Optimal Mass Transport over the Euler Equation

Presenter: Johan Karlsson

KTH Royal Institute of Technology, Sweden

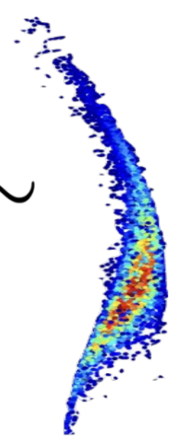
Authors: Charlie Yan (University of California Santa Cruz),
Iman Nodozi (University of California Santa Cruz),
Abhishek Halder (Iowa State University)

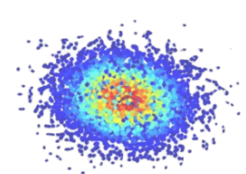
FrA22.6, CDC 2023, Marina Bay Sands, Singapore, December 15, 2023

Motivation

Steer stochastic spin subject to controlled dynamics + deadline constraints

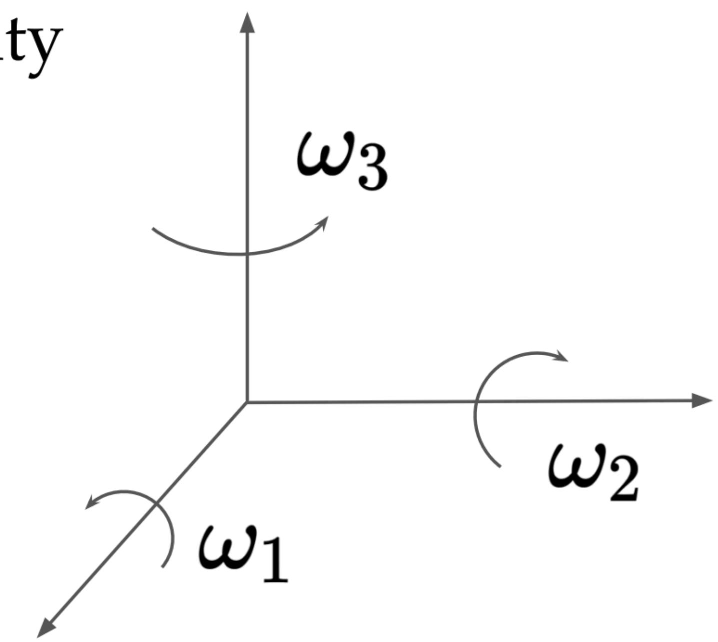
$\rightsquigarrow$

$\boldsymbol{\omega}(t=0) \sim$ 

$\boldsymbol{\omega}(t=T) \sim$ 

Stochastic state:

Angular velocity



Controlled dynamics: Euler equation (EE)

$$\boldsymbol{J}\dot{\boldsymbol{\omega}} = -[\boldsymbol{\omega}]^{\times}\boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{\tau}$$

Principal moment of inertia matrix

Control torque

$$[\boldsymbol{\omega}]^{\times} := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in \mathfrak{so}(3)$$

Rewrite as:

$$\dot{\boldsymbol{x}}^{\boldsymbol{u}} = \boldsymbol{\alpha} \odot \boldsymbol{f}(\boldsymbol{x}^{\boldsymbol{u}}) + \boldsymbol{\beta} \odot \boldsymbol{u}, i \in [\![3]\!] := \{1, 2, 3\}$$

where

$$\boldsymbol{f}(\boldsymbol{z}) := (z_2 z_3, z_3 z_1, z_1 z_2)^{\top} \text{ for } \boldsymbol{z} \in \mathbb{R}^3$$

$$\alpha_i := (J_{i+1 \mod 3} - J_{i+2 \mod 3})/J_i,\ \beta_i := 1/J_i,\ i \in [\![3]\!]$$

