Nonlinearity and Interaction in Schrödinger Bridges

Abhishek Halder

Department of Aerospace Engineering, Iowa State University
Department of Applied Mathematics, University of California Santa Cruz

Joint work with A. Eldesoukey, M.Z. Haider (ISU),
I. Napolitano (Scuola Superiore Meridionale),
G.A. Bondar (UC Santa Cruz), Y. Chen (Georgia Tech)

Session: Learning, Sampling, and Control
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This Talk is Based on

G.A. Bondar, A. Eldesoukey, Y. Chen, A. H.

Nonlinear Non-Gaussian Density Steering with Input and Noise Channel Mismatch. arXiv:2604.23370

A. Eldesoukey, Y. Chen, A.H.

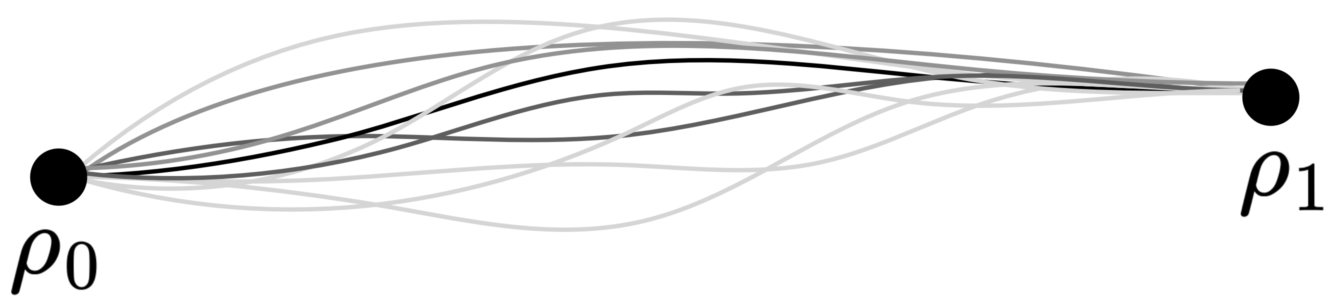
A Generalized Sinkhorn Algorithm for Mean-Field Schrödinger Bridge. *LCSS(10)*, 2026

A. Eldesoukey, M.Z. Haider, I. Napolitano, Y. Chen, A.H.

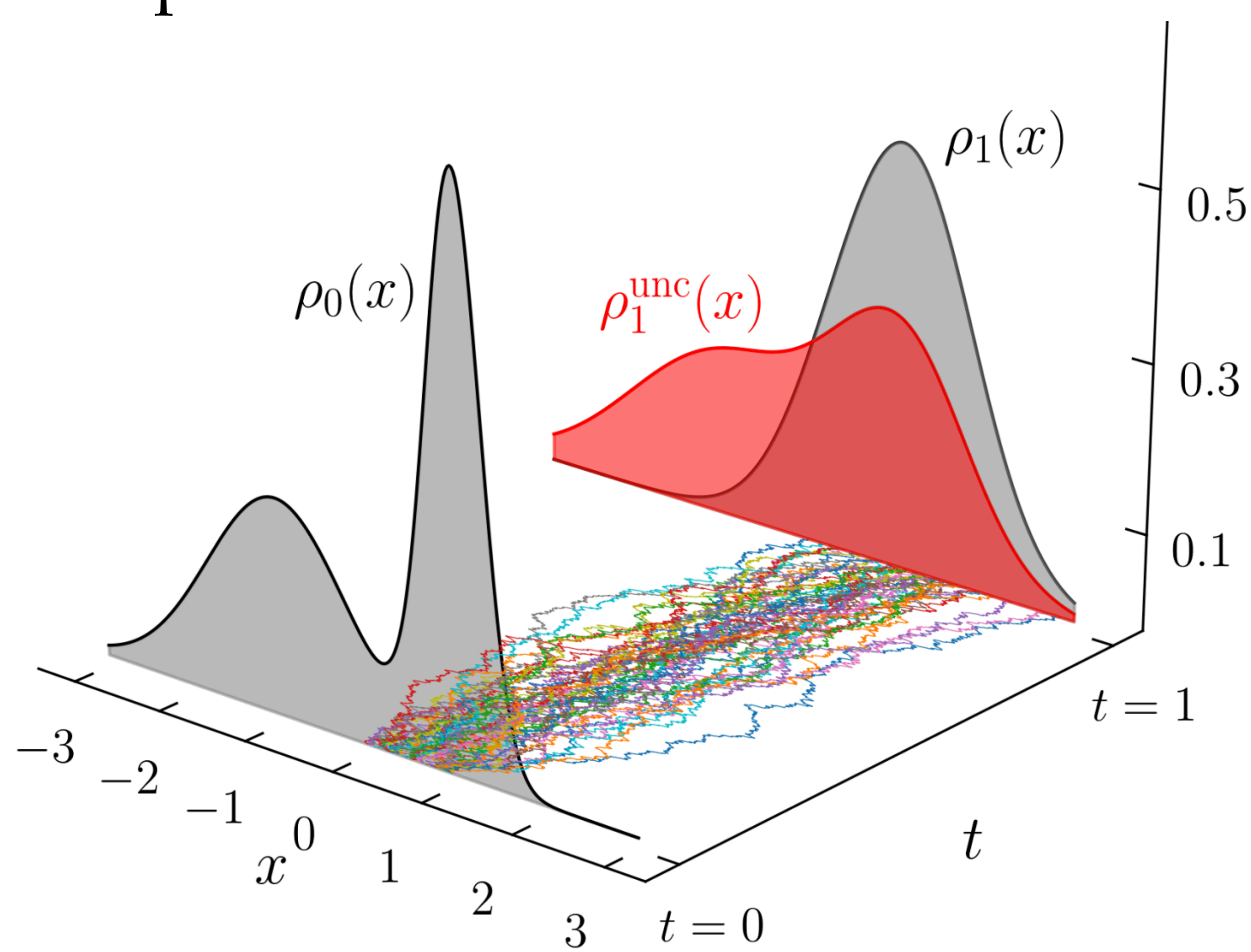
Schrödinger Bridges over Kinetic Swarming Models. arXiv:2608.25281

