On the Parameterized Computation of Minimum Volume Outer Ellipsoid of Minkowski Sum of Ellipsoids

Abhishek Halder

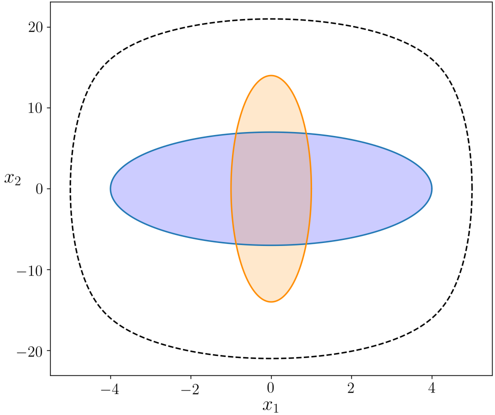
Department of Applied Mathematics
University of California, Santa Cruz
Santa Cruz, CA 95064

Minkowski Sum of Sets

Defn: $\mathcal{Z} = \mathcal{X} \dotplus \mathcal{Y} := \{z \mid z = x + y, x \in \mathcal{X}, y \in \mathcal{Y}\}$

Preserves convexity

Minkowski sum of two ellipsoids



Motivation: Outer Approximate Reach Sets

Linear control system: $\boldsymbol{x}^{+}(t) = \boldsymbol{F}(t)\boldsymbol{x}(t) + \boldsymbol{G}(t)\boldsymbol{u}(t)$

Set-valued (e.g., ellipsoidal) uncertainties: $\boldsymbol{x}(t_0) \in \mathcal{X}_0$, $\boldsymbol{x}(t_1) \in \mathcal{X}_1$, $\boldsymbol{u}(t) \in \mathcal{U}(t)$, $t_0 \leq t \leq t_1$

Forward ($\rightarrow$) and backward ($\leftarrow$) reach set in discrete time:

$$\overrightarrow{\mathcal{R}}\left(\mathcal{X}_0, t, t_0\right) = \boldsymbol{\Phi}\left(t, t_0\right)\mathcal{X}_0 \dot{+} \sum_{\tau=t_0}^{t-1} \boldsymbol{\Phi}\left(t, \tau+1\right)\boldsymbol{G}(\tau)\,\mathcal{U}(\tau)$$

$$\overleftarrow{\mathcal{R}}\left(\mathcal{X}_1, t, t_1\right) = \boldsymbol{\Phi}\left(t, t_1\right)\mathcal{X}_1 \dot{+} \sum_{\tau=t}^{t_1-1} -\boldsymbol{\Phi}\left(t, \tau\right)\boldsymbol{G}(\tau)\,\mathcal{U}(\tau)$$

Why Ellipsoids

Modeling: naturally describes norm bounded uncertainties

Fixed parameterization complexity: requires storing $n(n+3)/2$ reals in $\mathbb{R}^n$

Löwner-John Theorem: Minimum volume outer ellipsoid (MVOE) of any compact set is unique

Computing Löwner-John MVOE is Semi-infinite Program

Let $\mathcal{E}(\boldsymbol{A}, \boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\|_2 \leq 1\}$

$\mathcal{S}$ compact $\subset \mathbb{R}^n$

$$\mathcal{E}(\boldsymbol{A}_{\text{opt}}, \boldsymbol{b}_{\text{opt}}) = \underset{\boldsymbol{A} \succ 0, \boldsymbol{b} \in \mathbb{R}^n}{\arg\min} \; \log\det \boldsymbol{A}^{-1}$$

$$\text{s.t.} \; \sup_{\boldsymbol{x} \in \mathcal{S}} \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\|_2 \leq 1$$