Optimal steering of stochastic spin

Deterministic OMT-EE

strictly convex and superlinear

$$\inf_{\boldsymbol{u}\in\mathcal{U}} \int_0^T \mathbb{E}_{\mu^{\boldsymbol{u}}}[q(\boldsymbol{x}^{\boldsymbol{u}}) + r(\boldsymbol{u})]\mathrm{d}t$$

subject to
$$\dot{\boldsymbol{x}}^{\boldsymbol{u}} = \boldsymbol{\alpha} \odot \boldsymbol{f}(\boldsymbol{x}^{\boldsymbol{u}}) + \boldsymbol{\beta} \odot \boldsymbol{u}, \quad i \in \{1,2,3\},$$
$$\mu^{\boldsymbol{u}}(\boldsymbol{x}^{\boldsymbol{u}}, t=0) = \mu_0 \text{ (given)}, \quad \mu^{\boldsymbol{u}}(\boldsymbol{x}^{\boldsymbol{u}}, t=T) = \mu_T \text{ (given)}$$

Set of feasible policies $\mathcal{U} := \left\{ \boldsymbol{u} : \mathbb{R}^3 \times [0,T] \mapsto \mathbb{R}^3 \mid \int_0^T \mathbb{E}_{\mu^u}[r(\boldsymbol{u})]\mathrm{d}t < \infty \right\}$

Classical dynamic OMT: $q \equiv 0, \quad r(\cdot) \equiv \frac{1}{2}\|\cdot\|_2^2, \quad \boldsymbol{f} = \boldsymbol{0}, \quad \boldsymbol{\beta} = \boldsymbol{1}$

Optimal steering of stochastic spin (contd.)

Deterministic OMT-EE

$$\begin{aligned} &\inf_{\boldsymbol{u}\in\mathcal{U}} \int_0^T \mathbb{E}_{\mu^{\boldsymbol{u}}}[q(\boldsymbol{x}^{\boldsymbol{u}}) + r(\boldsymbol{u})]\mathrm{d}t \\ \text{subject to}\quad & \dot{\boldsymbol{x}}^{\boldsymbol{u}} = \boldsymbol{\alpha} \odot \boldsymbol{f}(\boldsymbol{x}^{\boldsymbol{u}}) + \boldsymbol{\beta} \odot \boldsymbol{u}, \quad i \in \{1,2,3\}, \\ & \mu^{\boldsymbol{u}}(\boldsymbol{x}^{\boldsymbol{u}}, t=0) = \mu_0 \text{ (given)}, \quad \mu^{\boldsymbol{u}}(\boldsymbol{x}^{\boldsymbol{u}}, t=T) = \mu_T \text{ (given)} \end{aligned}$$

Stochastic OMT-EE ⇝ generalized Schrödinger bridge problem

$$\begin{aligned} &\inf_{\boldsymbol{u}\in\mathcal{U}} \int_0^T \mathbb{E}_{\mu^{\boldsymbol{u}}}[q(\boldsymbol{x}^{\boldsymbol{u}}) + r(\boldsymbol{u})]\mathrm{d}t \\ \text{subject to}\quad & \mathrm{d}\boldsymbol{x}^{\boldsymbol{u}} = (\boldsymbol{\alpha} \odot \boldsymbol{f}(\boldsymbol{x}^{\boldsymbol{u}}) + \boldsymbol{\beta} \odot \boldsymbol{u})\,\mathrm{d}t + \sqrt{2\delta}\,\mathrm{d}\boldsymbol{w}, \quad i \in \{1,2,3\} \\ & \mu^{\boldsymbol{u}}(\boldsymbol{x}^{\boldsymbol{u}}, t=0) = \mu_0 \text{ (given)}, \quad \mu^{\boldsymbol{u}}(\boldsymbol{x}^{\boldsymbol{u}}, t=T) = \mu_T \text{ (given)} \end{aligned}$$

Static version of the OMT-EE

Deterministic:

$$\arg\inf_{\pi \in \Pi_2(\mu_0, \mu_T)} \int_{\mathbb{R}^3 \times \mathbb{R}^3} c(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\pi(\boldsymbol{x}, \boldsymbol{y})$$

