Generalized Gradient Flows for Stochastic Prediction, Filtering, Learning and Control

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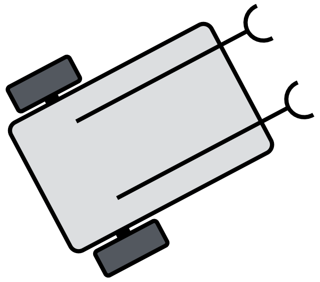
Overarching Theme

Systems-control theory and algorithms for densities

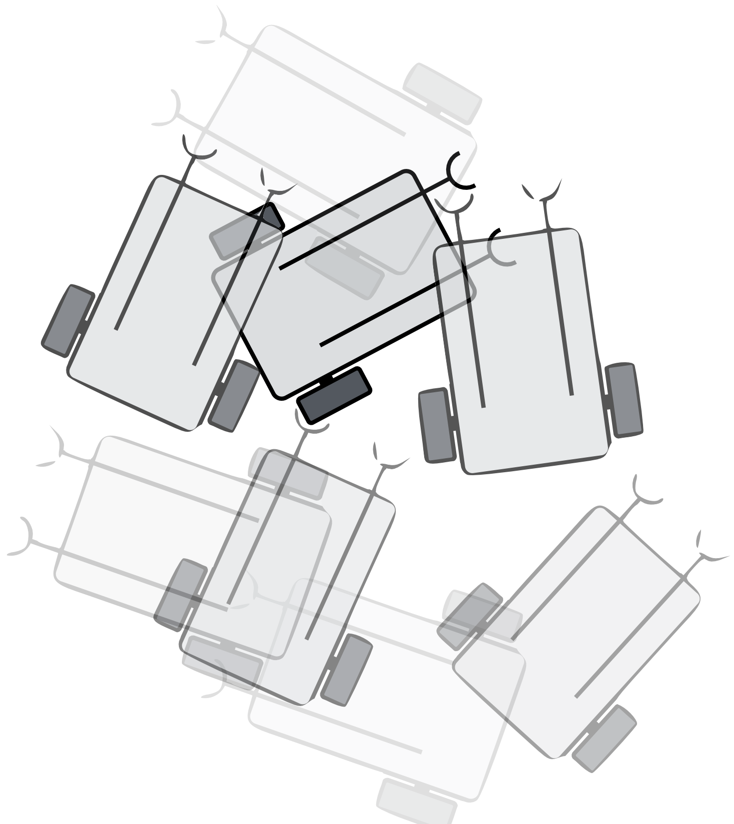
What is density?

Probability Density Fn.



$$\boldsymbol{x}(t) \in \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \in \mathcal{X} \equiv \mathbb{R}^2 \times \mathbb{S}^1$$

Probability Density Fn.

