaussian endpoints  $\mu_0 = \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_0), \quad \mu_1 = \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_1)$ 

→ coupled Riccati ODEs studied in Chen, Georgiu, Pavon, ACC 2015

**Non-Gaussian endpoints** 

 $\rightsquigarrow$  need  $\kappa$ 

arXiv:2406.00503

Closed-form derivation of  $\kappa$  using Hermite polynomials (elementary + painful + difficult to generalize)

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More disciplined alternative: Weyl calculus

#### New Idea: Weyl Calculus to Compute *k*

Originally developed for quantum mechanics

Appears less known to solve problems outside quantum mechanics

Many names: Weyl quantization, Wigner-Weyl transform

Proposed computational steps:

PDE  $\longrightarrow$  Weyl operator  $H(X, D) \longrightarrow$  Weyl symbol  $h(x, \xi) \longrightarrow$  Kernel  $\kappa$ 

Step 1: PDE  $\longrightarrow$  Weyl Operator H(X, D)

Let 
$$X_k := x_k, \quad D_k := rac{1}{\mathrm{i}} rac{\partial}{\partial x_k} \quad orall k \in [n]$$

$$oldsymbol{X} := egin{pmatrix} X_1 & \dots & X_n \end{pmatrix}^ op, \quad oldsymbol{D} := egin{pmatrix} D_1 & \dots & D_n \end{pmatrix}^ op$$

Write the semigroup  $\exp(-(t-t_0)\mathcal{L})$  in terms of X, D

**Example (Heat PDE).** 

$$\mathcal{L}\equiv -\Delta_{oldsymbol{x}}, \quad |oldsymbol{D}|^2:=\langle oldsymbol{D},oldsymbol{D}
angle=(-\mathrm{i})^2\langle
abla_{oldsymbol{x}},
abla_{oldsymbol{x}}
angle=-\Delta_{oldsymbol{x}}$$

Therefore,  $H_{\text{heat}}(\boldsymbol{X},\boldsymbol{D}) = \exp\left(-(t-t_0)|\boldsymbol{D}|^2\right)$ 

### **Step 1: PDE** $\longrightarrow$ **Weyl Operator** H(X, D)

**Reaction-diffusion PDE:** 

$$egin{aligned} rac{\partial \widehat{arphi}}{\partial t} &= \left( \Delta_{oldsymbol{z}} - rac{1}{2} oldsymbol{z}^ op oldsymbol{Q} z 
ight) \widehat{arphi}, \quad \widehat{arphi}(t_0, \cdot) &= \widehat{arphi}_0, \quad ext{w.l. o. g. }oldsymbol{Q} \succ oldsymbol{0} \ && egin{aligned} && rac{1}{2} oldsymbol{Q} = oldsymbol{V}^ op oldsymbol{\Delta} V \ && oldsymbol{z} \mapsto oldsymbol{x} := oldsymbol{V} oldsymbol{A} V \ && oldsymbol{z} \mapsto oldsymbol{x} := oldsymbol{V} oldsymbol{z}, \quad \hat{arphi}(t,oldsymbol{x}) := \widehat{arphi} oldsymbol{t}(t,oldsymbol{z} = oldsymbol{V}^ op oldsymbol{x} (t,oldsymbol{x}) := \widehat{arphi} oldsymbol{t}(t,oldsymbol{x}) := \widehat{arphi} oldsymbol{t}(t,oldsymbol{z} = oldsymbol{V}^ op oldsymbol{x} (t,oldsymbol{x}) := \widehat{arphi} oldsymbol{t}(t,oldsymbol{x}) :=$$

Weyl operator is a composition:

$$egin{aligned} Q_{oldsymbol{\Lambda}}(oldsymbol{X},oldsymbol{D}) &:= |oldsymbol{D}|^2 + \sum_{k=1}^n \lambda_k X_k^2 \ H_{oldsymbol{\Lambda}}(oldsymbol{X},oldsymbol{D}) &= \exp\left(-(t-t_0)Q_{oldsymbol{\Lambda}}(oldsymbol{X},oldsymbol{D})
ight) \end{aligned}$$

# Step 2: Weyl Operator $H(X, D) \longrightarrow$ Weyl Symbol $h(x, \xi)$

Derive a PDE IVP for the Weyl symbol

$$rac{\partial}{\partial t}h_{oldsymbol{\Lambda}}(oldsymbol{x},oldsymbol{\xi})=-\sum_{j=0}^{2}rac{1}{j!}\{q_{oldsymbol{\Lambda}},h_{oldsymbol{\Lambda}}\}_{j}(oldsymbol{x},oldsymbol{\xi}),\quad h_{oldsymbol{\Lambda}}ert_{t=t_{0}}=1$$

where  $q_{\mathbf{\Lambda}}(\boldsymbol{x}, \boldsymbol{\xi}) = |\boldsymbol{\xi}|^2 + \sum_{k=1}^n \lambda_k x_k^2$  and the *j*th order Poisson bracket

$$\{f,g\}_j(oldsymbol{x},oldsymbol{\xi}):=igg(rac{1}{2\mathrm{i}}igg)^jigg(\sum_{k=1}^nigg(rac{\partial^2}{\partial y_k\partial\xi_k}-rac{\partial^2}{\partial x_k\partial
u_k}igg)igg)^jf(oldsymbol{x},oldsymbol{\xi})g(oldsymbol{y},oldsymbol{\eta})igg|_{oldsymbol{y}=oldsymbol{x},\eta=oldsymbol{\xi}}\quadorall j=0,1,2,\dots$$