Stochastic:

$$\arg\inf_{\pi \in \Pi_2(\mu_0, \mu_T)} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left\{ c(\boldsymbol{x}, \boldsymbol{y}) + \delta \log \pi(\boldsymbol{x}, \boldsymbol{y}) \right\} \mathrm{d}\pi(\boldsymbol{x}, \boldsymbol{y}), \quad \delta > 0$$

entropic regularization

where

$$c(\boldsymbol{x}, \boldsymbol{y}) = \inf_{\gamma(\cdot) \in \Gamma_{\boldsymbol{x}\boldsymbol{y}}} \int_0^T L(t, \gamma(t), \dot{\gamma}(t)) \mathrm{d}t$$

with Lagrangian

$$L(t, \boldsymbol{\gamma}, \dot{\boldsymbol{\gamma}}) \equiv q(\boldsymbol{\gamma}) + r((\dot{\boldsymbol{\gamma}} - \boldsymbol{\alpha} \odot \boldsymbol{f}) \oslash \boldsymbol{\beta})$$

and

$$\Gamma_{\boldsymbol{x}\boldsymbol{y}} := \{ \boldsymbol{\gamma} : [0, T] \mapsto \mathbb{R}^n \mid \boldsymbol{\gamma}(\cdot) \text{ absolutely continuous}, \boldsymbol{\gamma}(0) = \boldsymbol{x}, \boldsymbol{\gamma}(T) = \boldsymbol{y} \}$$

Use of identified Lagrangian

Theorem: (informal)

Assume μ_0, μ_T are absolutely continuous with finite second moments.

Guaranteed existence-uniqueness of minimizer $\left(\rho^{\text{opt}}, \boldsymbol{u}^{\text{opt}}\right)$ for dynamic OMT-EE

Proof strategy: Show that the Lagrangian L is of weak Tonelli type

Then use Figalli's theorem [2007] on OMT costs derived from action functionals

Necessary conditions of optimality for OMT-EE

Stochastic:

Hamilton-Jacobi-Bellman PDE

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\|\boldsymbol{\beta} \odot \nabla_{\boldsymbol{x}^{\boldsymbol{u}}} \phi\|_2^2 + \langle \nabla_{\boldsymbol{x}^{\boldsymbol{u}}} \phi, \boldsymbol{\alpha} \odot \boldsymbol{f}(\boldsymbol{x}^{\boldsymbol{u}}) \rangle = -\delta \Delta_{\boldsymbol{x}^{\boldsymbol{u}}} \phi,$$

Fokker-Planck-Kolmogorov PDE

$$\frac{\partial \rho^{\mathrm{opt}}}{\partial t} + \nabla_{\boldsymbol{x}^{\boldsymbol{u}}} \cdot \left(\rho^{\mathrm{opt}} \left(\boldsymbol{\alpha} \odot \boldsymbol{f}(\boldsymbol{x}^{\boldsymbol{u}}) + \boldsymbol{\beta}^2 \odot \nabla_{\boldsymbol{x}^{\boldsymbol{u}}} \phi\right)\right) = \delta \Delta_{\boldsymbol{x}^{\boldsymbol{u}}} \rho^{\mathrm{opt}},$$

Endpoint constraints

$$\rho^{\mathrm{opt}}(\boldsymbol{x}^{\boldsymbol{u}}, t=0) = \rho_0, \quad \rho^{\mathrm{opt}}(\boldsymbol{x}^{\boldsymbol{u}}, t=T) = \rho_T,$$

Optimal control

$$\boldsymbol{u}^{\mathrm{opt}} = \boldsymbol{\beta} \odot \nabla_{\boldsymbol{x}^{\boldsymbol{u}}} \phi$$

value function

Deterministic: Solve above system of coupled PDE boundary value problem

Then pass to the limit $\delta \downarrow 0$

Case study: $q \equiv 0, \quad r(\,\cdot\,) \equiv \dfrac{1}{2}\|\cdot\|_2^2$

Numerically solve the coupled PDE boundary value problem

using modified physics informed neural network (PINN)