Schrödinger Bridge

Path space/large deviation viewpoint: max prob stochastic interpolant



Stochastic control viewpoint: min effort fixed horizon distribution steering



Control-affine Schrödinger Bridge Problem

$$\underset{(\rho^u, \boldsymbol{u}) \in \mathcal{P}_{01} \times \mathcal{U}}{\arg\inf} \int_{t_0}^{t_1} \mathbb{E}_{\rho^u} \left[q\left(t, \boldsymbol{x}^u\right) + \frac{1}{2} \|\boldsymbol{u}\|_2^2 \right] \mathrm{d}t$$

subject to

$$\mathrm{d}\boldsymbol{x}^{\boldsymbol{u}} = \left(\boldsymbol{f}\left(t, \boldsymbol{x}^{\boldsymbol{u}}\right) + \boldsymbol{g}\left(t, \boldsymbol{x}^{\boldsymbol{u}}\right)\boldsymbol{u}\right)\mathrm{d}t + \boldsymbol{\sigma}\left(t, \boldsymbol{x}^{\boldsymbol{u}}\right)\mathrm{d}\boldsymbol{w}$$

Control-affine drift coefficient

Diffusion coefficient

Diffusion tensor $\boldsymbol{\Sigma} := \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$

Input and noise channel mismatched if

$$\nexists \lambda > 0 \ \text{ such that } \ \lambda \boldsymbol{g}\boldsymbol{g}^\top - \boldsymbol{\Sigma} = \boldsymbol{0}$$

Conditions for Optimality

Primal PDE in controlled PDE $\rho^{\boldsymbol{u}}_{\text{opt}}$

$$\partial_t \rho^{\boldsymbol{u}}_{\text{opt}} + \nabla_{\boldsymbol{x}} \cdot \left(\rho^{\boldsymbol{u}}_{\text{opt}} \left(\boldsymbol{f} + \boldsymbol{g}\boldsymbol{g}^\top \nabla_{\boldsymbol{x}} S\right)\right) = \frac{1}{2} \Delta_{\boldsymbol{\Sigma}} \rho^{\boldsymbol{u}}_{\text{opt}}$$

Dual PDE in value function S

$$\partial_t S + \langle \nabla_{\boldsymbol{x}} S, \boldsymbol{f} \rangle + \frac{1}{2} \langle \nabla_{\boldsymbol{x}} S, \boldsymbol{g}\boldsymbol{g}^\top \nabla_{\boldsymbol{x}} S \rangle + \frac{1}{2} \langle \boldsymbol{\Sigma}, \text{Hess}_{\boldsymbol{x}}\, S \rangle = q$$

Boundary conditions

$$\rho^{\boldsymbol{u}}_{\text{opt}} (t = t_0, \cdot) = \rho_0(\cdot), \quad \rho^{\boldsymbol{u}}_{\text{opt}} (t = t_1, \cdot) = \rho_1(\cdot)$$

Optimal control

$$\boldsymbol{u}_{\text{opt}} = \boldsymbol{g}^\top \nabla_{\boldsymbol{x}} S$$

Standard Approach: Schrödinger Factors

Hopf-Cole transform

$$\left(\rho_{\mathrm{opt}}^{\boldsymbol{u}}, S\right) \mapsto (\widehat{\varphi}, \varphi) := \left(\rho_{\mathrm{opt}}^{\boldsymbol{u}} \exp(-S/\lambda), \exp(S/\lambda)\right)$$

Transformed PDE BVP

$$\partial_t \widehat{\varphi} + \nabla_{\boldsymbol{x}} \cdot \left(\widehat{\varphi}\,(\boldsymbol{f} + \boxed{\boldsymbol{f}_{\varphi}})\right) - \frac{1}{2}\Delta_{\boldsymbol{\Sigma}}\widehat{\varphi} + \left(\frac{q}{\lambda} + \boxed{q_{\varphi}}\right)\widehat{\varphi} = 0$$

$$\partial_t \varphi + \langle \nabla_{\boldsymbol{x}}\varphi, \boldsymbol{f} + \boxed{\boldsymbol{f}_{\varphi}} \rangle + \frac{1}{2}\langle \boldsymbol{\Sigma}, \mathrm{Hess}_{\boldsymbol{x}}\,\varphi \rangle - \left(\frac{q}{\lambda} + \boxed{q_{\varphi}}\right)\varphi = 0$$

$$\widehat{\varphi}\,(t_0, \cdot)\varphi\,(t_0, \cdot) = \rho_0(\cdot), \quad \widehat{\varphi}\,(t_1, \cdot)\varphi\,(t_1, \cdot) = \rho_1(\cdot)$$

where

$$\boldsymbol{f}_{\varphi} := \left(\lambda \boldsymbol{g}\boldsymbol{g}^{\top} - \boldsymbol{\Sigma}\right)\nabla_{\boldsymbol{x}} \log \varphi$$

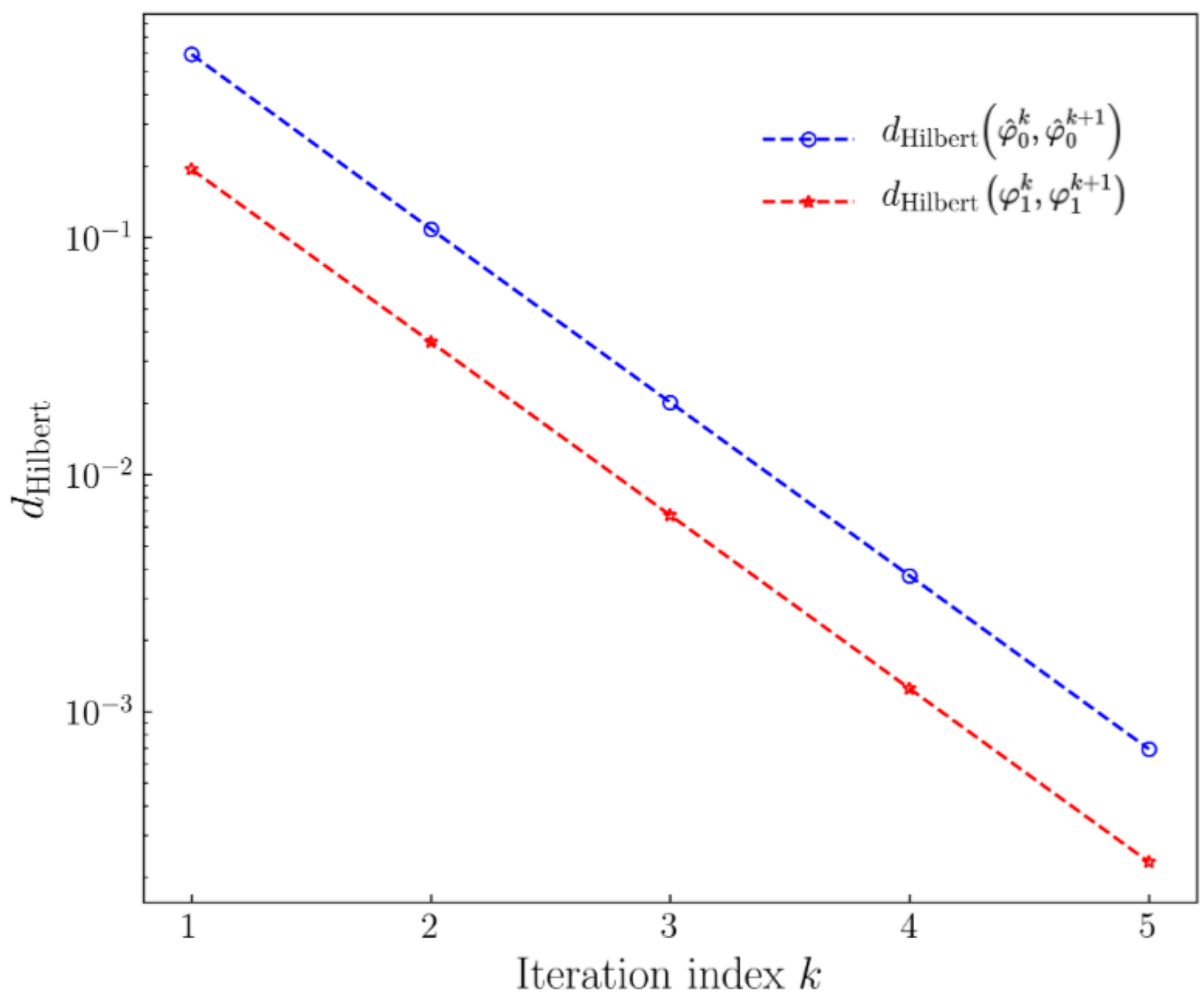
$$q_{\varphi} := \frac{1}{2}(\nabla_{\boldsymbol{x}} \log \varphi)^{\top} \left(\lambda \boldsymbol{g}\boldsymbol{g}^{\top} - \boldsymbol{\Sigma}\right)\nabla_{\boldsymbol{x}} \log \varphi$$