In our context, $\mathcal{S}$ is a Minkowski sum of ellipsoids

Computing MVOE of Minkowski Sum of Ellipsoids

In this case, no algorithm known to compute the Löwner-John MVOE

Standard approach: optimize over a parameterized family of outer ellipsoids

Parametric Description of Ellipsoid in $\mathbb{R}^d$

$(\boldsymbol{q}, \boldsymbol{Q})$ parameterization with $\boldsymbol{Q} \succ \mathbf{0}$:

$$\mathcal{E}\left(\boldsymbol{q}, \boldsymbol{Q}\right) = \{\boldsymbol{x} \in \mathbb{R}^d \mid (\boldsymbol{x} - \boldsymbol{q})^\top \boldsymbol{Q}^{-1}(\boldsymbol{x} - \boldsymbol{q}) \leq 1\}$$

$(\boldsymbol{A}, \boldsymbol{b}, c)$ parameterization with $\boldsymbol{A} \succ \mathbf{0}$:

$$\mathcal{E}(\boldsymbol{A}, \boldsymbol{b}, c) := \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{x}^\top \boldsymbol{b} + c \leq 0\}$$

$(\boldsymbol{q}, \boldsymbol{Q}) \leftrightarrow (\boldsymbol{A}, \boldsymbol{b}, c)$:

$$\boldsymbol{A} = \boldsymbol{Q}^{-1}, \quad \boldsymbol{b} = -\boldsymbol{Q}^{-1}\boldsymbol{q}, \quad c = \boldsymbol{q}^\top \boldsymbol{Q}^{-1}\boldsymbol{q} - 1$$

Parameterized Family of Outer Ellipsoids

Consider $\{\mathcal{E}_k\}_{k=1}^{K}$ in $\mathbb{R}^d$, $\mathcal{E}_k := \mathcal{E}(\boldsymbol{q}_k, \boldsymbol{Q}_k)$. Then

center of the Löwner-John ellipsoid

$$\boldsymbol{q}_{\mathrm{LJ}} = \boldsymbol{q}_1 + \boldsymbol{q}_2 + \ldots + \boldsymbol{q}_K$$

No formula for the shape matrix $\boldsymbol{Q}_{\mathrm{LJ}}$ known.

Durieu, Walter, Polyak (2001):

$$\mathcal{E}(\boldsymbol{q}_{\mathrm{LJ}}, \boldsymbol{Q}_{\mathrm{LJ}}) \subseteq \mathcal{E}(\boldsymbol{q}_{\mathrm{LJ}}, \boldsymbol{Q}(\boldsymbol{\alpha}))$$

$$\boldsymbol{Q}(\boldsymbol{\alpha}) = \sum_{k=1}^{K} \alpha_k^{-1} \boldsymbol{Q}_k, \quad \boldsymbol{\alpha} \in \mathbb{R}_+^K, \quad \mathbf{1}^\top \boldsymbol{\alpha} = 1$$

For $K = 2$ Ellipsoids

$$\alpha_2 = 1 - \alpha_1, \quad \alpha_1/(1-\alpha_1) \mapsto \beta$$

$$\mathcal{E}(\boldsymbol{q}_{\text{LJ}}, \boldsymbol{Q}_{\text{LJ}}) \subseteq \mathcal{E}(\boldsymbol{q}_{\text{LJ}}, \boldsymbol{Q}(\beta))$$
$$\boldsymbol{Q}(\beta) = (1 + 1/\beta)\, \boldsymbol{Q}_1 + (1+\beta)\, \boldsymbol{Q}_2, \; \beta > 0$$

minimum volume parametric optimization:

$$\underset{\beta > 0}{\text{minimize}} \; \log\det\left(\boldsymbol{Q}(\beta)\right)$$

Let $\lambda_i = \text{eig}(\boldsymbol{Q}_1^{-1}\boldsymbol{Q}_2)$. First order optimality:

β_{opt} is unique positive root of $\displaystyle\sum_{i=1}^{d} \frac{1 - \beta^2\lambda_i}{1 + \beta\lambda_i} = 0$

New Algorithm

First order condition can be rewritten as:

$$\beta^2 \sum_{i=1}^{d} \lambda_i / \left(1 + \beta\lambda_i\right) = \sum_{i=1}^{d} 1 / \left(1 + \beta\lambda_i\right)$$