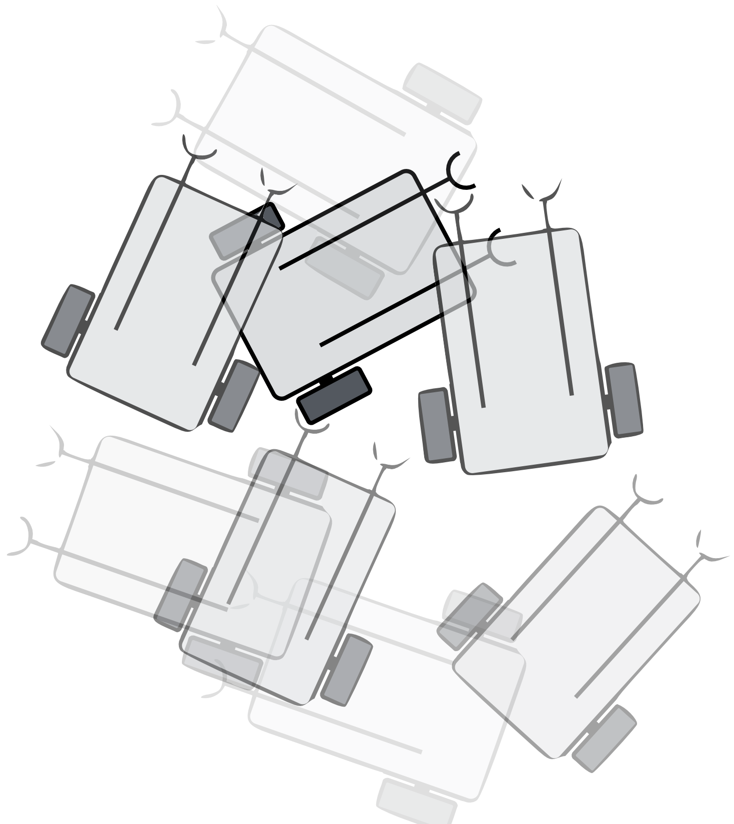


$$\boldsymbol{x}(t) \in \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \in \mathcal{X} \equiv \mathbb{R}^2 \times \mathbb{S}^1$$

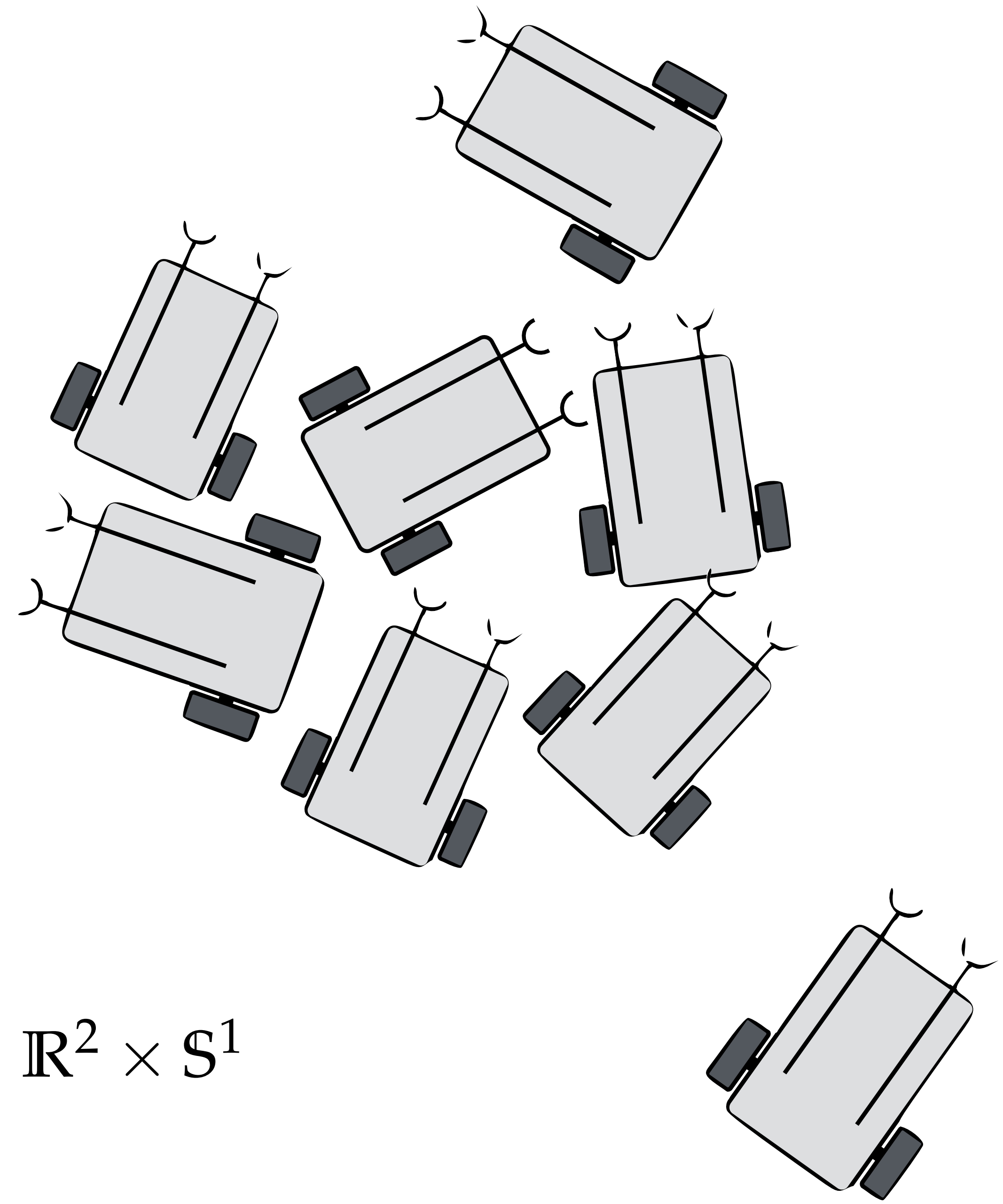
$$\rho\left(\boldsymbol{x}, t\right) : \mathcal{X} \times [0, \infty) \mapsto \mathbb{R}_{\geq 0}$$