Channel Matching Case is Linearly Solvable

The case $\boldsymbol{g}\boldsymbol{g}^\top \propto \boldsymbol{\Sigma}$: dynamic Sinkhorn recursion

$$\widehat{\varphi}_0 \xrightarrow{\mathcal{F}} \widehat{\varphi}_1 \xrightarrow{\rho_1/\widehat{\varphi}_1} \varphi_1 \xrightarrow{\mathcal{B}} \varphi_0 \xrightarrow{\rho_0/\varphi_0} \left(\widehat{\varphi}_0\right)_{\text{next}}$$

where $\quad \widehat{\varphi}_i(\cdot) := \widehat{\varphi}(t = t_i, \cdot), \quad \varphi_i(\cdot) := \varphi(t = t_i, \cdot) \quad \forall i \in \{0, 1\}$



Channel Mismatched Case: Sinkhorn w. Memory

Idea: use memory from the backward pass

$$\begin{tikzcd}
\varphi_1 \arrow[r, "\mathcal{B}"] & \varphi_0 \arrow[r, "\rho_0/\varphi_0"] & \widehat{\varphi}_0 \arrow[r, "\mathcal{F}_\varphi"] & \widehat{\varphi}_1 \arrow[r, "\rho_1/\widehat{\varphi}_1"] & (\varphi_1)_{\mathrm{next}} \\
& \boxed{(\varphi_t)_{t\in[t_0,t_1]}} & & &
\end{tikzcd}$$

We prove: this algorithm is locally stable (w.r.t. Hilbert ball of estimated radius)

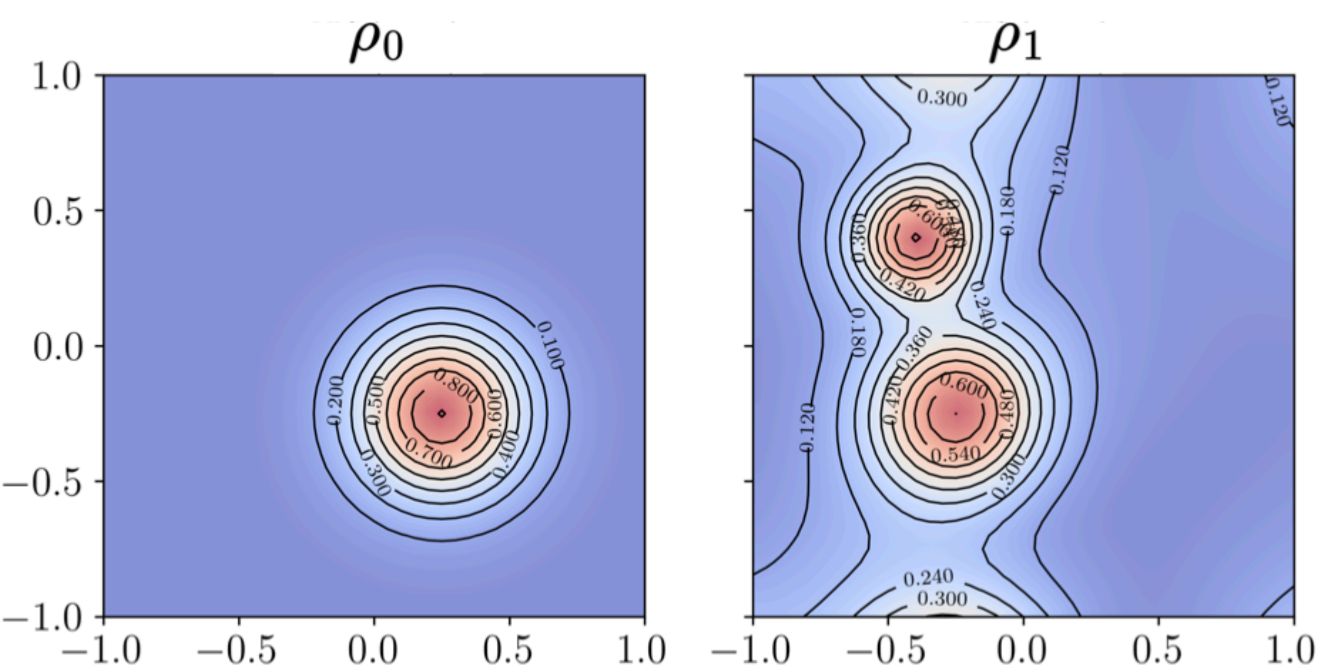
Channel Mismatched Case: Numerical Example

