Prediction and Optimal Feedback Control of Probability Densities in Power System Dynamics

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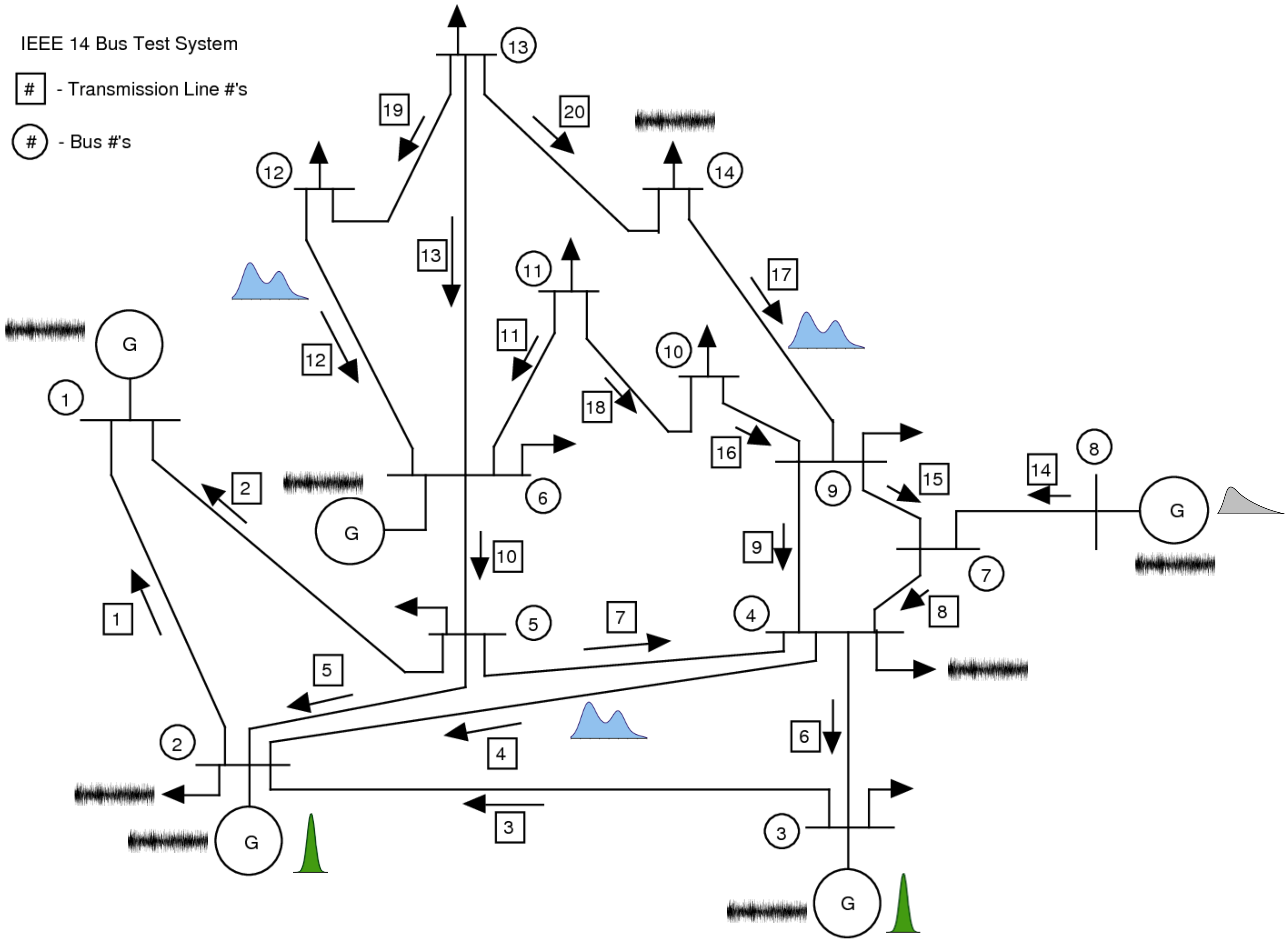
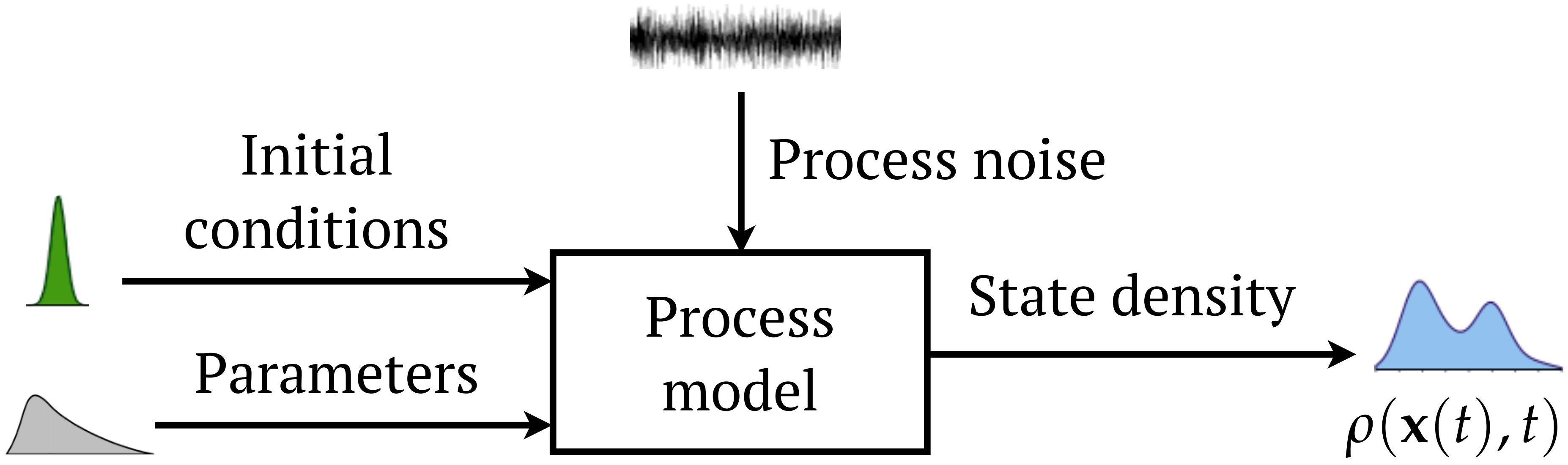
Uncertainty management in power systems

Probability density function (PDF)

Prediction + optimal control of PDFs



Joint PDF prediction



Trajectory flow:

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}(\mathbf{X}, t)\,\mathrm{d}t + \mathbf{g}(\mathbf{X}, t)\,\mathrm{d}\mathbf{w}(t), \quad \mathrm{d}\mathbf{w}(t) \sim \mathcal{N}(0, \mathbf{Q}\mathrm{d}t)$$

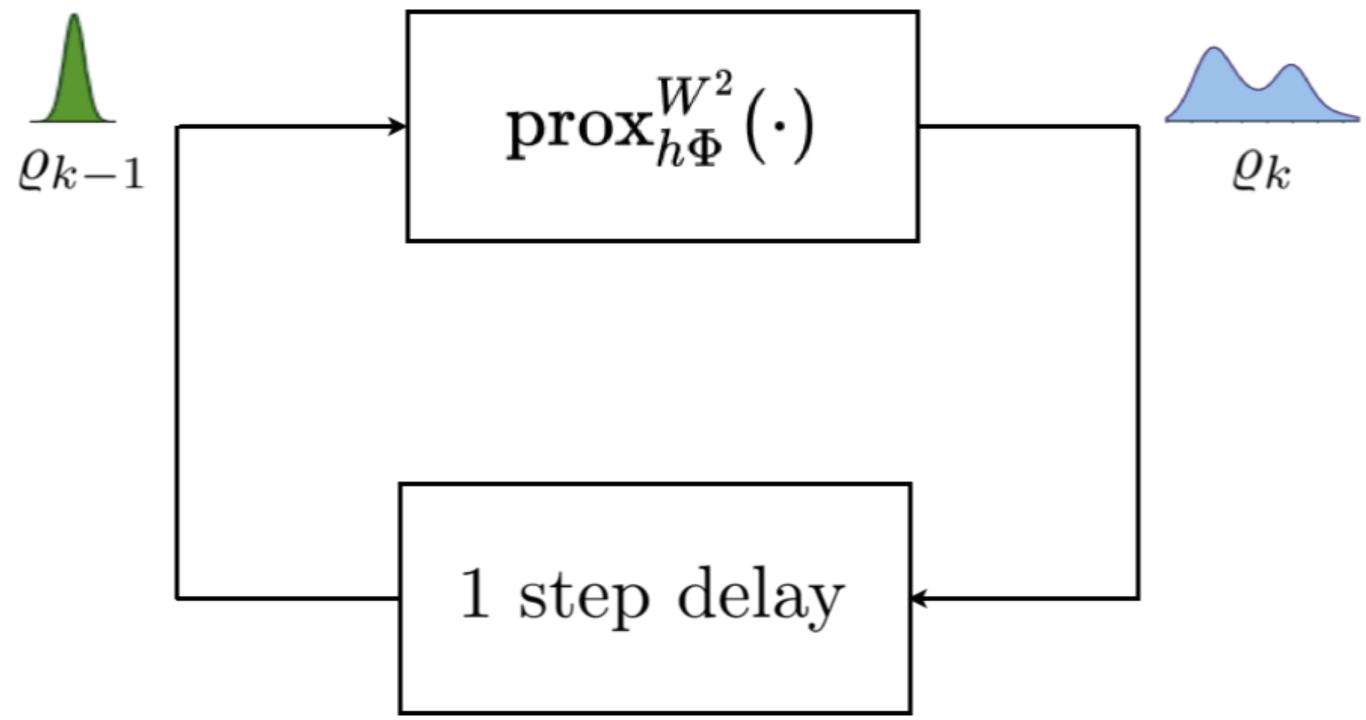
Density flow:

$$\frac{\partial \rho}{\partial t} = \mathcal{L}_{\mathrm{FP}}(\rho) := -\nabla \cdot (\rho \mathbf{f}) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left(\left(\mathbf{g}\mathbf{Q}\mathbf{g}^\top \right)_{ij} \rho \right)$$

Proposed computation: grid-less non-parametric

Main idea: Solve $\dfrac{\partial \rho}{\partial t} = \mathcal{L}_{\mathrm{FP}}\rho, \;\; \rho(x, t=0) = \rho_0$ as gradient flow in $\mathcal{P}_2(\mathcal{X})$

Infinite dimensional variational recursion:

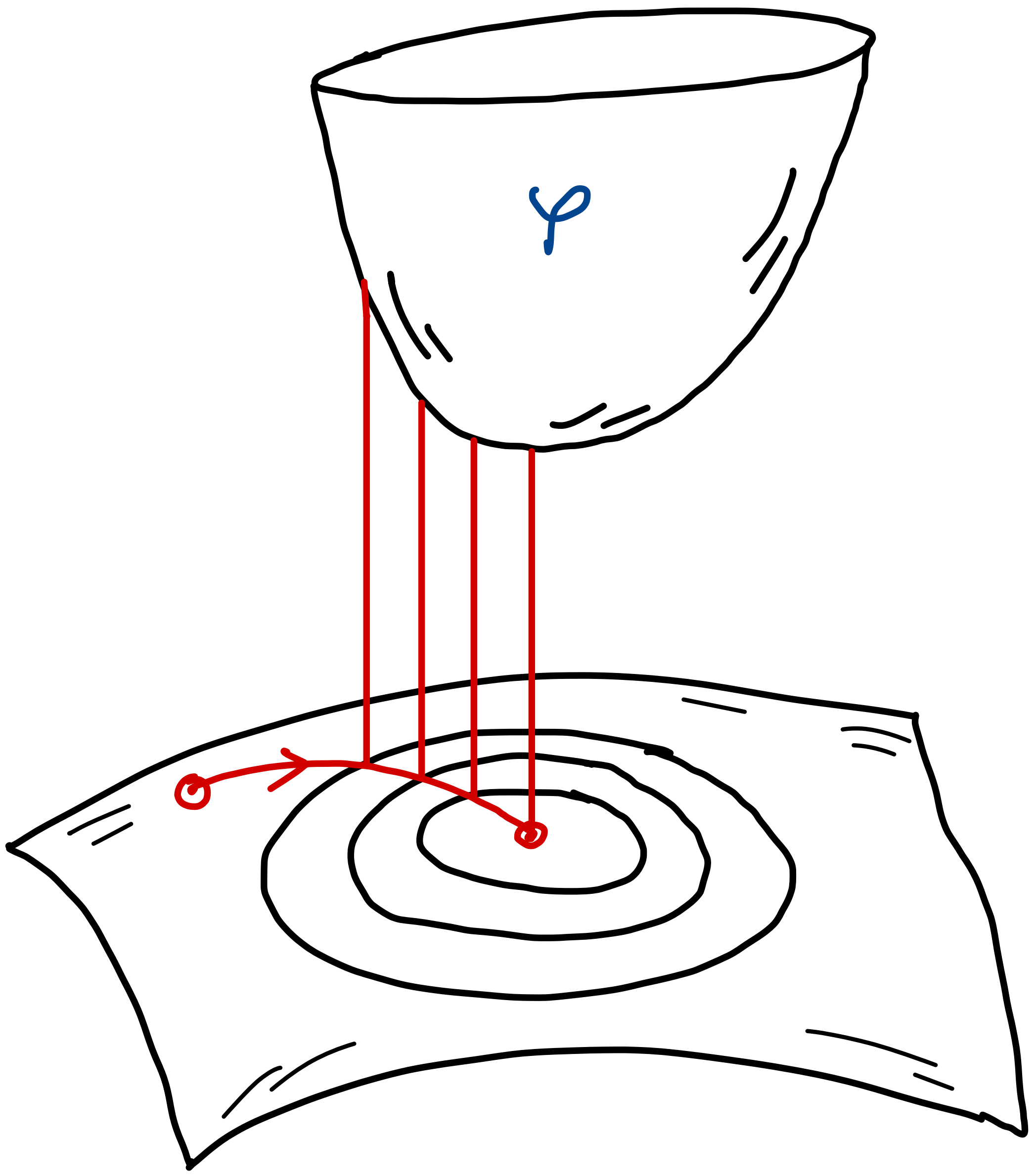


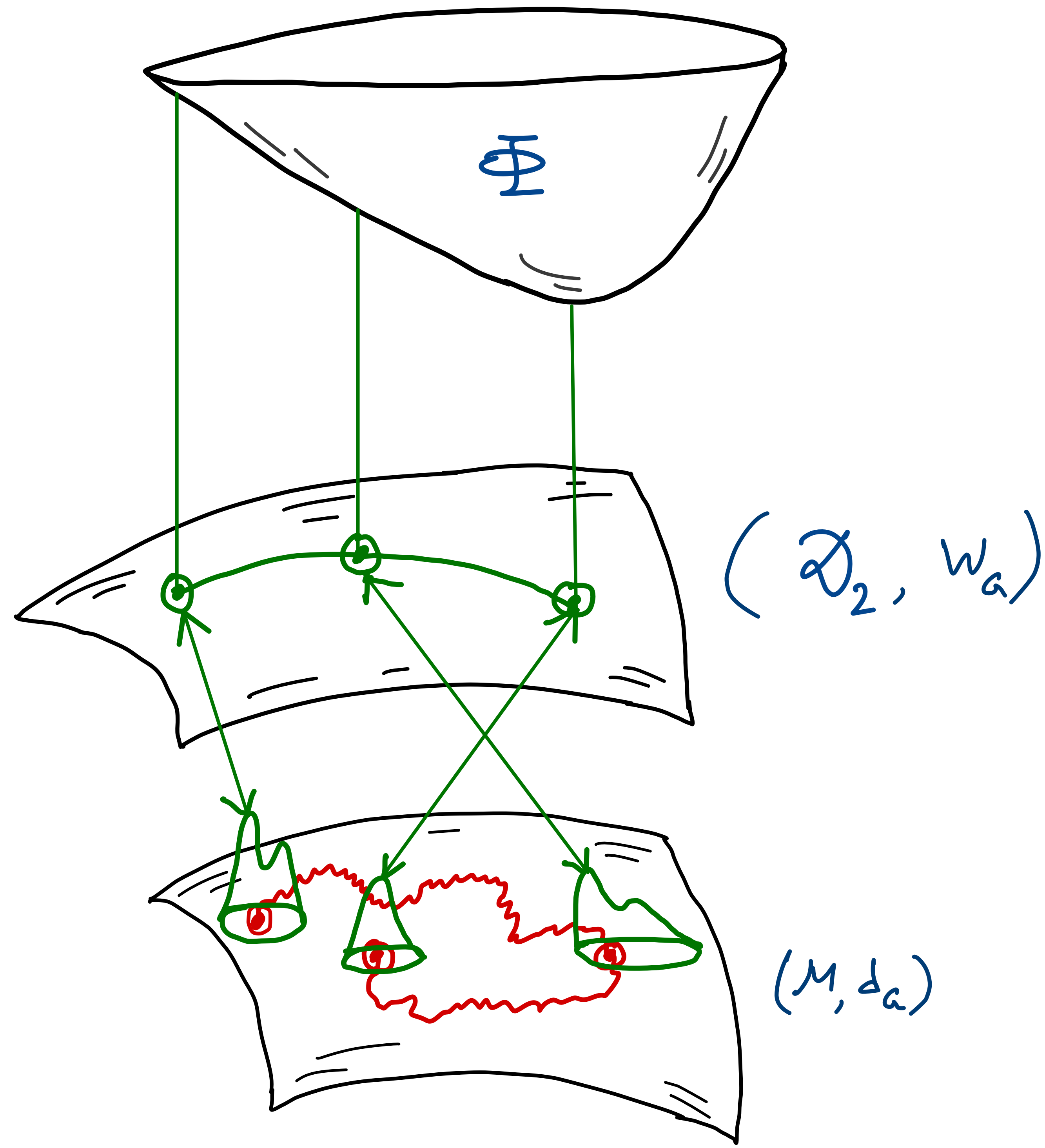
Proximal operator: $\varrho_k = \mathrm{prox}_{h\Phi}^{W^2}(\varrho_{k-1}) := \underset{\varrho \in \mathcal{P}_2(\mathcal{X})}{\arg\inf} \left\{ \frac{1}{2} W^2(\varrho, \varrho_{k-1}) + h\Phi(\varrho) \right\}$

Optimal transport cost: $W^2(\varrho, \varrho_{k-1}) := \displaystyle\inf_{\pi \in \Pi(\varrho, \varrho_{k-1})} \int_{\mathcal{X} \times \mathcal{X}} c(x, y) \, \mathrm{d}\pi(x, y)$

Free energy functional: $\Phi(\varrho) := \displaystyle\int_{\mathcal{X}} \psi \varrho \, \mathrm{d}x + \beta^{-1} \int_{\mathcal{X}} \varrho \log \varrho \, \mathrm{d}x$

Generalized gradient flow with base manifold

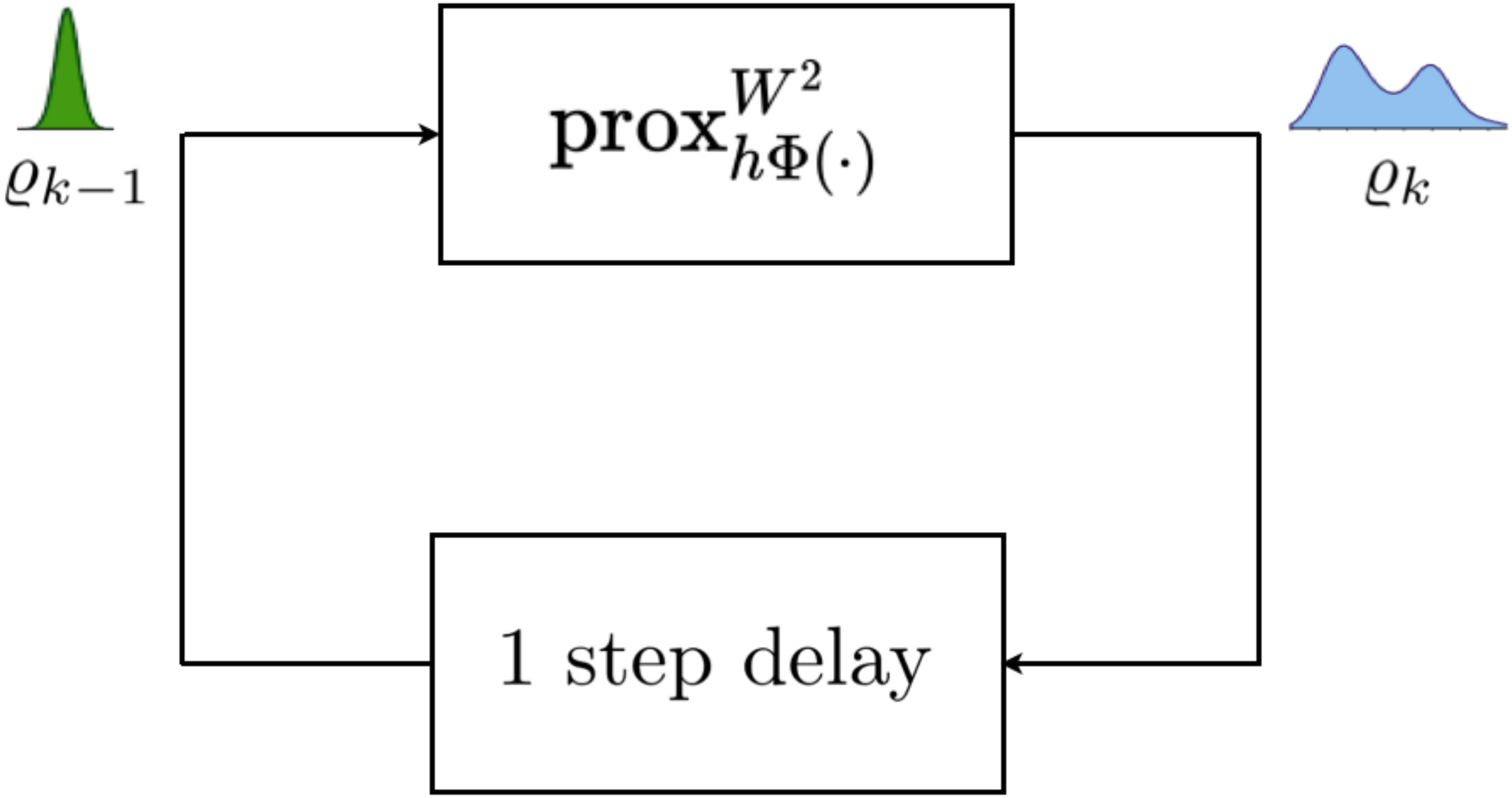




Design pair (W_G, Φ)

Algorithm: gradient ascent on the dual space

Uncertainty propagation via point clouds



No spatial discretization or function approximation

Algorithm: gradient ascent on the dual space

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla \psi \rho) + \beta^{-1} \Delta \rho$$

$\Updownarrow$ **Proximal Recursion**

$$\rho_k = \rho(\boldsymbol{x}, t = kh) = \underset{\rho \in \mathcal{P}_2(\mathbb{R}^n)}{\arg\inf} \left\{ \frac{1}{2} W^2(\rho, \rho_{k-1}) + h\, \Phi(\rho) \right\}$$

$\Downarrow$ **Discrete Primal Formulation**

$$\boldsymbol{\varrho}_k = \arg\min_{\boldsymbol{\varrho}} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + h \langle \boldsymbol{\psi}_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \rangle \right\}$$

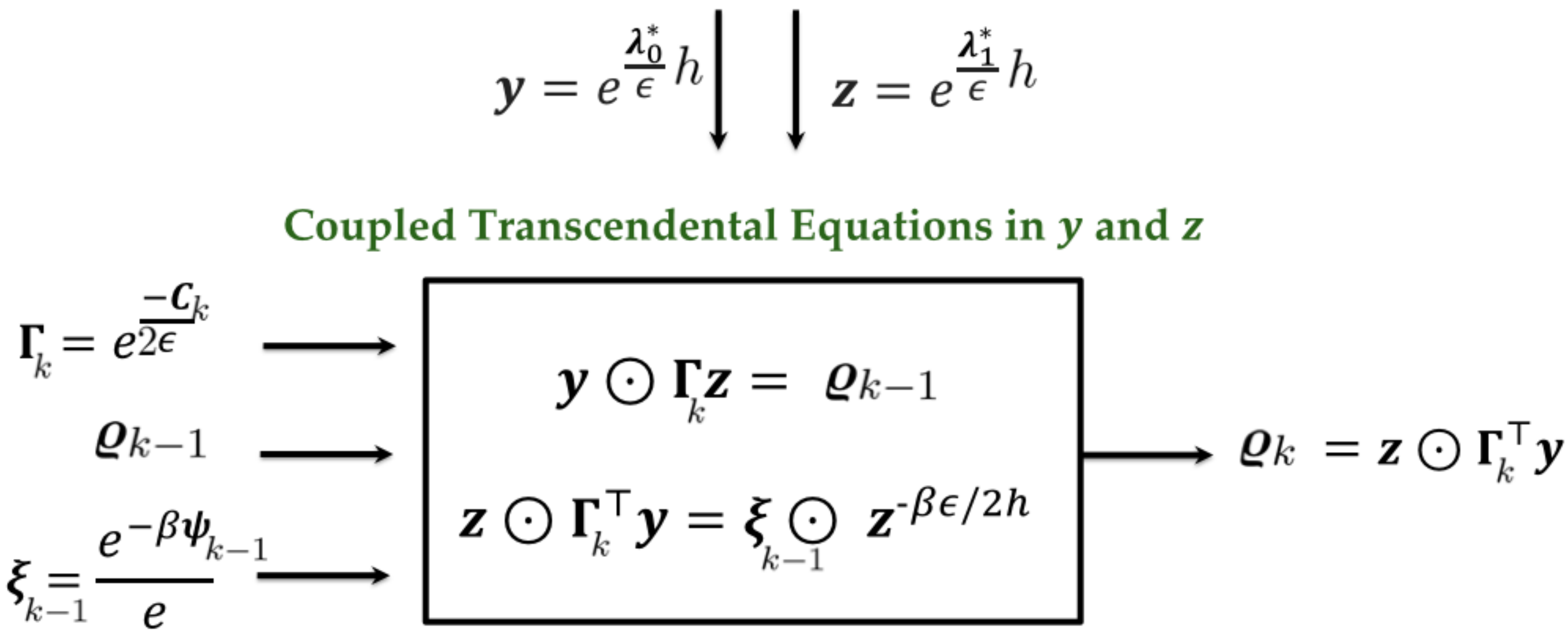
$\Downarrow$ **Entropic Regularization**

$$\boldsymbol{\varrho}_k = \arg\min_{\boldsymbol{\varrho}} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + \epsilon H(\boldsymbol{M}) + h \langle \boldsymbol{\psi}_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \rangle \right\}$$

$\Updownarrow$ **Dualization**

$$\boldsymbol{\lambda}_0^{\text{opt}}, \boldsymbol{\lambda}_1^{\text{opt}} = \underset{\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1 \geq 0}{\arg\max} \left\{ \langle \boldsymbol{\lambda}_0, \boldsymbol{\varrho}_{k-1} \rangle - F^\star(-\boldsymbol{\lambda}_1) - \frac{\epsilon}{h} \left(\exp(\boldsymbol{\lambda}_0^\top h/\epsilon) \exp(-\boldsymbol{C}_k/2\epsilon) \exp(\boldsymbol{\lambda}_1 h/\epsilon) \right) \right\}$$