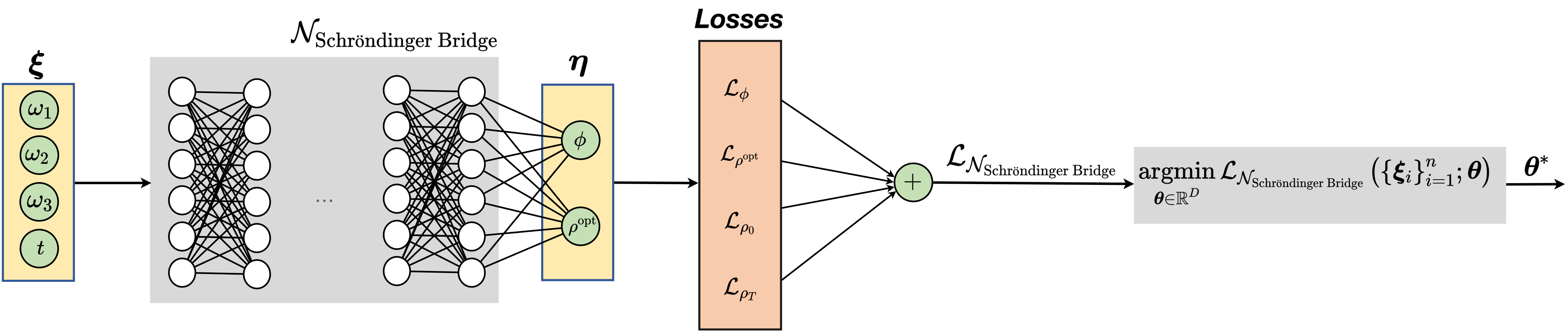
Network input: $\boldsymbol{\xi} := (\omega_1, \omega_2, \omega_3, t)$

Network output: $\boldsymbol{\eta} := (\phi, \rho^{\text{opt}})$

Train network parameters $\boldsymbol{\theta} \in \mathbb{R}^D$ such that $\boldsymbol{\eta}(\boldsymbol{\xi}) \approx \mathcal{N}_{\text{Schrödinger Bridge}}(\boldsymbol{\xi}; \boldsymbol{\theta})$

Compare the optimally controlled vs. uncontrolled PDF evolution

Modified PINN architecture



HJB PDE loss: $\mathcal{L}_\phi$

FPK PDE loss: $\mathcal{L}_{\rho^{\text{opt}}}$

Sinkhorn loss: $W_\varepsilon^2(\mu_0,\mu_1) := \inf\limits_{\pi\in\Pi_2(\mu_0,\mu_T)} \int_{\mathbb{R}^n\times\mathbb{R}^n} \left\{\|\boldsymbol{x}-\boldsymbol{y}\|_2^2 + \varepsilon\log\pi(\boldsymbol{x},\boldsymbol{y})\right\}\mathrm{d}\pi(\boldsymbol{x},\boldsymbol{y})$

Sinkhorn losses for boundary conditions: $\mathcal{L}_{\rho_i} := W_\varepsilon^2\left(\rho_i, \rho_i^{\text{epoch index}}(\boldsymbol{\theta})\right)$

Implementation friendly: $\texttt{Autodiff}_{\boldsymbol{\theta}} W_\varepsilon^2\left(\rho_i, \rho_i^{\text{epoch index}}(\boldsymbol{\theta})\right) \quad \forall i\in\{0,T\}$

Uncontrolled PDF evolution for Euler equation

Uncontrolled (unc) Liouville PDE IVP:

$$\frac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot \left(\rho \boldsymbol{\alpha} \odot \boldsymbol{f}(\boldsymbol{x})\right) = 0\,, \;\; \rho(\boldsymbol{x}, t=0) = \rho_0 \; (\text{given})$$

Because $\boldsymbol{f}$ is divergence-free, IVP solution: $\rho^{\text{unc}}(\boldsymbol{x}, t) = \rho_0(\boldsymbol{x}_0(\boldsymbol{x}, t))$

inverse flow map

For axisymmetric rigid body $(J_1 = J_2 \neq J_3)$

$$\rho^{\text{unc}}(x_1, x_2, x_3, t) = \rho_0\left(\left(\frac{x_1^2 + x_2^2}{1+\gamma^2}\right)^{\frac{1}{2}}, \gamma\left(\frac{x_1^2 + x_2^2}{1+\gamma^2}\right)^{\frac{1}{2}}, x_3\right)$$

$$\gamma := \frac{x_2 - x_1 \tan(\alpha_2 x_3 t)}{x_1 + x_2 \tan(\alpha_2 x_3 t)}$$