Proposed fixed point iteration:

$$\beta_{n+1} = g\left(\beta_n\right) := \left(\frac{\sum_{i=1}^{d} \frac{1}{1+\beta_n\lambda_i}}{\sum_{i=1}^{d} \frac{\lambda_i}{1+\beta_n\lambda_i}} \right)^{\frac{1}{2}}, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$$

Theorem:

$g(\cdot)$ is contractive in Hilbert metric on $\mathbb{R}_+$

Numerical Results: comparison with SDP Relaxation via $\mathcal{S}$-procedure

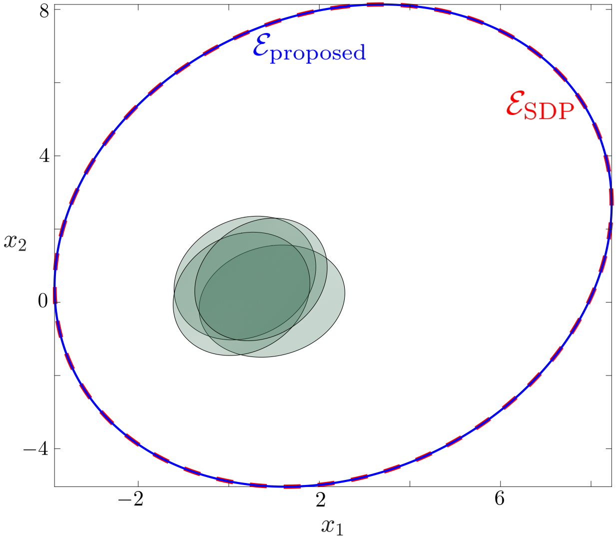
input: $\mathcal{E}(\boldsymbol{A}_i, \boldsymbol{b}_i, c_i)$ in $\mathbb{R}^d$, $i = 1, \ldots, K$

$$
\begin{aligned}
&\underset{\boldsymbol{A}_0, \boldsymbol{b}_0, \tau_1, \ldots, \tau_K}{\text{minimize}} && \log\det \boldsymbol{A}_0^{-1} \\
&\text{s.t.} && \boldsymbol{A}_0 \succ \mathbf{0},\ \tau_k \geq 0, k = 1, \ldots, K,
\end{aligned}
$$

$$
\begin{bmatrix} \boldsymbol{E}_0^\top \boldsymbol{A}_0 \boldsymbol{E}_0 & \boldsymbol{E}_0^\top \boldsymbol{b}_0 & \mathbf{0} \\ \boldsymbol{b}_0^\top \boldsymbol{E}_0 & -1 & \boldsymbol{b}_0^\top \\ \mathbf{0} & \boldsymbol{b}_0 & -\boldsymbol{A}_0 \end{bmatrix} - \sum_{k=1}^{K} \tau_k \begin{bmatrix} \widetilde{\boldsymbol{A}}_k & \widetilde{\boldsymbol{b}}_k & \mathbf{0} \\ \widetilde{\boldsymbol{b}}_k^\top & c_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \preceq \mathbf{0}
$$

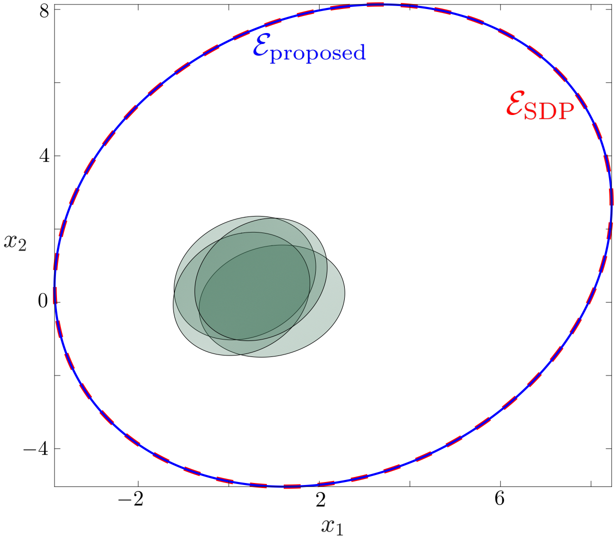
output: $\mathcal{E}(-(\boldsymbol{A}_0^*)^{-1} \boldsymbol{b}_0^*, (\boldsymbol{A}_0^*)^{-1})$