Recursion on the cone

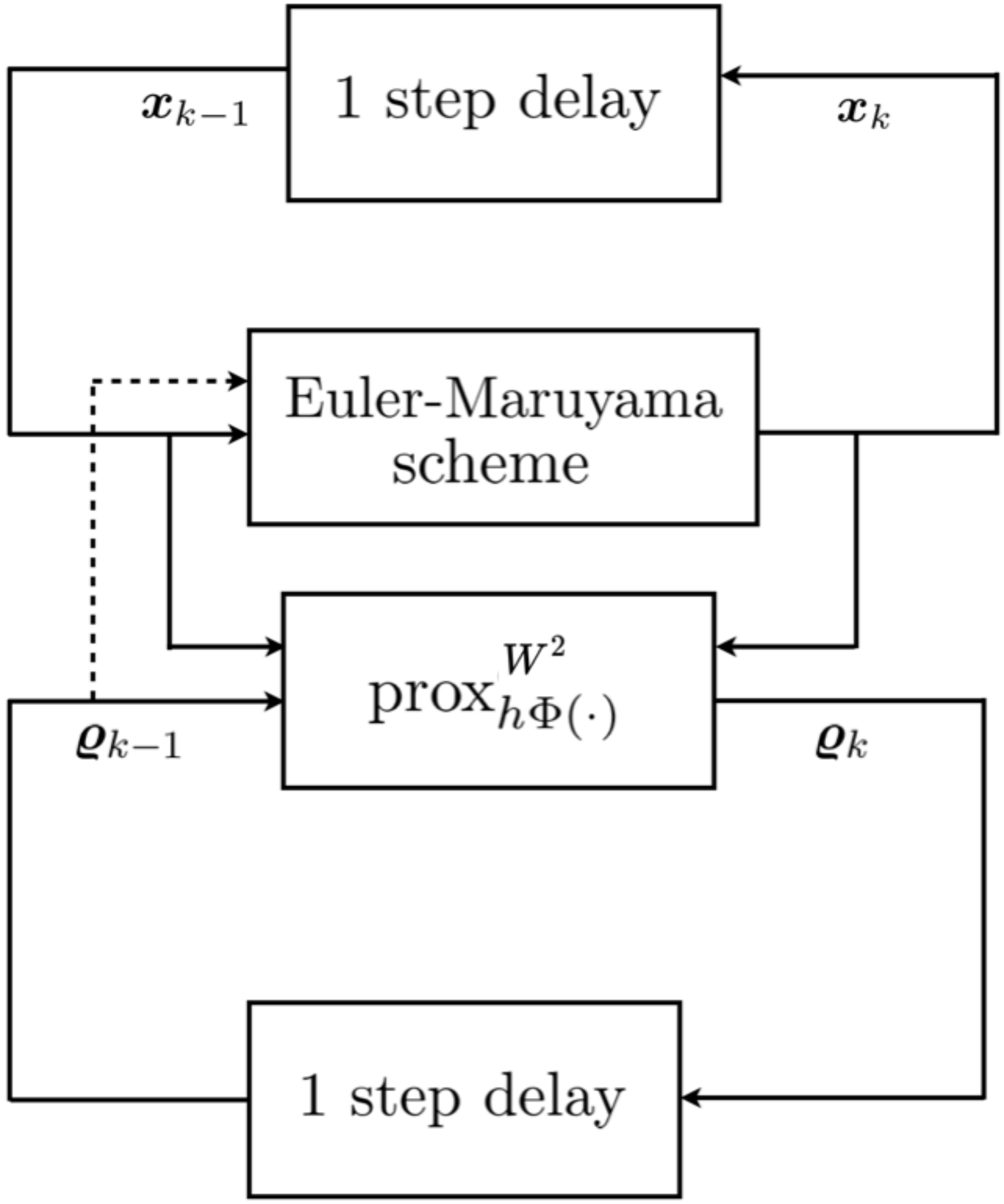


Theorem: Consider the recursion on the cone $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0}$

$$\boldsymbol{y} \odot (\boldsymbol{\Gamma}_k \boldsymbol{z}) = \varrho_{k-1}, \quad \boldsymbol{z} \odot \left(\boldsymbol{\Gamma}_k{}^\top \boldsymbol{y}\right) = \boldsymbol{\xi}_{k-1} \odot \boldsymbol{z}^{-\frac{\beta\epsilon}{h}},$$

Then the solution $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ gives the proximal update $\varrho_k = \boldsymbol{z}^* \odot (\boldsymbol{\Gamma}_k{}^\top \boldsymbol{y}^*)$

Algorithmic setup

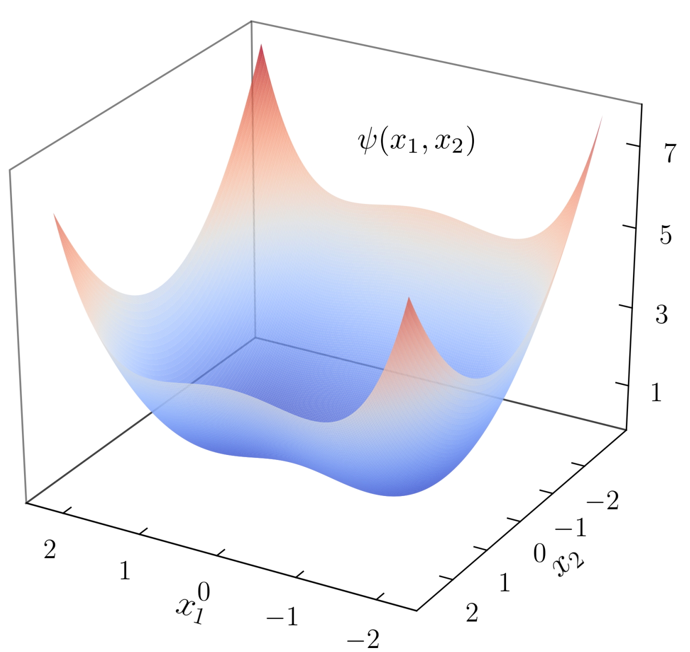


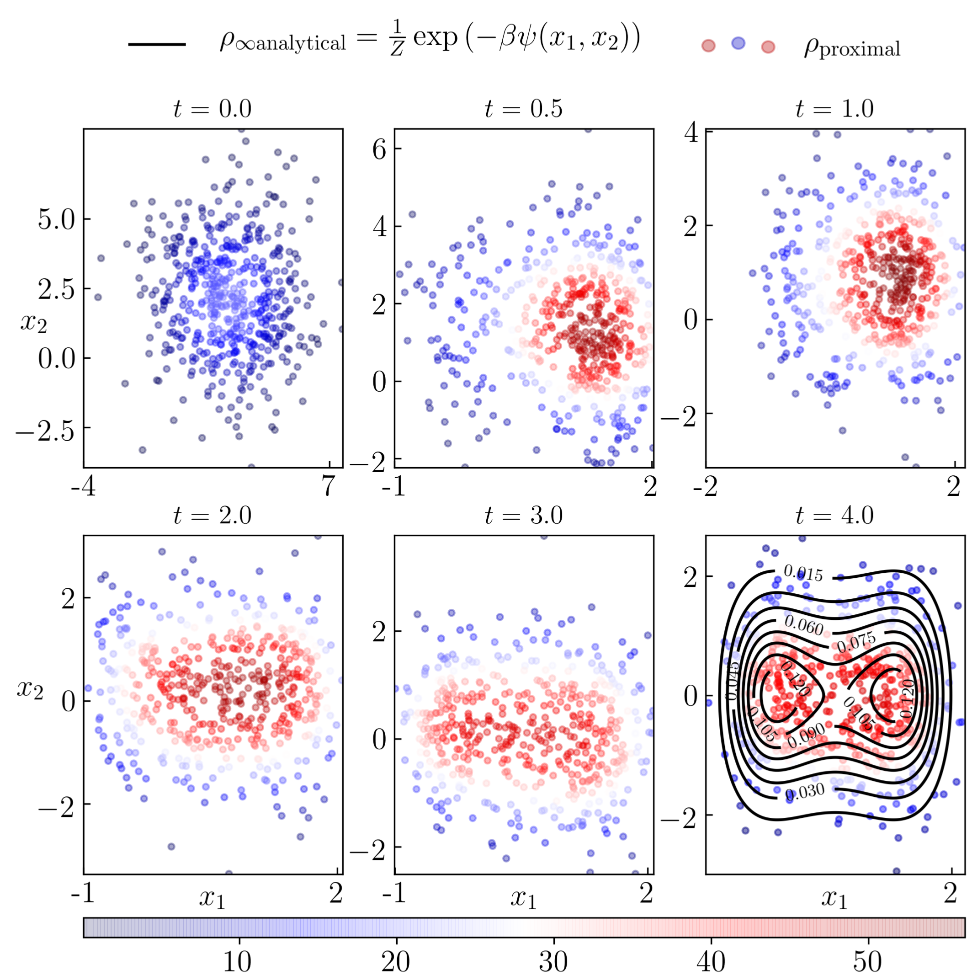
Theorem: Block co-ordinate iteration of $(\mathbf{y}, \mathbf{z})$ recursion is contractive on $\mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0}$.

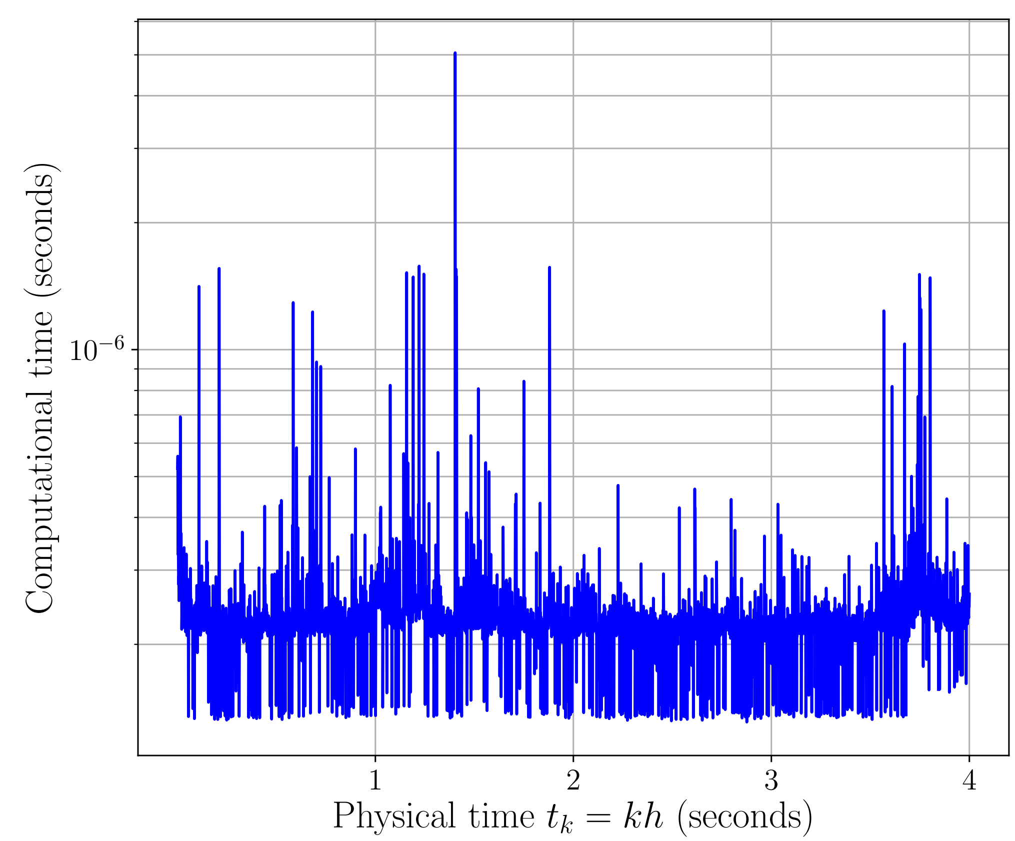
Details + more numerical case studies + extension to mean field models

K.F. Caluya, and A. H., Gradient flow algorithms for density propagation in stochastic systems , *IEEE Trans. Automatic Control*, 65(10), pp. 3991–4004, 2019.

K.F. Caluya, and A. H., Proximal recursion for solving the Fokker-Planck equation, *Proc. American Control Conference*, pp. 4098–4103, 2019.