Non-axisymmetric case in terms of Jacobi elliptic functions

Numerical simulation

$$\rho_0 = \mathcal{N}((2,2,2), 0.5\boldsymbol{I}_3), \qquad \rho_T = \mathcal{N}((0,0,0), 0.5\boldsymbol{I}_3)$$

3 hidden layers, 70 neurons in each, tanh activation, ADAM

80k epochs, 100k domain samples (mini-batched 35k of every 40k epoch)
$+$ 1250 boundary condition samples

Sinkhorn loss regularizer $\varepsilon = 0.1$

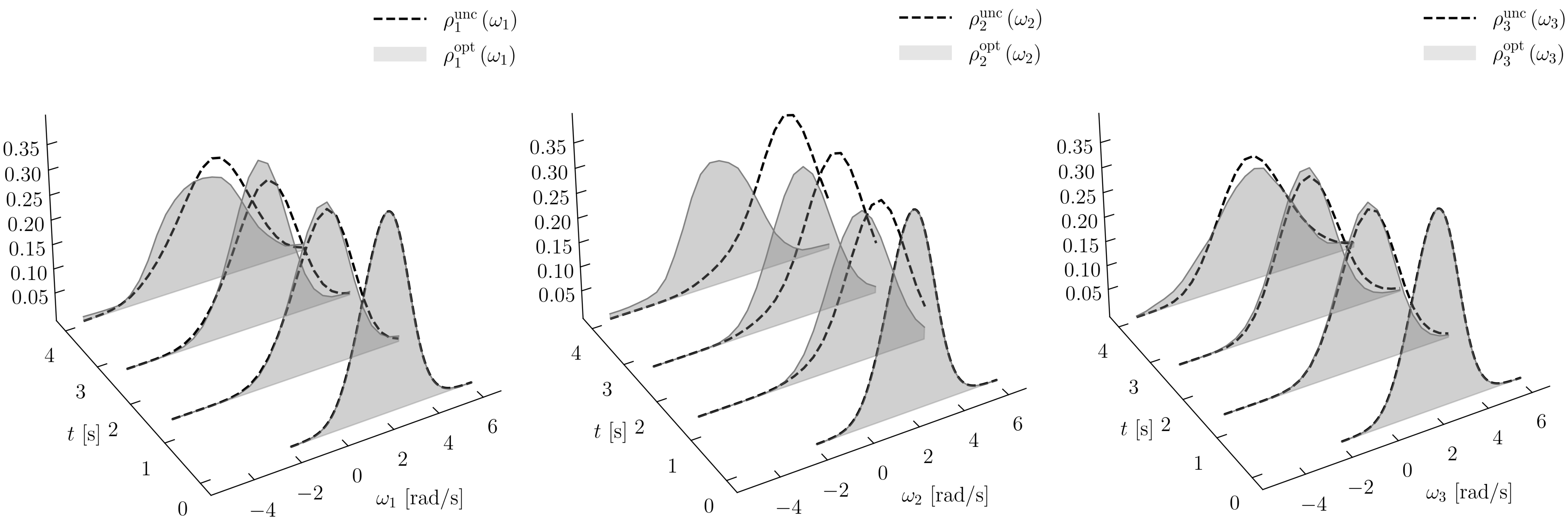
Principal moments of inertia: $J_1 = 0.45$, $J_2 = 0.50$, $J_3 = 0.55$