$$\int_{\mathcal{X}} \rho \, \mathrm{d}\boldsymbol{x} = 1 \quad \text{for all } t \in [0, \infty)$$

Probability Density Fn.



Population Density Fn.



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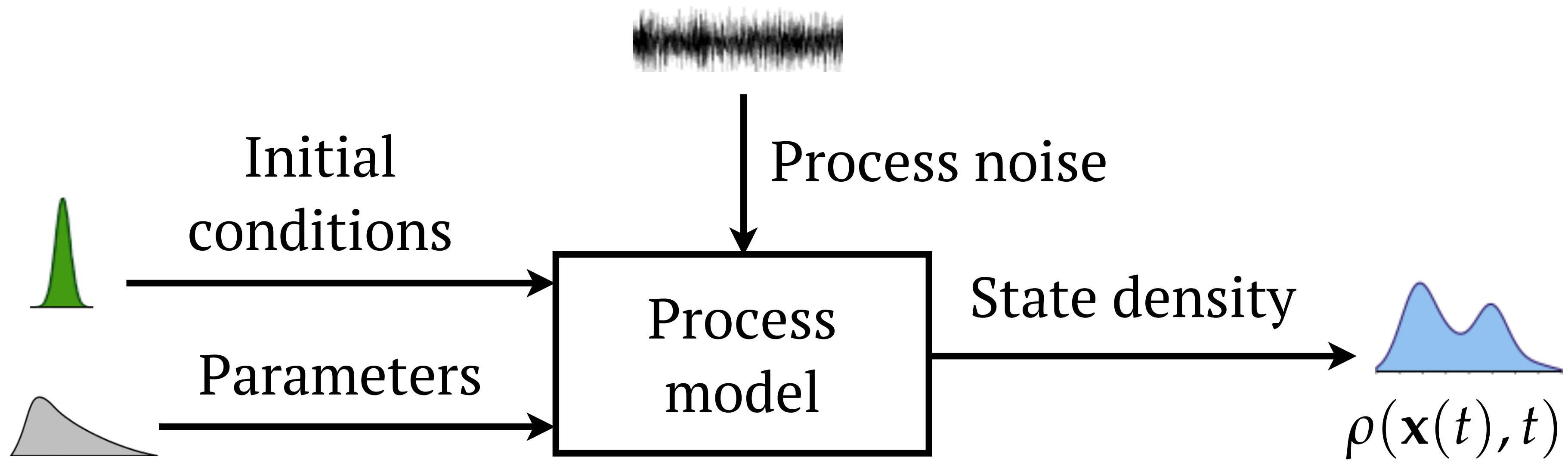
$$\int_{\mathcal{X}} \rho \, \mathrm{d}\boldsymbol{x} = 1 \quad \text{for all } t \in [0, \infty)$$

Why care about densities?