Network reduced power system model

Noisy nonuniform Kuramoto model (after Kron reduction)

$$m_i\ddot{\theta}_i + \gamma_i\dot{\theta}_i = P_i - \sum_{j=1}^{n} k_{ij}\sin(\theta_i - \theta_j - \varphi_{ij}) + \sigma_i \times \text{stochastic forcing}$$

Mixed Conservative-Dissipative SDE over state variables $(\boldsymbol{\theta}, \boldsymbol{\omega}) \in \mathbb{T}^n \times \mathbb{R}^n$

$$\begin{pmatrix} \mathrm{d}\boldsymbol{\theta} \\ \mathrm{d}\boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\omega} \\ -\boldsymbol{M}^{-1}\nabla_{\boldsymbol{\theta}}V(\boldsymbol{\theta}) - \boldsymbol{M}^{-1}\boldsymbol{\Gamma}\boldsymbol{\omega} \end{pmatrix}\mathrm{d}t + \begin{pmatrix} \boldsymbol{0}_{n\times n} \\ \boldsymbol{M}^{-1}\boldsymbol{\Sigma} \end{pmatrix}\mathrm{d}\boldsymbol{w}$$

Potential function $V : \mathbb{T}^n \mapsto \mathbb{R}$

$$V(\boldsymbol{\theta}) := -\sum_{i=1}^{n} P_i\theta_i + \sum_{i<j} k_{ij}(1 - \cos(\theta_i - \theta_j - \varphi_{ij}))$$

Network reduced power system model

Kron reduced admittance matrix

$$\boldsymbol{Y} = [Y_{ij}]_{i,j=1}^{n} \in \mathbb{C}^{n\times n}, \quad Y_{ij} = Y_{ji}$$

Internal voltage and current for generator i

$$E_i, I_i \in \mathbb{C}$$

Parameters in the dynamics

$$P_i = P_i^{\text{mech}} - P_i^{\text{load}} - |E_i|^2 \Re(Y_{ii}) + \Re(E_i \cdot I_i^*),$$

$$\varphi_{ij} = \begin{cases} -\arctan\left(\dfrac{\Re(Y_{ij})}{\Im(Y_{ij})}\right), & \text{if } i \neq j, \\ 0, & \text{otherwise,} \end{cases}$$

$$k_{ij} = \begin{cases} |E_i||E_j||Y_{ij}|, & \text{if } i \neq j, \\ 0, & \text{otherwise,} \end{cases}$$

Not straightforward to apply the prox recursion

Kinetic Fokker-Planck PDE

$$\frac{\partial \rho}{\partial t} = -\langle \boldsymbol{\omega}, \nabla_{\theta} \rho \rangle + \nabla_{\boldsymbol{\omega}} \cdot \left(\rho \left(\boldsymbol{M}^{-1} \boldsymbol{\Gamma} \boldsymbol{\omega} + \boldsymbol{M}^{-1} \nabla_{\boldsymbol{\theta}} V(\boldsymbol{\theta}) + \frac{1}{2} \boldsymbol{M}^{-1} \boldsymbol{\Sigma} \boldsymbol{\Sigma}^{\top} \boldsymbol{M}^{-1} \nabla_{\boldsymbol{\omega}} \log \rho \right) \right)$$

Einstein relation does not hold …

$$\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\top} = \beta^{-1} \left(\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^{\top} \right) \qquad \text{for some } \beta > 0$$

Consider change of variable $\begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix} := \boldsymbol{\Psi} \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{pmatrix}$

where invertible linear map $\boldsymbol{\Psi} := \boldsymbol{I}_2 \otimes \left(\boldsymbol{M} \boldsymbol{\Sigma}^{-1} \right)$

From anisotropic to isotropic diffusion

Transformed SDE

$$\begin{pmatrix} \mathrm{d}\boldsymbol{\xi} \\ \mathrm{d}\boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\eta} \\ -\boldsymbol{\Upsilon}\nabla_{\boldsymbol{\xi}} U(\boldsymbol{\xi}) - \nabla_{\boldsymbol{\eta}} F(\boldsymbol{\eta}) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \mathbf{0}_{n\times n} \\ \boldsymbol{I}_n \end{pmatrix} \mathrm{d}\boldsymbol{w}$$

Set up

$$\tilde{\varrho}_k = \mathrm{prox}_{h\widetilde{\Phi}}^{\widetilde{W}}(\tilde{\varrho}_{k-1}) \equiv \underset{\tilde{\rho}\in\mathcal{P}_2}{\arg\inf} \frac{1}{2}\widetilde{W}^2(\tilde{\varrho},\tilde{\varrho}_{k-1}) + h\widetilde{\Phi}(\tilde{\rho}), \quad \tilde{\varrho}_0 := \tilde{\rho}_0,$$

where

$$\widetilde{\Phi}(\tilde{\rho}) := \int_{\mathbb{T}^n\times\mathbb{R}^n} \left(F(\boldsymbol{\eta}) + \frac{1}{2}\log\tilde{\rho}\right)\tilde{\rho}\,\mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}$$

From anisotropic to isotropic diffusion

Consistency guarantee

$$\tilde{\varrho}_k(\boldsymbol{\xi}, \boldsymbol{\eta}) \xrightarrow{h \downarrow 0} \tilde{\varrho}(t = kh, \boldsymbol{\xi}, \boldsymbol{\eta}) \text{ in } L^1(\mathbb{T}^n \times \mathbb{R}^n)$$

Evolve weighted point cloud

where

$$\tilde{\boldsymbol{x}}_k^i := \left(\boldsymbol{\xi}_k^i, \boldsymbol{\eta}_k^i\right)^\top, \quad i = 1, \ldots, N, \quad \boldsymbol{k} \in \mathbb{N}$$

and then push forward via inverse map to come back to

$(\boldsymbol{\theta}, \boldsymbol{\omega})$ **coordinate**

IEEE 14 bus case study

Parameters from MATPOWER and ANDES

Nominal case (Case I): $P_i^{\text{mech}}, P_i^{\text{load}}$ from steady state power flow

Post-contingency case (Case II): Line 13 fails at $t = 0$

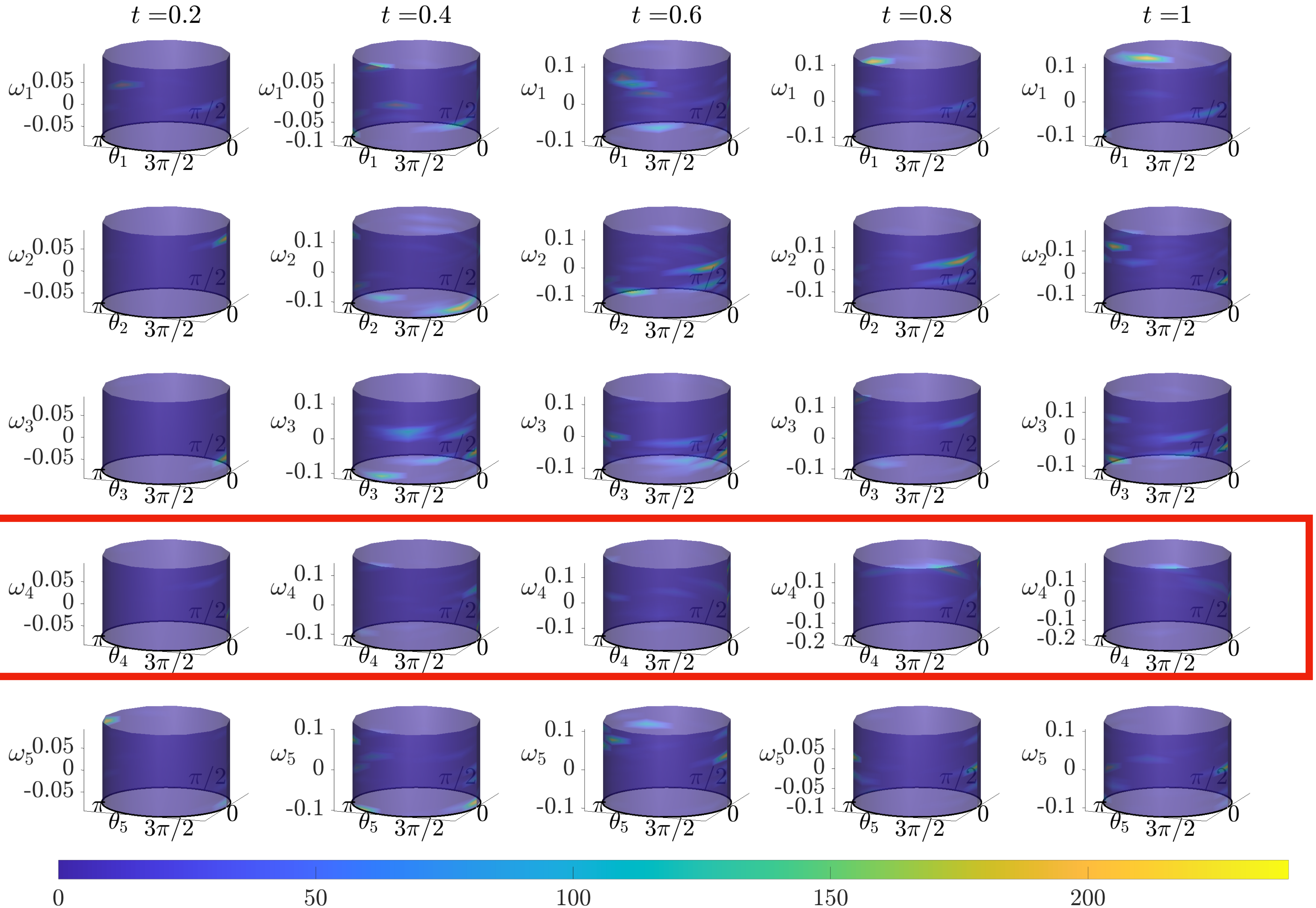
State space: $\mathbb{T}^5 \times \mathbb{R}^5$

Can account time-varying $P_i^{\text{mech}}, P_i^{\text{load}}$

Initial $\mathbb{T}^n$ marginal as product von Mises with mean angles from the steady state AC power flow

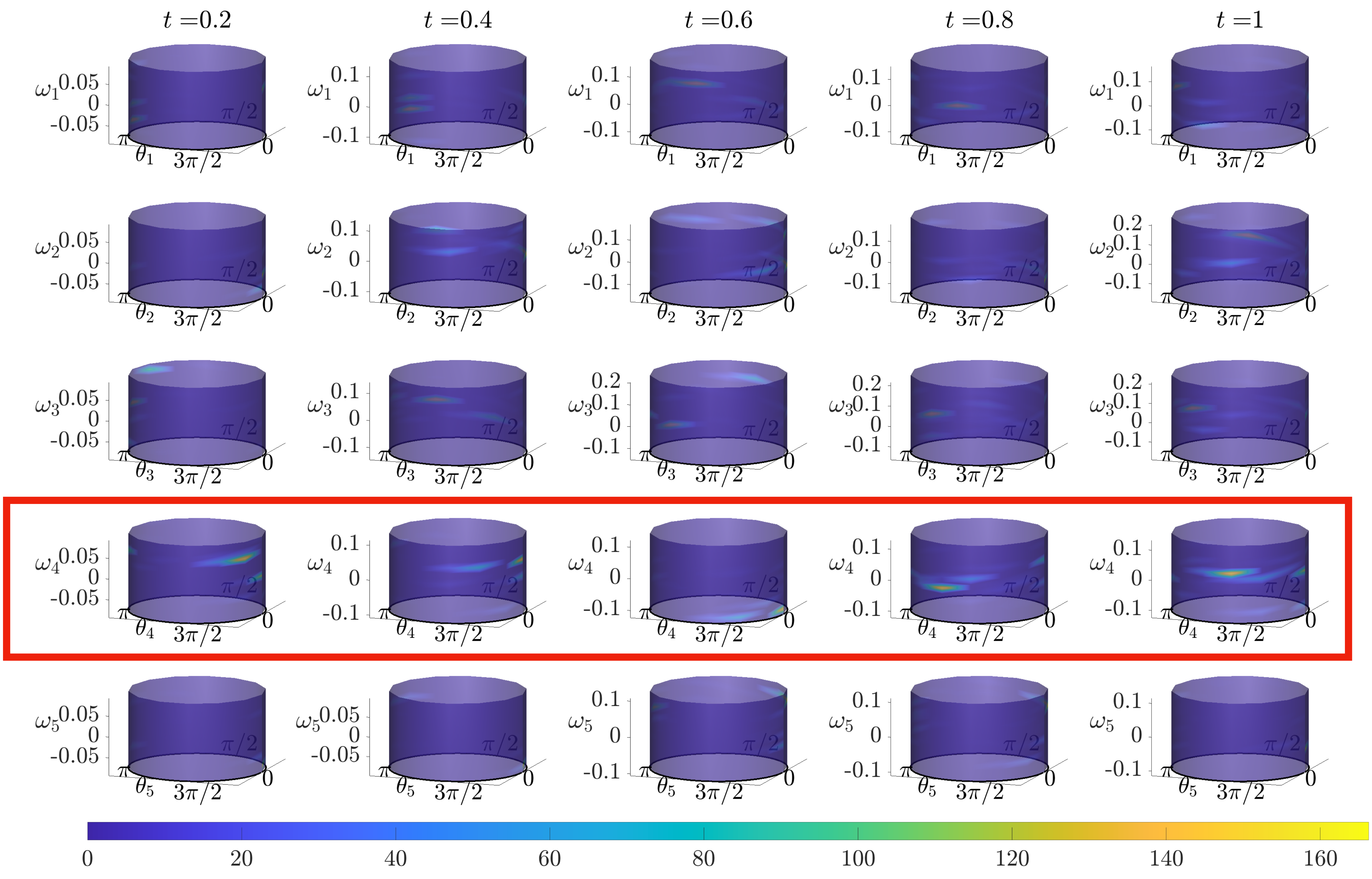
IEEE 14 bus case study: bivariate (θ, ω) marginals

Nominal case:

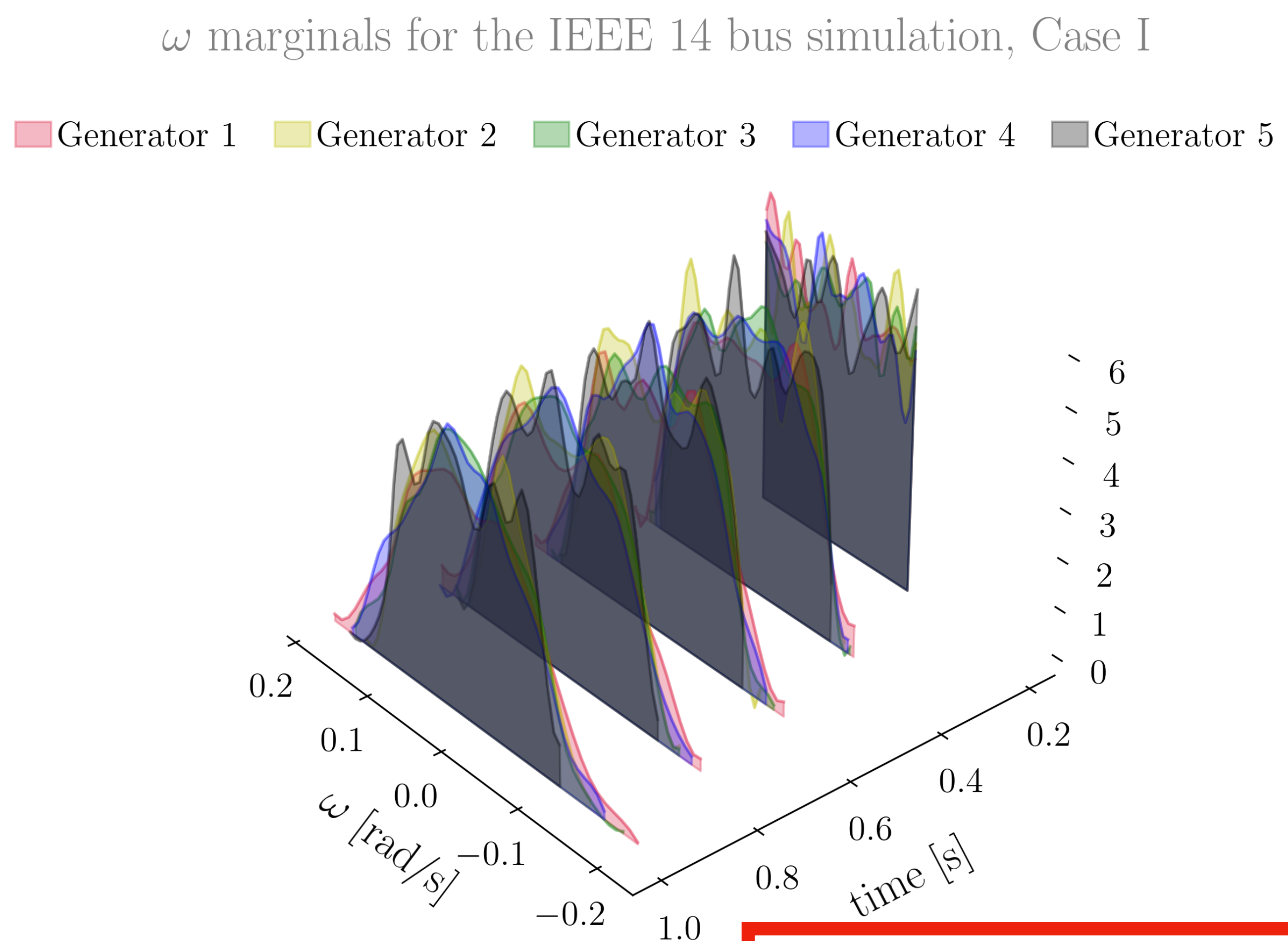


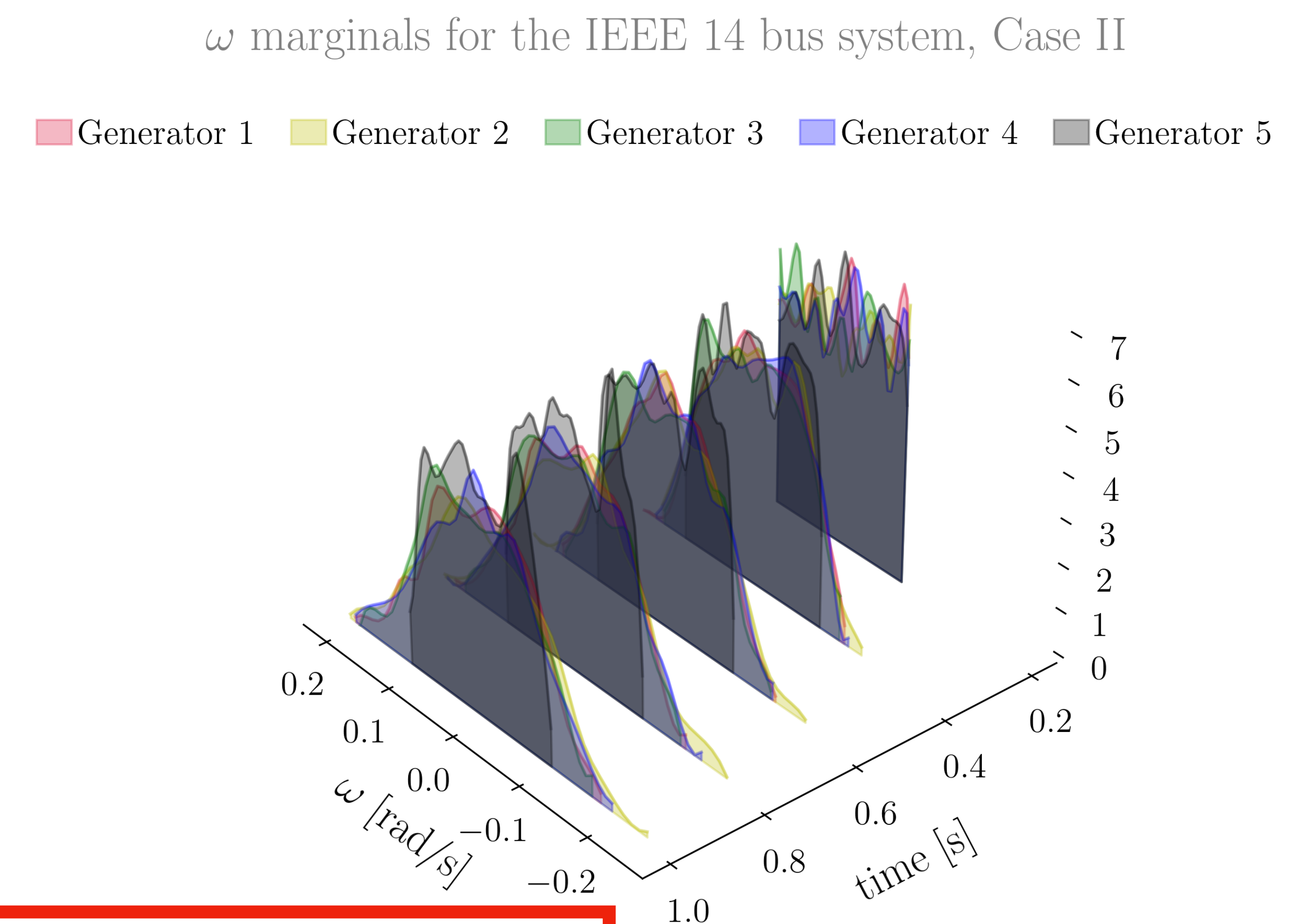
IEEE 14 bus case study: bivariate (θ, ω) marginals

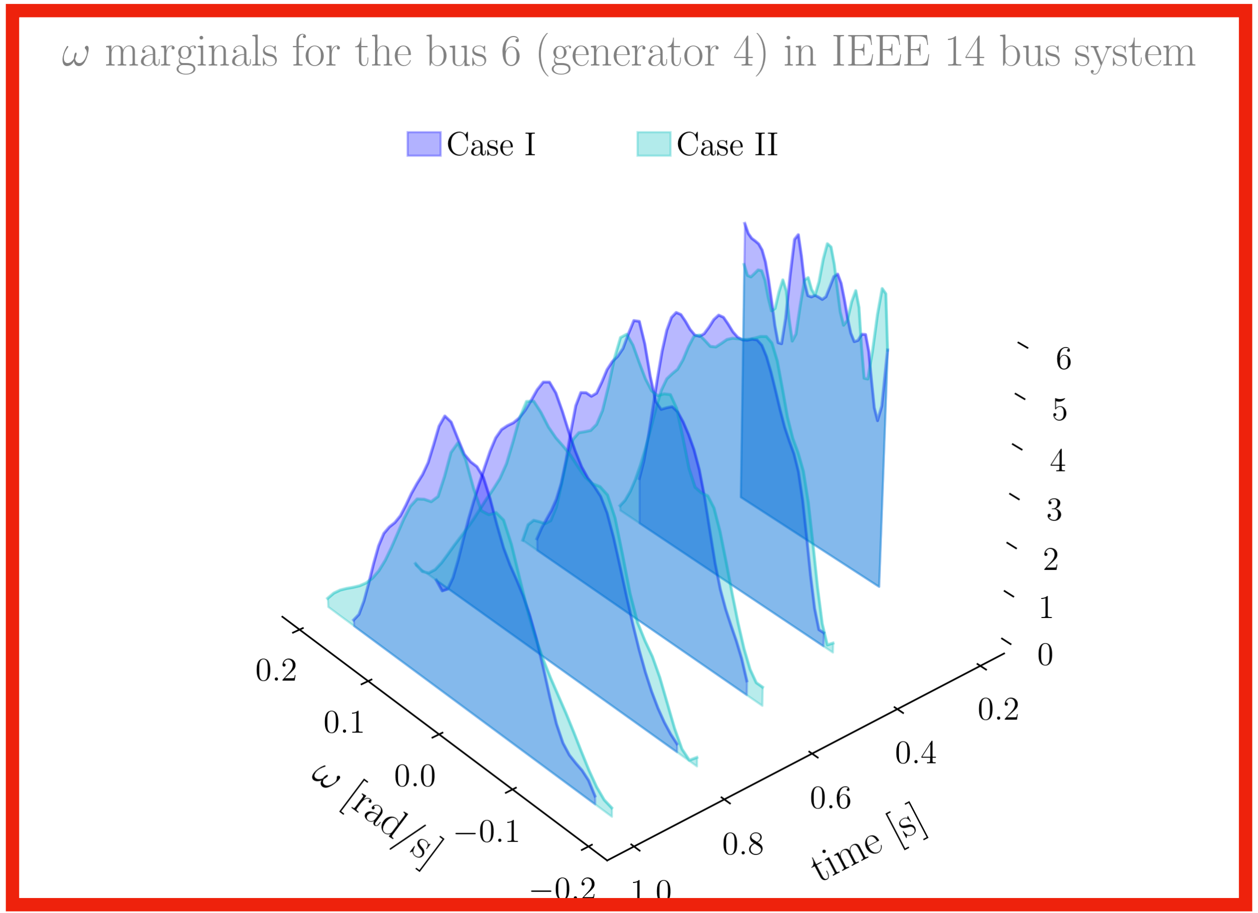
Post-contingency case:



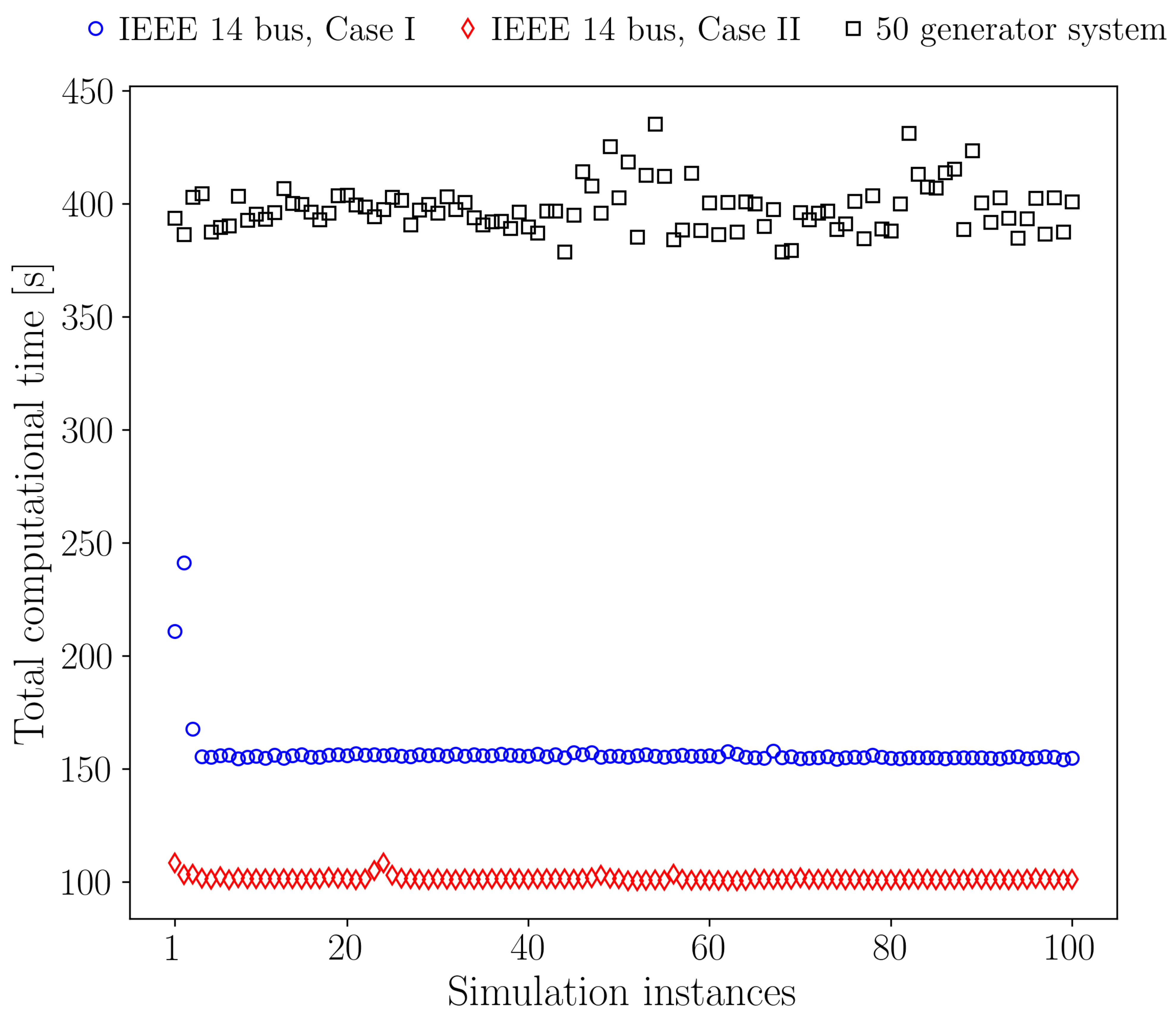
IEEE 14 bus case study: univariate ω marginals