Cubic spring-mass-damper with process noise

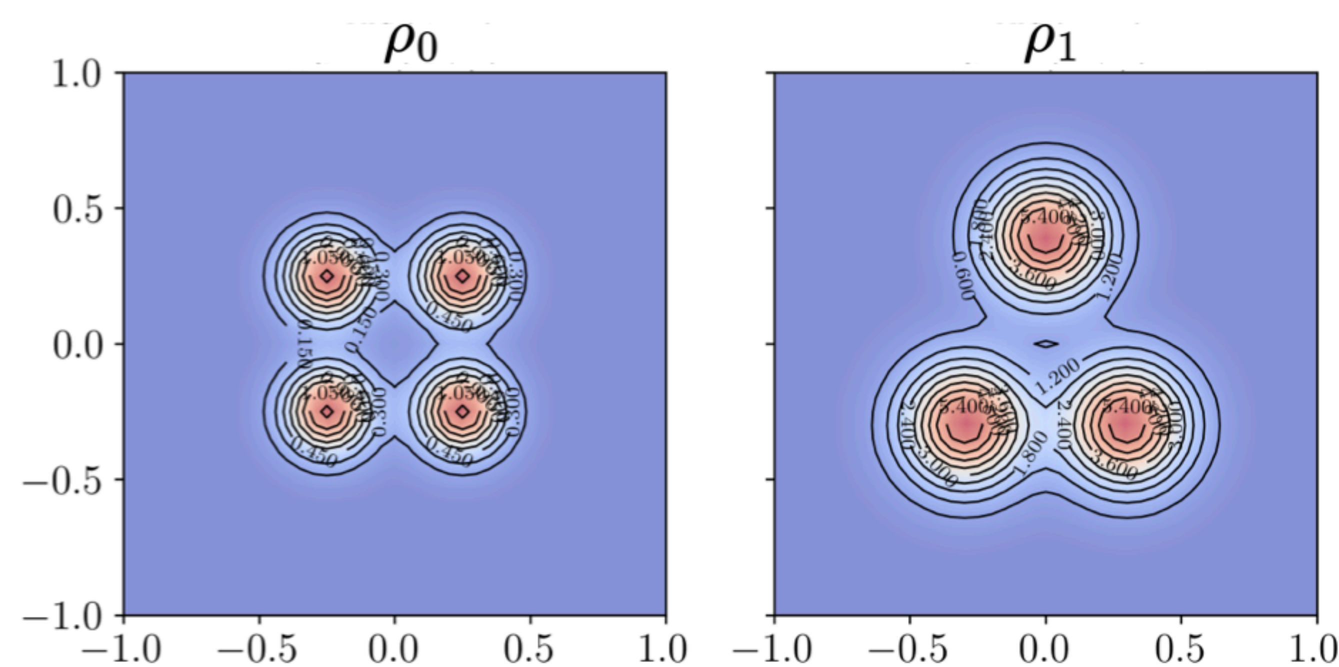
$$\boldsymbol{f}(t, x_1, x_2) = \begin{pmatrix} x_2 \\ -\dfrac{\partial}{\partial x_1} V(x_1) - \gamma x_2 \end{pmatrix}, \; V(\cdot) = \frac{1}{4}(\cdot)^4, \; \gamma = 1$$

$$\boldsymbol{g}(t, x_1, x_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; \boldsymbol{\sigma}(t, x_1, x_2) = \boldsymbol{I}$$