Numerical Results: 2D Example, $K = 4$



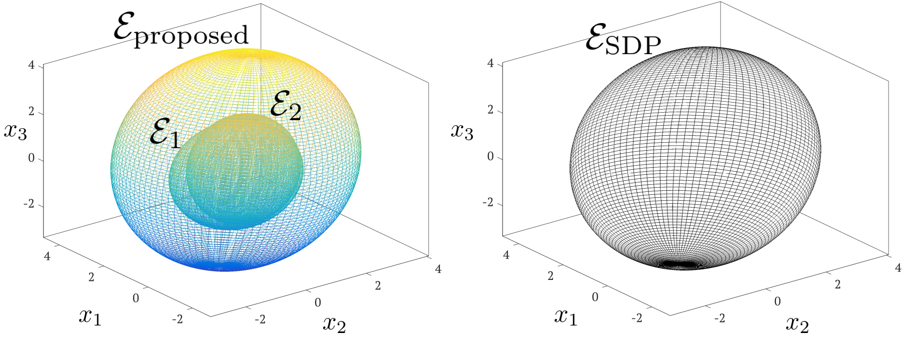
$$\mathrm{vol}\left(\mathcal{E}_{\text{proposed}}\right) = 40.1885, \mathrm{vol}\left(\mathcal{E}_{\text{SDP}}\right) = 40.1884$$

Numerical Results: 2D Example, $K = 4$



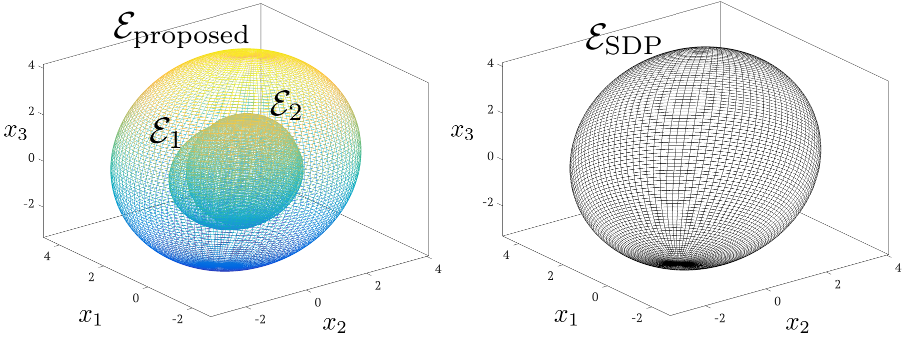
$t_{\text{proposed}} = 0.009184$ sec, $t_{\text{SDP}} = 1.513608$ sec

Numerical Results: 3D Example, $K = 2$



$$\text{vol}\left(\mathcal{E}_{\text{proposed}}\right) = 49.0122, \text{vol}\left(\mathcal{E}_{\text{SDP}}\right) = 49.0121$$

Numerical Results: 3D Example, $K = 2$



$$t_{\text{proposed}} = 0.007521 \text{ sec}, \; t_{\text{SDP}} = 1.687587 \text{ sec}$$

Numerical Results: 2D Forward Reach Set in Discrete Time

$$\boldsymbol{x}^{+}(t) = \boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{G}\boldsymbol{u}(t)$$