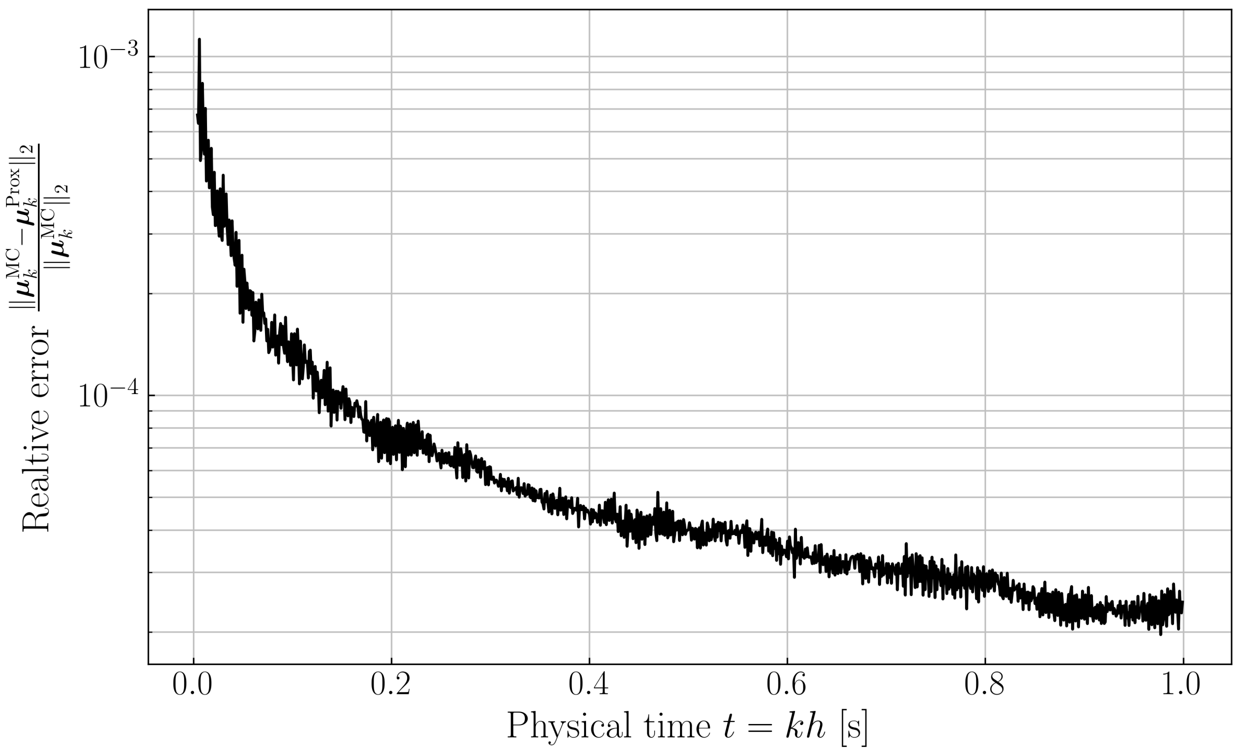


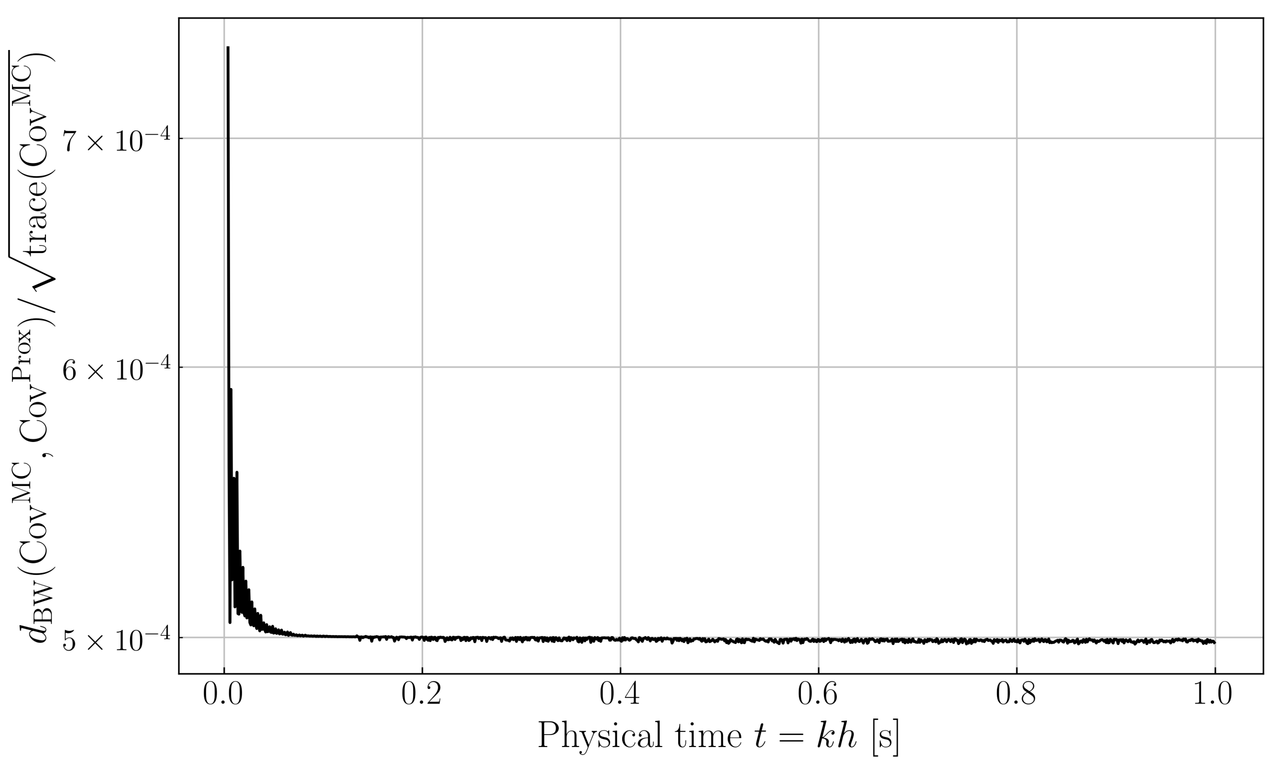


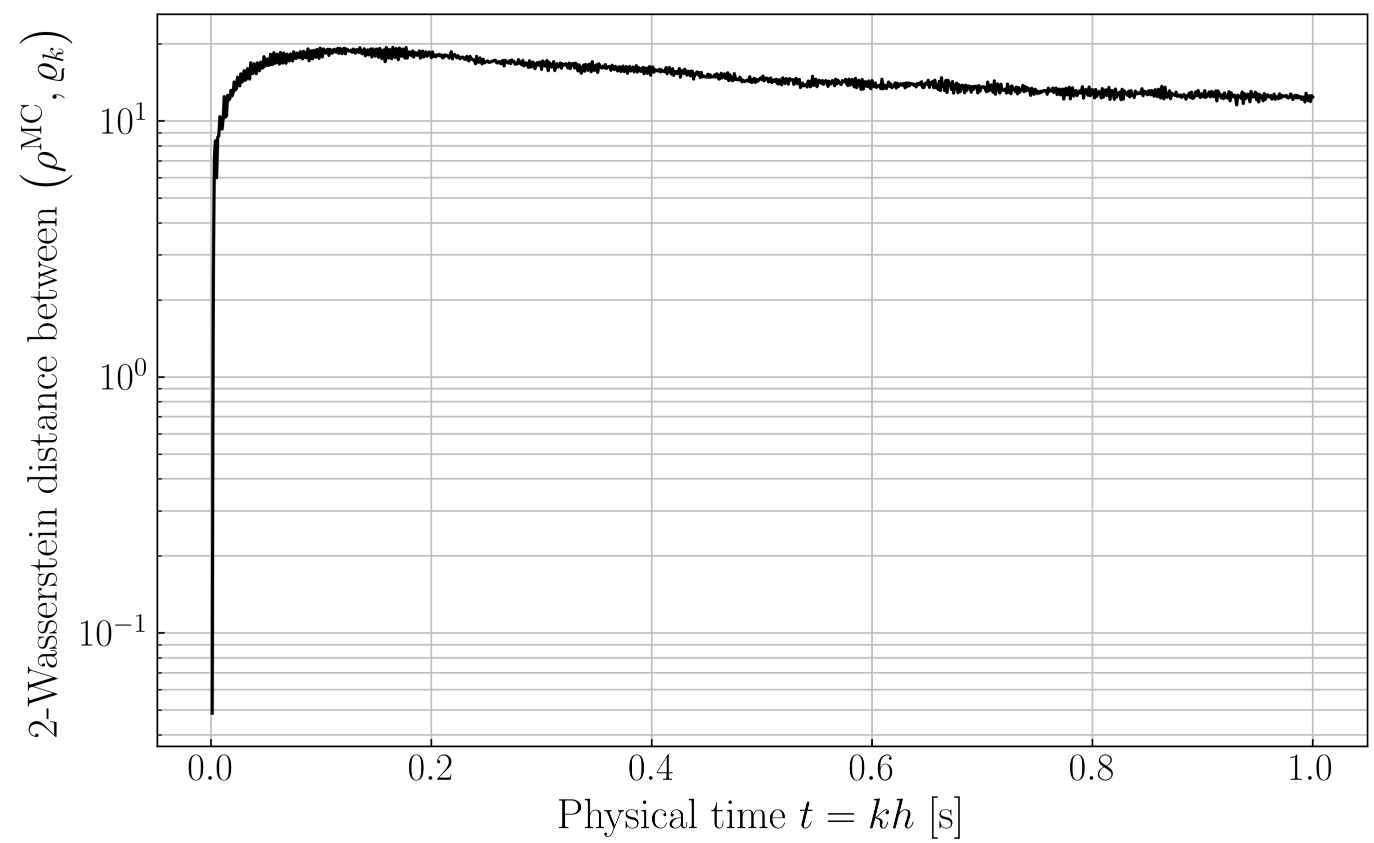
Total computational time for 1 min sim. horizon



Statistical consistency w.r.t. Monte Carlo: 50 gen.







A. H., K.F. Caluya, P. Ojaghi, and X. Geng, Stochastic uncertainty propagation in power system dynamics using measure-valued proximal recursions, arXiv:2108.13405

Finite horizon optimal steering of joint PDFs

$$\inf_{\boldsymbol{u}\in\mathcal{U}} \mathbb{E}_{\mu^u}\left[\int_0^T \|\boldsymbol{u}\|_2^2 \,\mathrm{d}t\right]$$

subject to

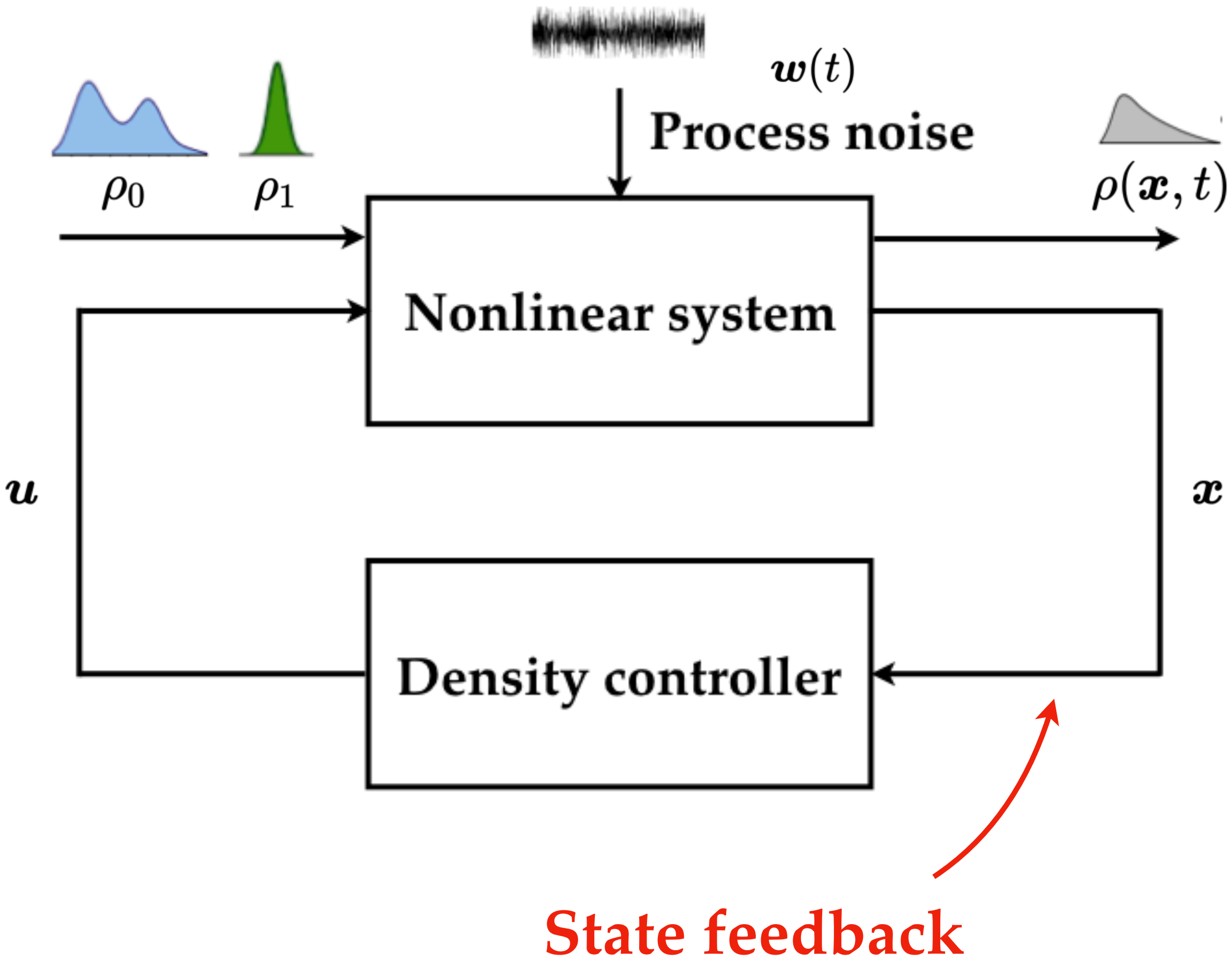
$$\mathrm{d}\boldsymbol{\theta} = (-\nabla_{\boldsymbol{\theta}} V(\boldsymbol{\theta}) + \boldsymbol{S}\boldsymbol{u})\mathrm{d}t + \sqrt{2}\boldsymbol{S}\mathrm{d}\boldsymbol{w}$$

$$\boldsymbol{\theta}(t=0) \sim \rho_0(\text{ given }), \boldsymbol{\theta}(t=T) \sim \rho_T(\text{ given })$$

or

$$\begin{pmatrix}\mathrm{d}\boldsymbol{\theta}\\ \mathrm{d}\boldsymbol{\omega}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\omega}\\ -\boldsymbol{M}^{-1}\nabla_{\boldsymbol{\theta}} V(\boldsymbol{\theta}) - \boldsymbol{M}^{-1}\boldsymbol{\Gamma}\boldsymbol{\omega} + \boldsymbol{M}^{-1}\boldsymbol{S}\boldsymbol{u}\end{pmatrix}\mathrm{d}t + \begin{pmatrix}\boldsymbol{0}_{n\times 1}\\ \sqrt{2}\boldsymbol{M}^{-1}\boldsymbol{S}\mathrm{d}\boldsymbol{w}\end{pmatrix}$$

$$\begin{pmatrix}\boldsymbol{\theta}(t=0)\\ \boldsymbol{\omega}(t=0)\end{pmatrix} \sim \rho_0(\text{ given }), \begin{pmatrix}\boldsymbol{\theta}(t=T)\\ \boldsymbol{\omega}(t=T)\end{pmatrix} \sim \rho_T(\text{ given })$$