Two endpoint pairs



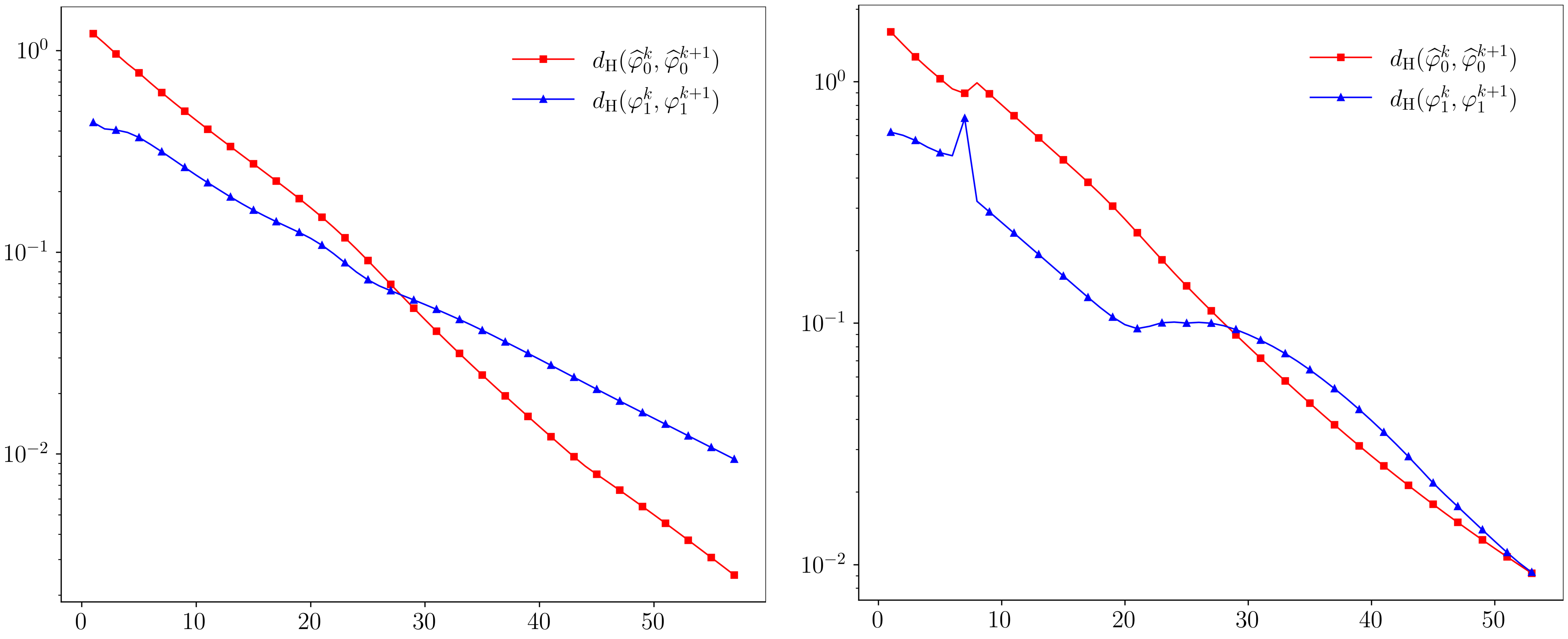
The first pair



The second pair

Channel Mismatched Case: Numerical Example

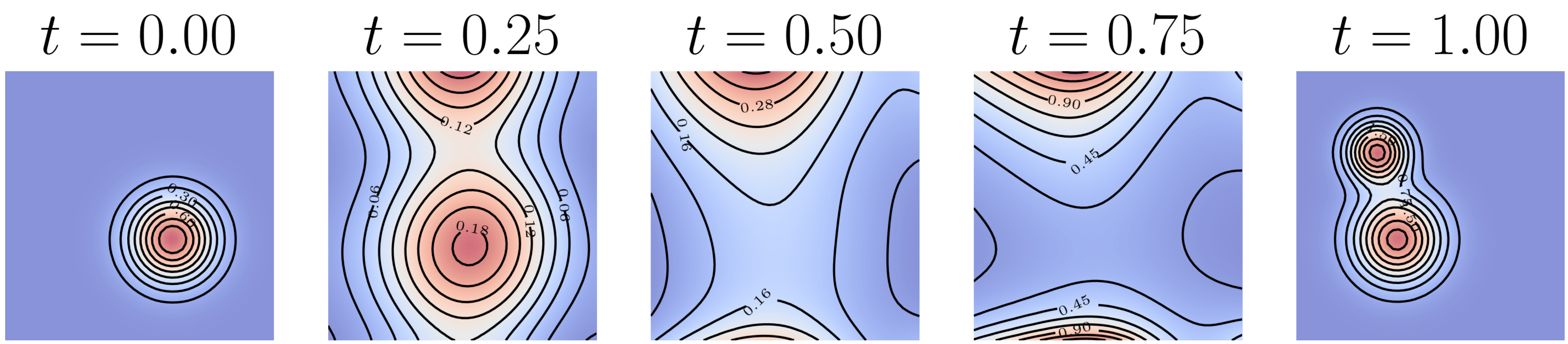
Convergence



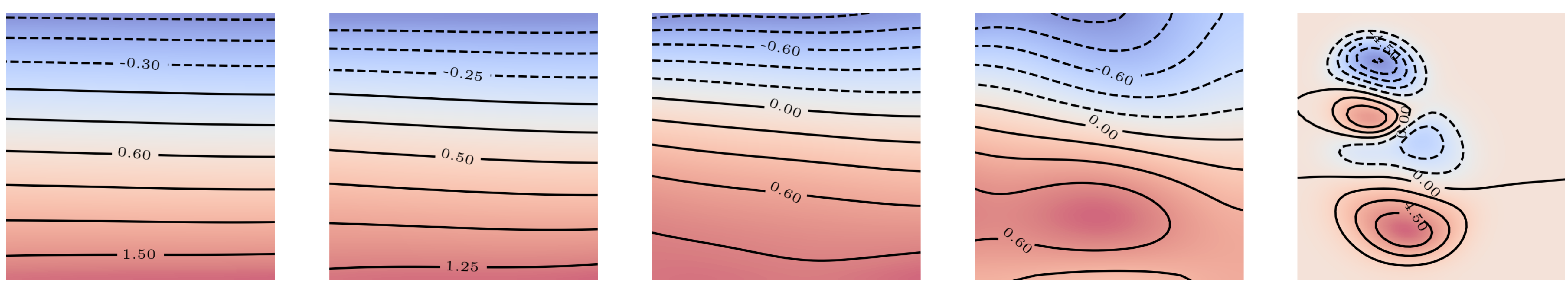
Channel Mismatched Case: Numerical Example

Solution for the first pair of endpoints

Optimally controlled joint state PDF



Optimal control

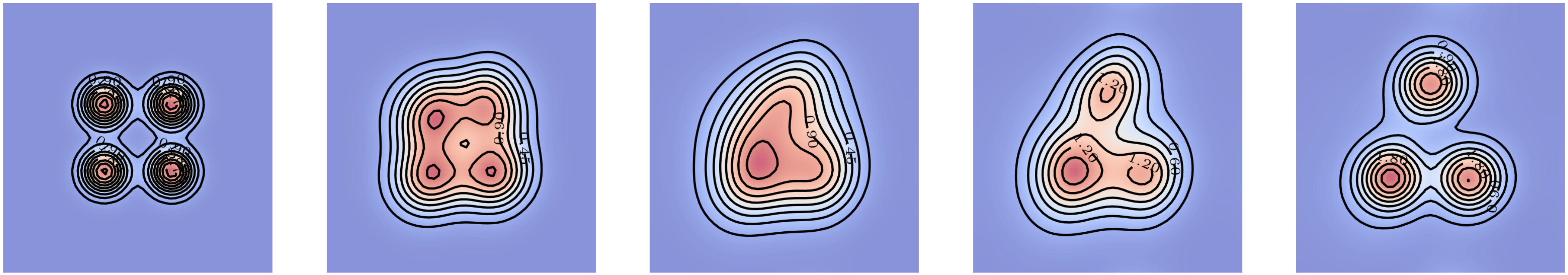


Channel Mismatched Case: Numerical Example

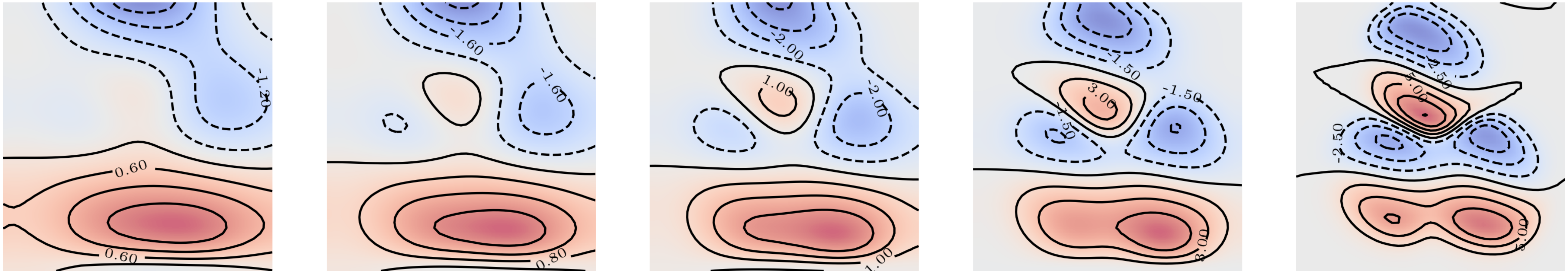
Solution for the second pair of endpoints

Optimally controlled joint state PDF

$t = 0.00$ $\quad$ $t = 0.25$ $\quad$ $t = 0.50$ $\quad$ $t = 0.75$ $\quad$ $t = 1.00$



Optimal control



1st Order Mean-Field Schrödinger Bridge

Prior kinematics for first order interacting agents

$$\mathrm{d}X_t^i = \frac{1}{N}\sum_{j=1, j\neq i}^{N} \Gamma(X_t^i - X_t^j)\,\mathrm{d}t + \sigma^i\,\mathrm{d}B_t^i$$