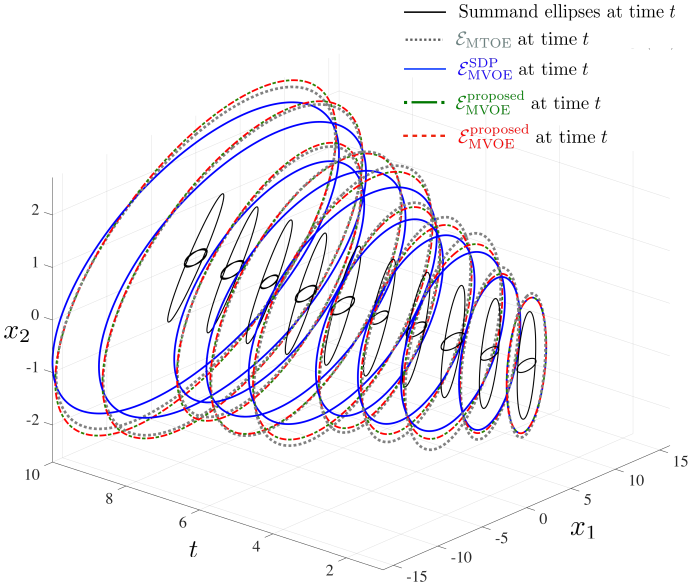
$$\boldsymbol{F} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \boldsymbol{G} = \begin{pmatrix} h & h^2/2 \\ 0 & h \end{pmatrix}, h = 0.3$$

$$\mathcal{X}_0 = \mathcal{E}\left(\boldsymbol{0}, Q_0\right), \mathcal{U}(t) = \mathcal{E}\left(\boldsymbol{0}, \boldsymbol{U}(t)\right)$$

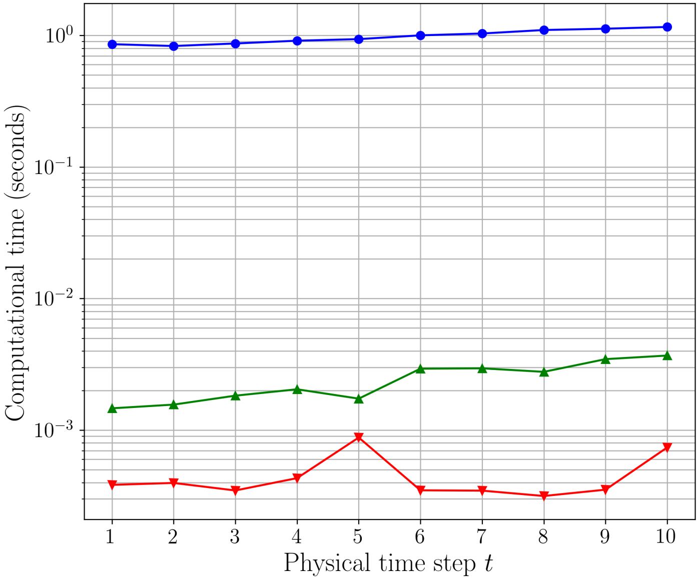
$$\boldsymbol{Q}_0 = \boldsymbol{I}_2, \boldsymbol{U}(t) = \left(1 + \cos^2(t)\right) \mathrm{diag}([10, 0.1])$$

$$\overrightarrow{\mathcal{R}}\left(\mathcal{X}_0, t, t_0\right) = \boldsymbol{F}^t \mathcal{E}\left(\boldsymbol{0}, \boldsymbol{Q}_0\right) + \sum_{k=0}^{t-1} \boldsymbol{F}^{t-k-1} \boldsymbol{G} \mathcal{E}\left(\boldsymbol{0}, \boldsymbol{U}(t)\right)$$

Numerical Results: 2D Forward Reach Set in Discrete Time



Numerical Results: 2D Forward Reach Set in Discrete Time



Recap

– New fixed point algorithm for computing the parameterized MVOE of Minkowski sum of ellipsoids

– Guaranteed convergence, rate is fast due to contractive properties on the cone

– Orders of magnitude speed-up in computational time compared to the standard SDP relaxation

Thank You

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