Atypical stochastic control problem

$$\inf_{(\rho^u,\boldsymbol{u})} \int_0^T \int_{\mathcal{X}} \|\boldsymbol{u}(\boldsymbol{x},t)\|_2^2 \rho^{\boldsymbol{u}}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t$$

subject to either

$$\frac{\partial \rho^{\boldsymbol{u}}}{\partial t} = -\nabla_{\boldsymbol{\theta}} \cdot (\rho^{\boldsymbol{u}}(\boldsymbol{S}\boldsymbol{u} - \nabla_{\boldsymbol{\theta}} V)) + \langle \boldsymbol{D}, \mathbf{Hess}(\rho^{\boldsymbol{u}}) \rangle$$

$$\text{or } \frac{\partial \rho^{\boldsymbol{u}}}{\partial t} = \nabla_{\boldsymbol{\omega}} \cdot \big(\rho^{\boldsymbol{u}} \big(\boldsymbol{M}^{-1} \nabla_{\boldsymbol{\theta}} V(\boldsymbol{\theta}) + \boldsymbol{M}^{-1} \boldsymbol{\Gamma}\boldsymbol{\omega} - \boldsymbol{M}^{-1}\boldsymbol{S}\boldsymbol{u} + \boldsymbol{M}^{-1}\boldsymbol{D}\boldsymbol{M}^{-1} \nabla_{\boldsymbol{\omega}} \log \rho^{\boldsymbol{u}}\big) - \langle \boldsymbol{\omega}, \nabla_{\boldsymbol{\theta}} \rho^{\boldsymbol{u}} \rangle$$

$$\boxed{\rho^u(\boldsymbol{x}, t=0) = \rho_0 \text{ (given)}, \quad \rho^u(\boldsymbol{x}, t=T) = \rho_T \text{ (given)}}$$

Endpoint PDF constraints

Conditions of optimality:

Coupled nonlinear PDEs in (ρ^u, ψ)

Optimal joint state PDF (controlled forward Kolmogorov PDE) — ρ^u

Value function (HJB PDE) — ψ

Solution of the stochastic control problem

Existence and uniqueness guaranteed for compactly supported ρ_0, ρ_T

Hopf-Cole transform: $(\rho^u, \psi) \mapsto (\hat{\varphi}, \varphi)$

Schrödinger factors

$$\frac{\partial \hat{\varphi}}{\partial t} = \mathscr{L}_{\text{forward}} \hat{\varphi}$$

$$\frac{\partial \varphi}{\partial t} = \mathscr{L}_{\text{backward}} \varphi$$

$$\hat{\varphi}_0 \varphi_0 = \rho_0$$

$$\hat{\varphi}_T \varphi_T = \rho_T$$

Recover the optimal solution:

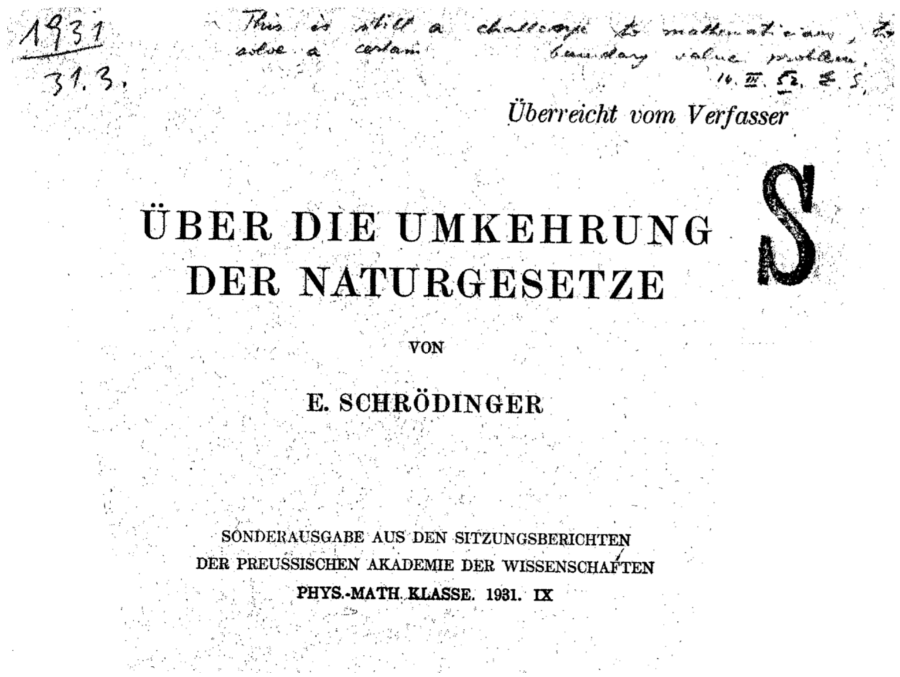
$$\rho^u(\boldsymbol{x}, t) = \left(\prod_{i=1}^{n} \frac{m_i^2}{\sigma_i^2} \right) \hat{\varphi} \varphi$$

$$u^{\text{opt}}(\boldsymbol{x}, t) = \left(\boldsymbol{I}_2 \otimes \boldsymbol{S} \boldsymbol{M}^{-1} \right) \nabla_{\boldsymbol{x}} \log \varphi$$

2 coupled nonlinear PDEs $\longrightarrow$ boundary-coupled linear PDEs!!

Feedback Synthesis via the Schrödinger System

Schrödinger's (until recently) forgotten papers:



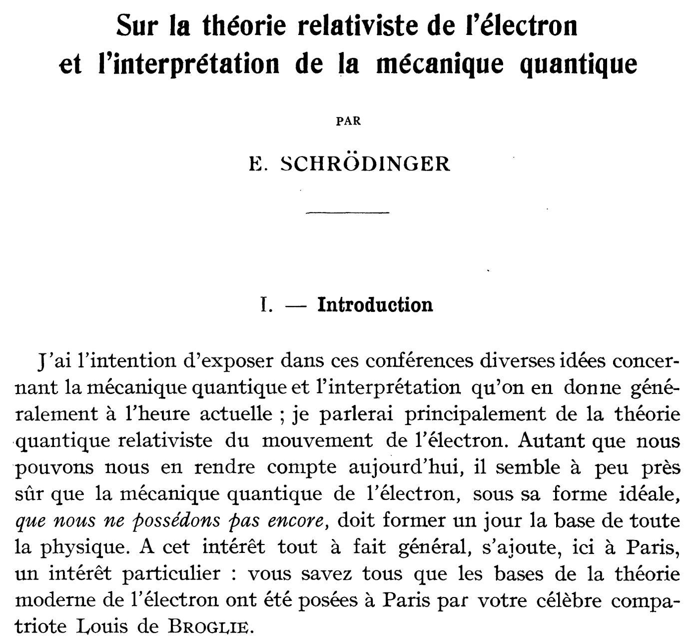
Überreicht vom Verfasser

ÜBER DIE UMKEHRUNG DER NATURGESETZE

VON

E. SCHRÖDINGER

SONDERAUSGABE AUS DEN SITZUNGSBERICHTEN DER PREUSSISCHEN AKADEMIE DER WISSENSCHAFTEN PHYS.-MATH. KLASSE. 1931. IX



Sur la théorie relativiste de l'électron et l'interprétation de la mécanique quantique

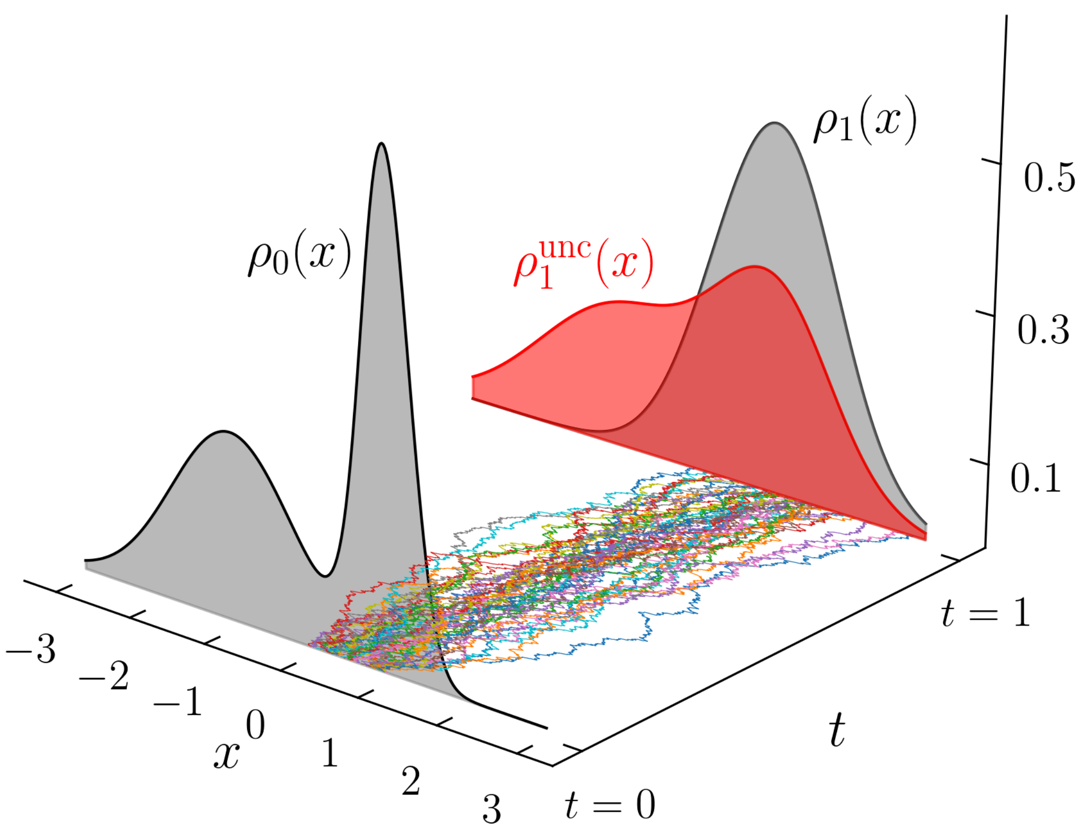
PAR

E. SCHRÖDINGER

I. — Introduction

J'ai l'intention d'exposer dans ces conférences diverses idées concernant la mécanique quantique et l'interprétation qu'on en donne généralement à l'heure actuelle ; je parlerai principalement de la théorie quantique relativiste du mouvement de l'électron. Autant que nous pouvons nous en rendre compte aujourd'hui, il semble à peu près sûr que la mécanique quantique de l'électron, sous sa forme idéale, *que nous ne possédons pas encore*, doit former un jour la base de toute la physique. A cet intérêt tout à fait général, s'ajoute, ici à Paris, un intérêt particulier : vous savez tous que les bases de la théorie moderne de l'électron ont été posées à Paris par votre célèbre compatriote Louis de Broglie.