Controlled mean-field SDE

$$\mathrm{d}X_t = (\underbrace{\mathcal{I}_t[\rho_t]}_{\int_{\mathbb{R}^d} \Gamma\,(x-\tilde{x})\rho_t(\tilde{x})\mathrm{d}\tilde{x}}(X_t) + u_t(X_t))\mathrm{d}t + \sigma\,\mathrm{d}B_t$$

Stochastic control problem

$$\underset{u_\cdot,\rho_\cdot}{\text{Minimize}} \quad \frac{1}{2\sigma^2}\int_0^T \int_{\mathbb{R}^d} \|u_t(x)\|^2 \rho_t(x)\,\mathrm{d}x\,\mathrm{d}t,$$

subject to

Controlled Mckean-Vlasov integro-PDE

$$\partial_t\rho_t + \nabla_x\cdot\big((\mathcal{I}_t[\rho_t] + u_t)\rho_t\big) = \frac{\sigma^2}{2}\Delta_x\rho_t,$$

$$\rho_0(x) = \rho_{\text{initial}}(x), \quad \rho_T(x) = \rho_{\text{final}}(x).$$

2nd Order Mean-Field Schrödinger Bridge

Prior kinetics for second order interacting agents

$$\mathrm{d}X_t^i = V_t^i \,\mathrm{d}t$$
$$\mathrm{d}V_t^i = \frac{1}{N} \sum_{j=1, j\neq i}^{N} \Gamma\left(X_t^i - X_t^j, V_t^i - V_t^j\right)\mathrm{d}t + \sigma^i \mathrm{d}B_t^i$$

Controlled mean-field SDE

$$\mathrm{d}X_t = V_t \mathrm{d}t$$
$$\mathrm{d}V_t = (\mathcal{I}_t[\mu_t](X_t, V_t) + u_t(X_t, V_t))\mathrm{d}t + \sigma \,\mathrm{d}B_t$$

Stochastic control problem

$$\underset{u_\cdot, \mu_\cdot}{\text{Minimize}} \quad \frac{1}{2\sigma^2} \int_0^T \int_{\mathbb{R}^{2d}} \|u_t(x,v)\|^2 \mu_t(x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$$

subject to

Controlled Vlasov-Fokker-Planck intergo-PDE

$$\partial_t \mu_t + \langle v, \nabla_x \mu_t \rangle + \nabla_v \cdot \left(\mu_t \left(\mathcal{I}_t[\mu_t] + u_t\right)\right) = \frac{\sigma^2}{2} \Delta_v \mu_t$$

$$\mu_0(x,v) = \mu_{\text{initial}}(x,v), \quad \mu_T(x,v) = \mu_{\text{final}}(x,v)$$

2nd Order Mean-Field Schrödinger Bridge

Prior kinetics for second order interacting agents

$$\mathrm{d}X_t^i = V_t^i \,\mathrm{d}t$$
$$\mathrm{d}V_t^i = \frac{1}{N}\sum_{j=1, j\neq i}^{N} \Gamma\left(X_t^i - X_t^j, V_t^i - V_t^j\right)\mathrm{d}t + \sigma^i \mathrm{d}B_t^i$$

Controlled mean-field SDE

$$\mathrm{d}X_t = V_t \mathrm{d}t$$
$$\mathrm{d}V_t = (\mathcal{I}_t[\mu_t](X_t, V_t) + u_t(X_t, V_t))\mathrm{d}t + \sigma \,\mathrm{d}B_t$$

Stochastic control problem

$$\underset{u_\cdot, \mu_\cdot}{\text{Minimize}} \quad \frac{1}{2\sigma^2}\int_0^T \int_{\mathbb{R}^{2d}} \|u_t(x,v)\|^2 \mu_t(x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$$

subject to

$$\partial_t \mu_t + \langle v, \nabla_x \mu_t \rangle + \nabla_v \cdot \left(\mu_t \left(\mathcal{I}_t[\mu_t] + u_t\right)\right) = \frac{\sigma^2}{2}\Delta_v \mu_t$$

Partial information constraints

$$\int_{\mathbb{R}^d} \mu_0(x,v)\,\mathrm{d}v = \rho_{\text{initial}}(x), \qquad \int_{\mathbb{R}^d} \mu_T(x,v)\,\mathrm{d}v = \rho_{\text{final}}(x)$$

Mean-Field Schrödinger Bridge: Results

Derived dynamic Sinkhorn-like recursions to solve all these problem variants

Idea: delayed evaluations and nested updates for nonlinear, nonlocal PDEs

Proved local convergence

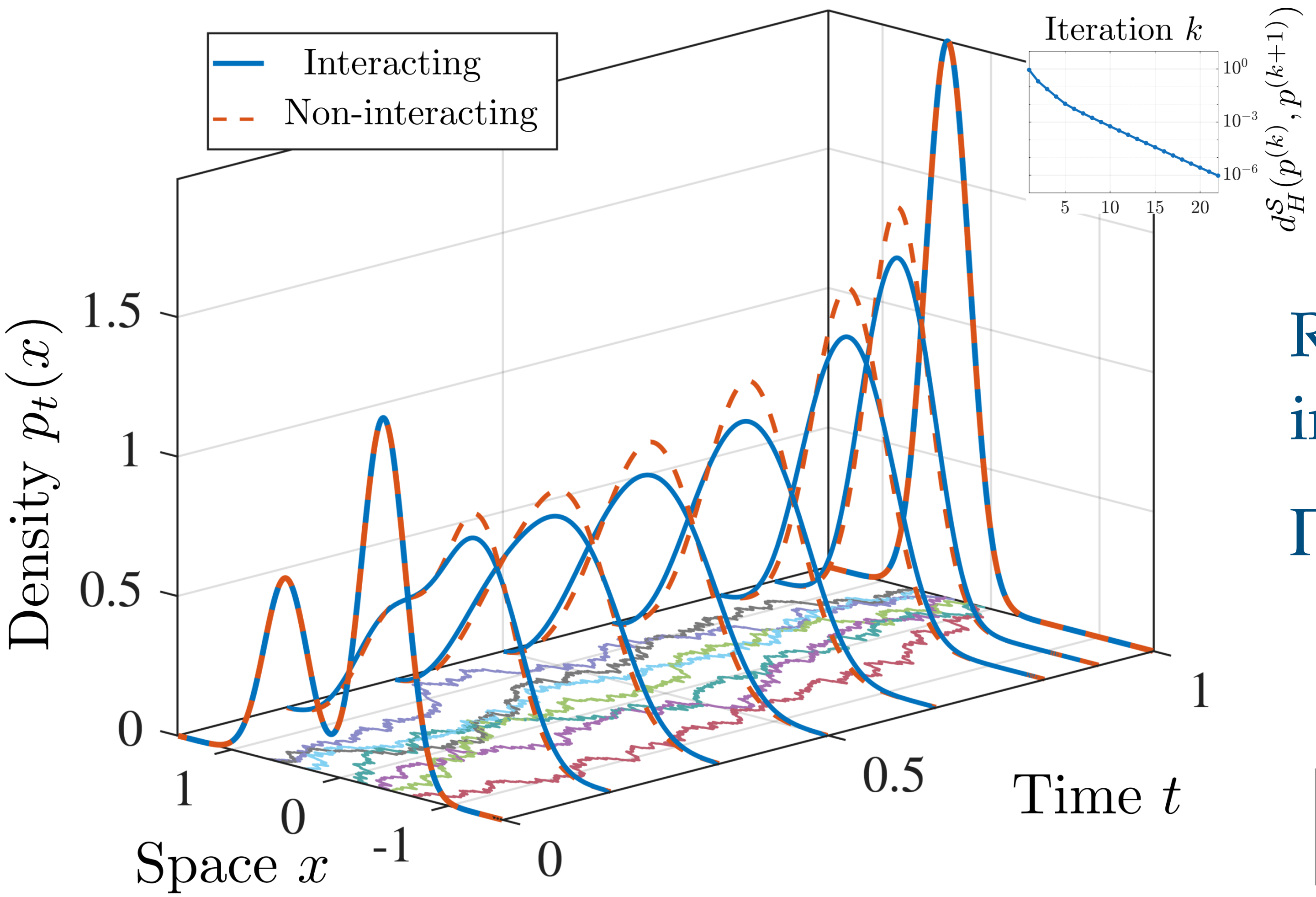
Details for 1st order MFSB:

A. Eldesoukey, Y. Chen, A.H.
A Generalized Sinkhorn Algorithm for Mean-Field Schrödinger Bridge.
LCSS(10), 2026

Details for 2nd order MFSB:

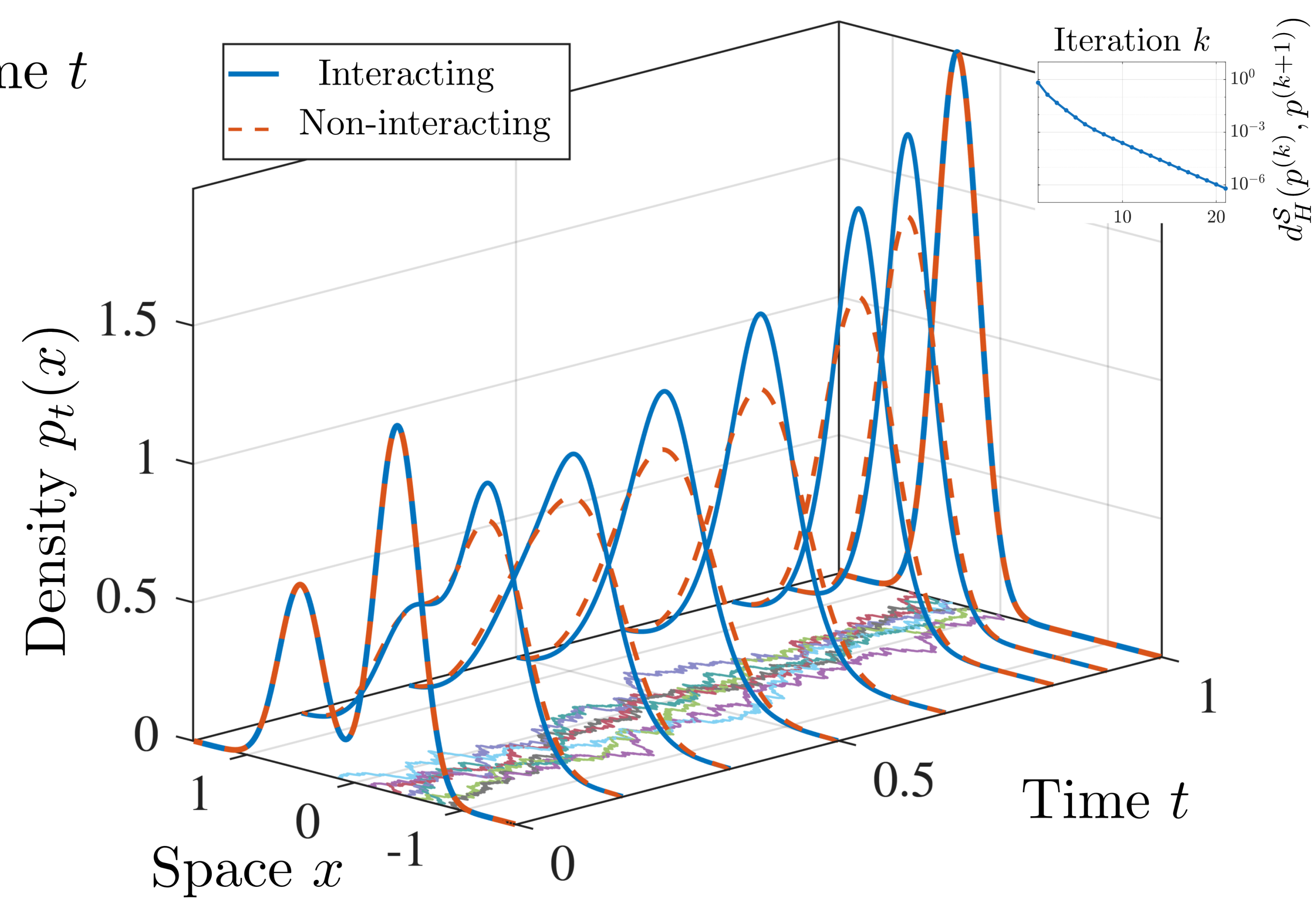
A. Eldesoukey, M.Z. Haider, I. Napolitano, Y. Chen, A.H.
Schrödinger Bridges over Kinetic Swarming Models. arXiv:2608.25281