Final time $T = 4$

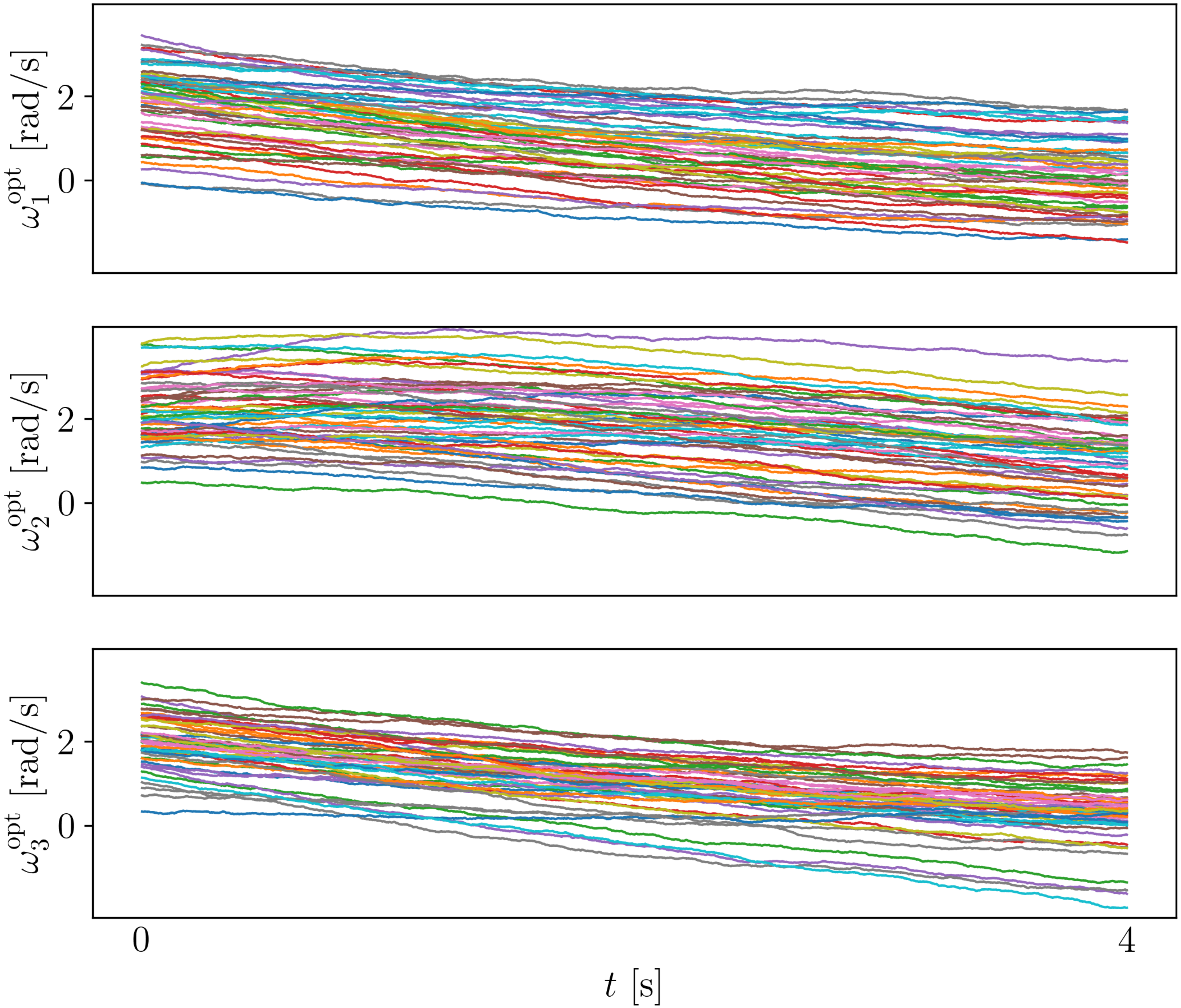
PINN space-time collocation domain: $[-5,5]^3 \times [0,4]$

Univariate marginals of optimally controlled joint

- Four snapshots
- Uncontrolled (--) vs controlled ($\boxed{}$) for $\omega_1, \omega_2, \omega_3$



50 optimal closed-loop state sample paths



Euler-Maruyama integration with noise strength 0.1

Summary of contributions

OMT-EE: formulation, existence-uniqueness of solution, conditions for optimality

Modified PINN for numerical solution of the coupled PDE system

Ongoing work

Stochastic steering of attitude-spin over tangent bundle $\mathcal{T}\mathbf{SO}(3) \simeq \mathbf{SO}(3) \times \mathbb{R}^{3}$

Thank You

Acknowledgment: NSF 2112755