Hopf-Cole transform: $(\rho^{\text{opt}}, \psi) \mapsto (\varphi, \hat{\varphi})$

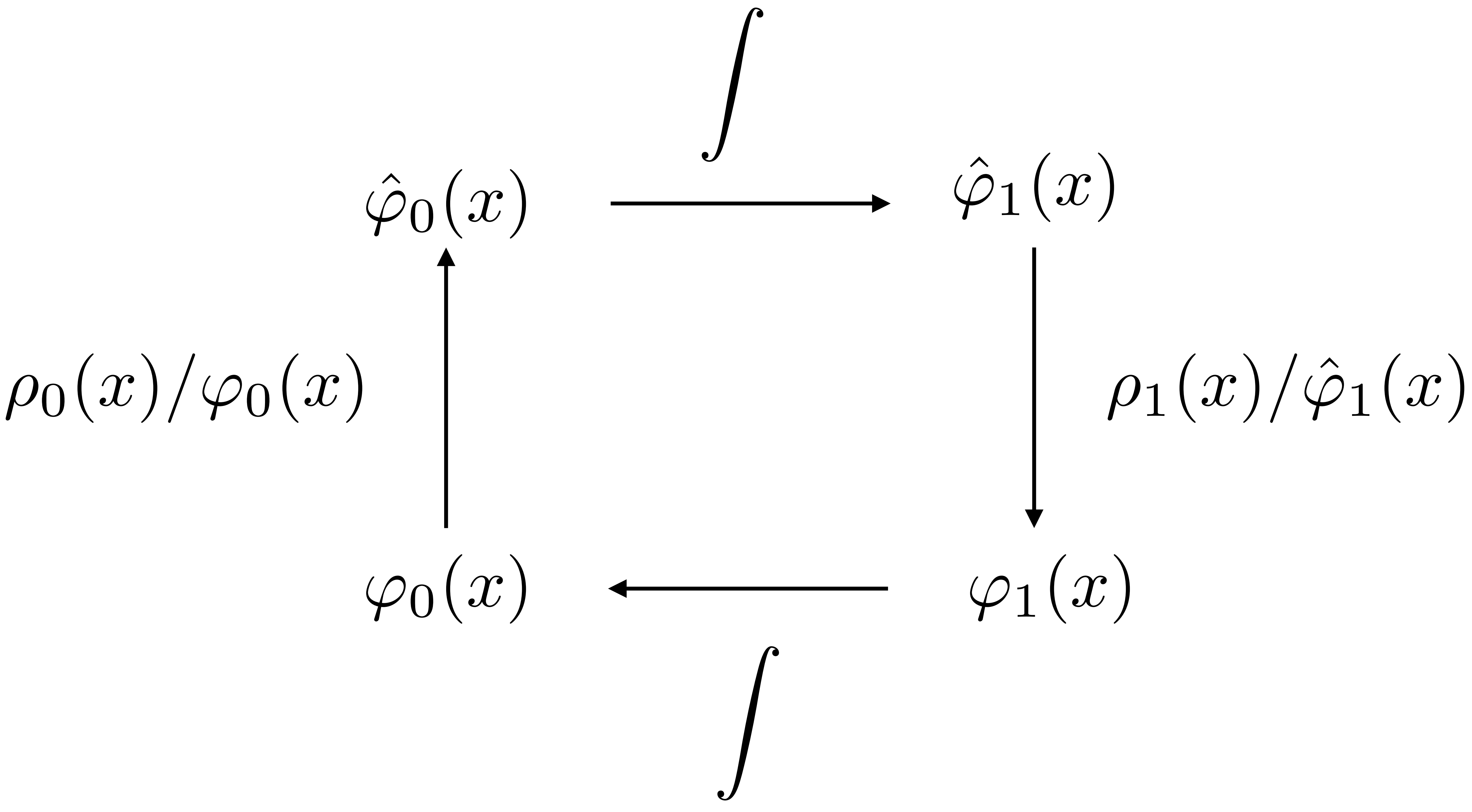


$$\varphi(x,t) = \exp\left(\frac{\psi(x,t)}{2\epsilon}\right),$$

$$\hat{\varphi}(x,t) = \rho^{\text{opt}}(x,t)\exp\left(-\frac{\psi(x,t)}{2\epsilon}\right),$$

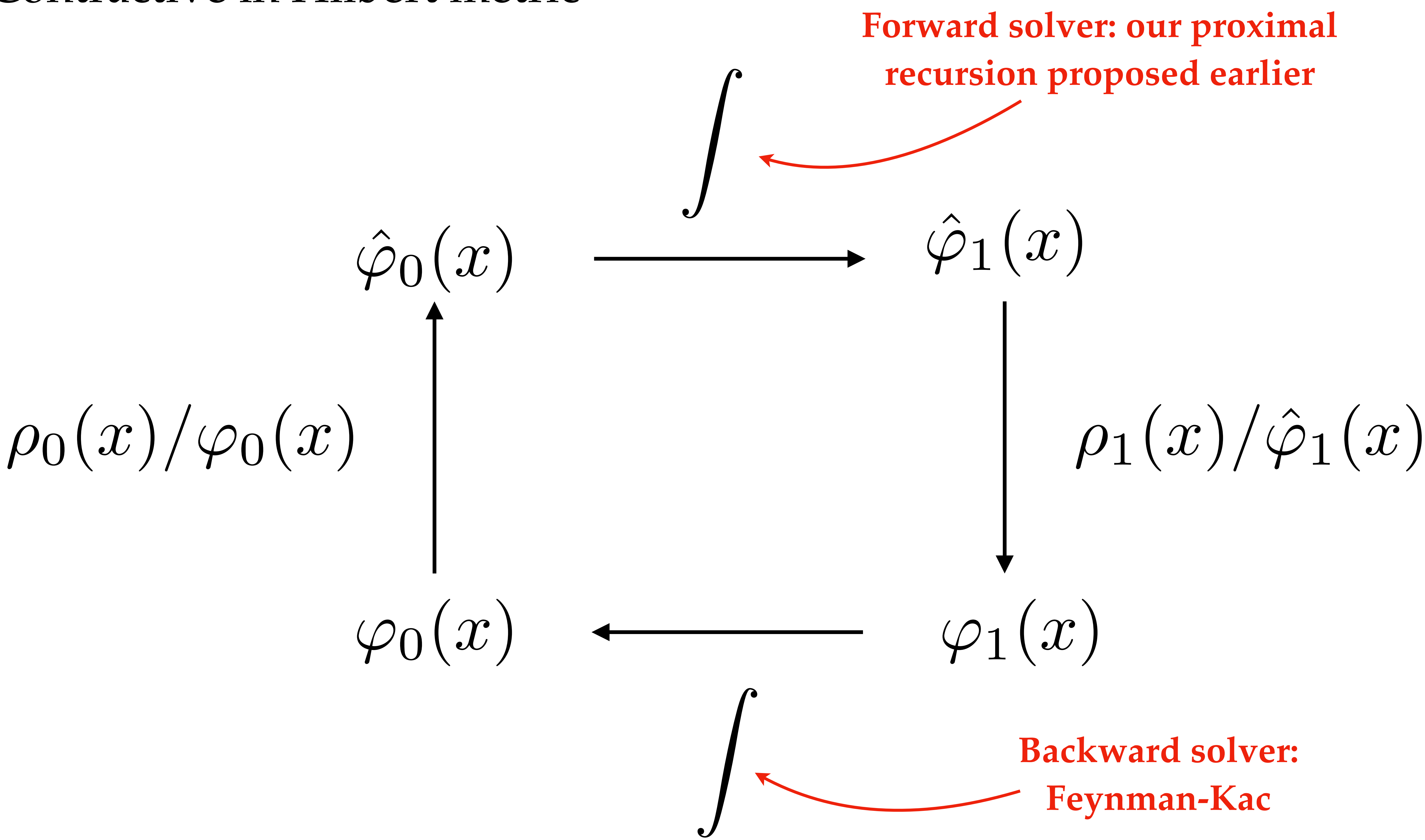
Fixed point recursion over $(\hat{\varphi}_0, \varphi_1)$

Contractive in Hilbert metric



Fixed point recursion over $(\hat{\varphi}_0, \varphi_1)$

Contractive in Hilbert metric

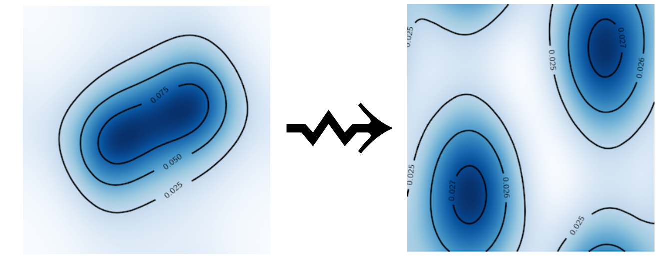


Example on $\mathbb{T}^2 \times \mathbb{R}^2$

Prescribed endpoint joints:

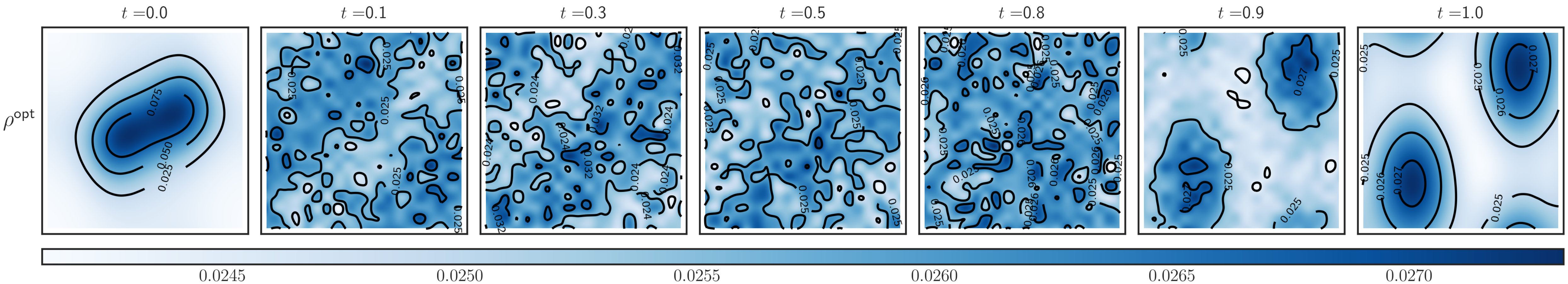
$$\rho_0(\boldsymbol{\theta},\boldsymbol{\omega}) = \bar{\rho}_0(\boldsymbol{\theta}) \times \text{Unif}\left([0,0.2]^2\right)$$

$$\rho_1(\boldsymbol{\theta},\boldsymbol{\omega}) = \bar{\rho}_1(\boldsymbol{\theta}) \times \text{Unif}\left([0,0.2]^2\right)$$

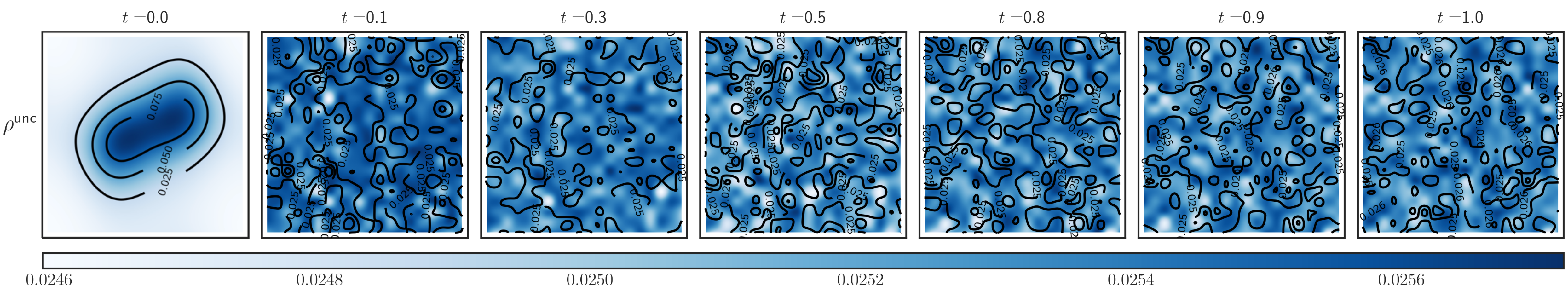


$\bar{\rho}_0(\boldsymbol{\theta})$ $\bar{\rho}_1(\boldsymbol{\theta})$

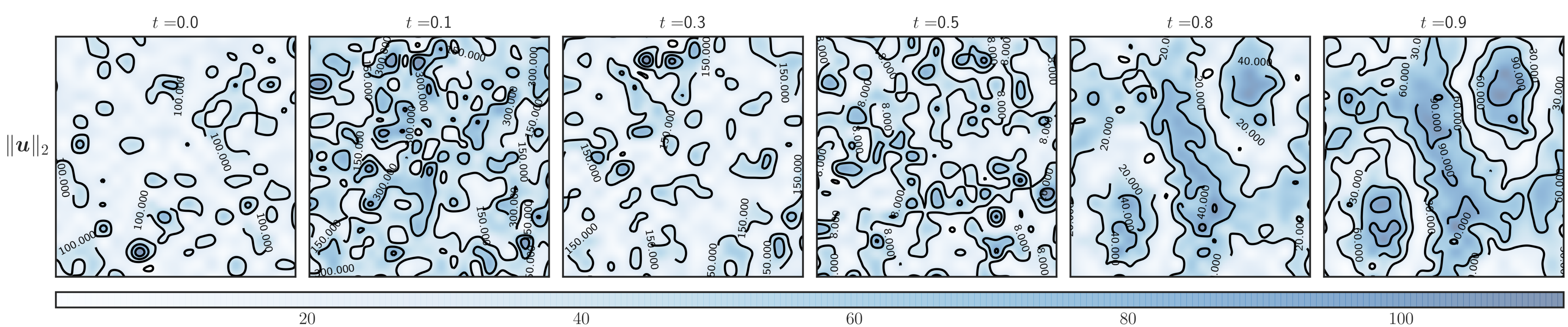
$\boldsymbol{\theta}$ marginals of the optimally controlled joint PDFs:



$\boldsymbol{\theta}$ marginals of the uncontrolled joint PDFs:



Magnitude of the optimal control policy:



Details on the optimal control of joint PDFs

I. Nodozi, and A. H., Schrödinger meets Kuramoto via Feynman-Kac: minimum effort distribution steering for noisy nonuniform Kuramoto oscillators, arXiv:2202.09734.

K.F. Caluya, and A. H., Wasserstein proximal algorithms for the Schrödinger bridge problem: density control with nonlinear drift, *IEEE Trans. Automatic Control*, 67(3), pp. 1163–1178, 2022.

K.F. Caluya, and A. H., Reflected Schrödinger bridge: density control with path constraints, *Proc. of the American Control Conference*, 2021.

K.F. Caluya, and A. H., Finite horizon density steering for multi-input state feedback linearizable systems, *Proc. of the American Control Conference*, 2020.

Summary

Fast proximal recursions for joint PDF propagation subject to power system dynamics

Minimum effort finite horizon steering of joint PDFs via state feedback subject to power system dynamics

Ongoing effort for distributed computation:

I. Nodozi, and A. H., A distributed algorithm for measure-valued optimization with additive objective, arXiv:2202.08930.

Thank You