1st order MFSB: Numerical Examples

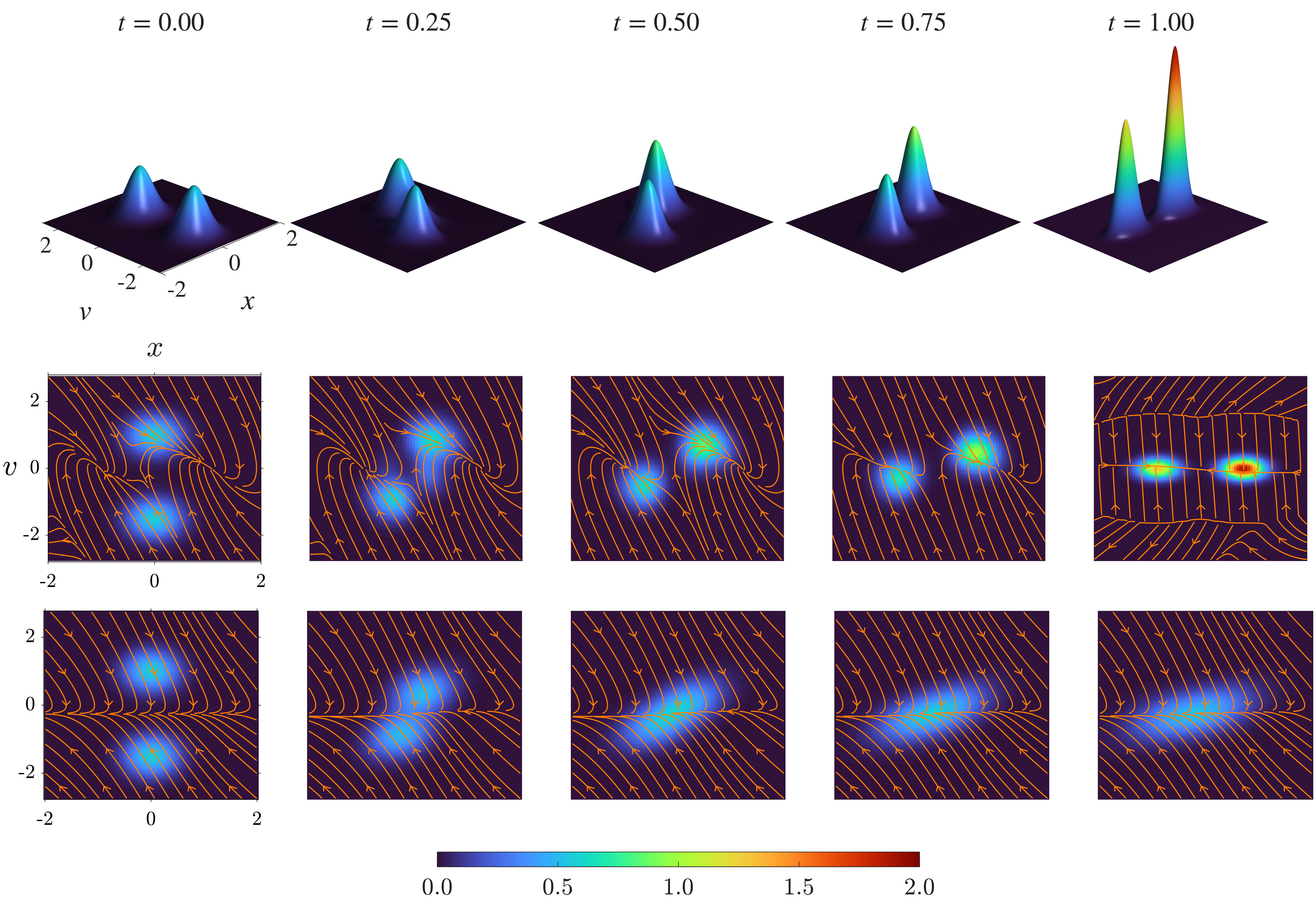


Repulsive (left) and attractive (bottom) interaction kernels

$$\Gamma(x) = \pm\, 1.4/\left(\sqrt{x^2 + 0.01^2}\right)^{0.2}$$

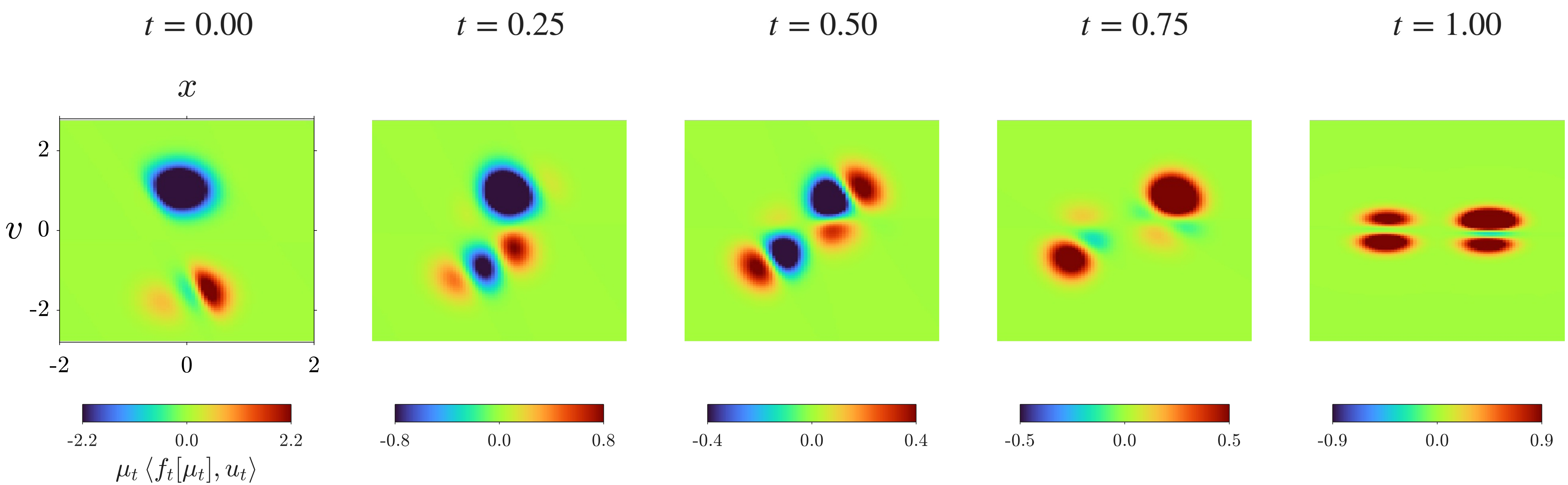


2nd order MFSB: Steering Cucker-Smale Kinetics



2nd order MFSB: Steering Cucker-Smale Kinetics

Inner product of interaction drift and control



Optimal control initially fights and then leverages nonlocal interaction

Summary

Nonlinearity

- Sinkhorn w. memory solves control-affine SB with input vs noise channel mismatch

- local stability proof for the algorithm

Interaction

- generalized Sinkhorn-like algorithms for 1st and 2nd order MFSB

- can handle full or partial information for endpoint marginals

- local convergence proof for the algorithms

Thank You

2112755, 2111688, 2450377