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where  $q_{\mathbf{\Lambda}}(\boldsymbol{x}, \boldsymbol{\xi}) = |\boldsymbol{\xi}|^2 + \sum_{k=1}^n \lambda_k x_k^2$  and the *j*th order Poisson bracket

$$\{f,g\}_j(oldsymbol{x},oldsymbol{\xi}) := igg(rac{1}{2\mathrm{i}}igg)^j igg(\sum_{k=1}^n igg(rac{\partial^2}{\partial y_k \partial \xi_k} - rac{\partial^2}{\partial x_k \partial 
u_k}igg)igg)^j f(oldsymbol{x},oldsymbol{\xi})g(oldsymbol{y},oldsymbol{\eta})igg|_{oldsymbol{y}=oldsymbol{x},\eta=oldsymbol{\xi}} \quad orall j=0,1,2,\dots$$

Solution of this PDE IVP:

$$h_{oldsymbol{\Lambda}}(oldsymbol{x},oldsymbol{\xi}) = \left(\prod_{k=1}^n rac{1}{\coshig(\sqrt{\lambda_k}(t-t_0)ig)}
ight) \expig(-\sum_{k=1}^n rac{\lambda_k x_k^2 + oldsymbol{\xi}_k^2}{\sqrt{\lambda_k}} anhig(\sqrt{\lambda_k}(t-t_0)ig)ig)$$

# Step 3: Weyl Symbol $h(x, \xi) \longrightarrow$ Kernel $\kappa$

In general,

non-unitary inverse Fourier transform of  $\boldsymbol{\xi} \mapsto h\left((\boldsymbol{x} + \boldsymbol{y})/2, \boldsymbol{\xi}\right)$ 

Applying this to our Weyl symbol  $h_{\Lambda}$  gives

$$egin{split} &\kappa_{oldsymbol{\Lambda}}(t_0,oldsymbol{x},t,oldsymbol{y})\ &= \left(\prod_{k=1}^n rac{\lambda_k^{1/4}}{\sqrt{2\pi\sinh\left(2\sqrt{\lambda_k}(t-t_0)
ight)}}
ight)\ & imes\exp\left(-\sum_{k=1}^n rac{\sqrt{\lambda_k}}{2}(x_k^2+y_k^2)rac{\cosh\left(2\sqrt{\lambda_k}(t-t_0)
ight)}{\sinh\left(2\sqrt{\lambda_k}(t-t_0)
ight)}
ight)\ & imes\exp\left(\sum_{k=1}^n \sqrt{\lambda_k}x_ky_kigg(rac{1}{\sinh\left(2\sqrt{\lambda_k}(t-t_0)
ight)}igg)
ight) \end{split}$$

### Summary for the Derived $\kappa_{\Lambda}$

Recovers the formula in

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Limit  $\frac{1}{2}Q = \Lambda \downarrow 0$  recovers the heat kernel

Special case  $\frac{1}{2}Q = \Lambda = I_n$  recovers the Mehler kernel in quantum mechanics

#### Summary for the Derived $\kappa_{\Lambda}$ (contd.)

Permits generalization: 
$$q(z) = \frac{1}{2}z^{\mathsf{T}}Qz + r^{\mathsf{T}}z + s$$
,  $Q > 0$ 

$$egin{split} &\kappa_{(\mathbf{\Lambda},m{r},s)}(t_0,m{x},t,m{y}) \ &= \left(\prod_{k=1}^n rac{\lambda_k^{1/4} \exp\left(-c_k(t-t_0)
ight)}{\sqrt{2\pi\sinh\left(2\sqrt{\lambda_k}(t-t_0)
ight)}}
ight) \ & imes \exp\left(\sum_{k=1}^n -rac{\sqrt{\lambda_k}}{2}(x_k^2+y_k^2)rac{\cosh\left(2\sqrt{\lambda_k}(t-t_0)
ight)}{\sinh\left(2\sqrt{\lambda_k}(t-t_0)
ight)} + rac{\sqrt{\lambda_k}x_ky_k}{\sinh\left(2\sqrt{\lambda_k}(t-t_0)
ight)} - rac{\left(rac{1}{2}m{r}^ opm{v}_k(x_k+y_k) + rac{1}{4\lambda_k}ig(m{r}^ opm{v}_kig)^2
ight) ext{tanh}(\sqrt{\lambda_k}(t-t_0))}{\sqrt{\lambda_k}}
ight) \end{split}$$

$$c_k := rac{1}{4\lambda_k} (oldsymbol{r}^ opoldsymbol{v}_k)^2 - rac{s}{n}, \quad oldsymbol{v}_k ext{ is } k ext{th column of }oldsymbol{V}^ op$$

Helps in solving the Schrödinger bridge problem with quadratic state cost:

$$egin{aligned} \widehat{arphi}(t,oldsymbol{z}) &= \hat{
u}(t,oldsymbol{x})ig|_{oldsymbol{x}=oldsymbol{V}oldsymbol{z}} = \int_{\mathbb{R}^n} &\kappa_{(oldsymbol{\Lambda},oldsymbol{r},s)}(t_0,oldsymbol{V}oldsymbol{z},t,oldsymbol{y}) arphi_0ig(oldsymbol{V}^ opoldsymbol{y}ig) \mathrm{d}oldsymbol{y} \end{aligned}$$

## Thank You

Details:





\*\*\*\*\*\*