Prediction Problem

Compute
joint state PDF
$\rho\left(x, t\right)$



Trajectory flow:

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}(\mathbf{X}, t)\,\mathrm{d}t + \mathbf{g}(\mathbf{X}, t)\,\mathrm{d}\mathbf{w}(t), \quad \mathrm{d}\mathbf{w}(t) \sim \mathcal{N}(0, \mathbf{Q}\mathrm{d}t)$$

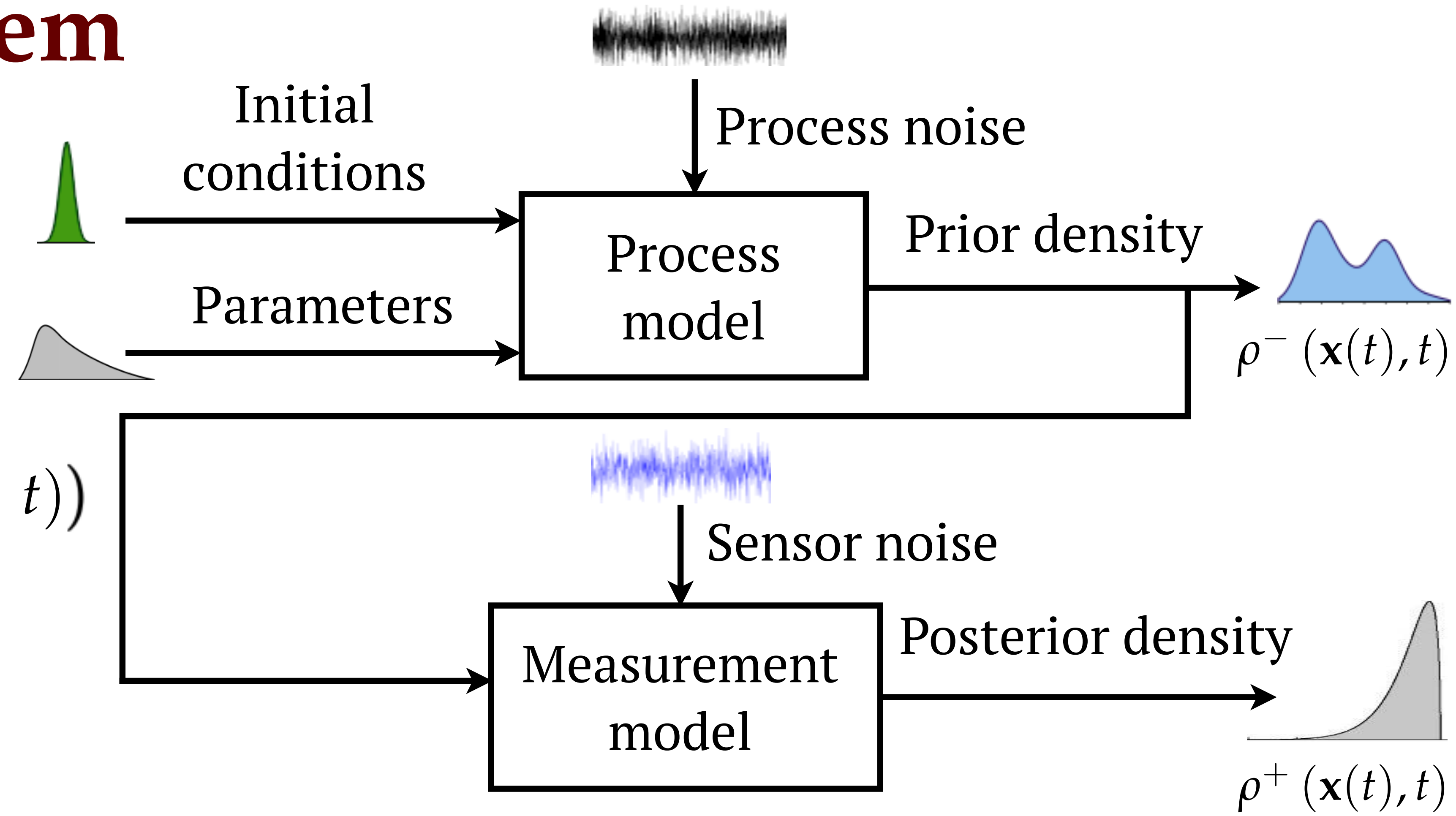
Density flow:

$$\frac{\partial \rho}{\partial t} = \mathcal{L}_{\mathrm{FP}}(\rho) := -\nabla \cdot (\rho \mathbf{f}) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left(\left(\mathbf{g}\mathbf{Q}\mathbf{g}^{\top} \right)_{ij} \rho \right)$$

Filtering Problem

Compute conditional joint state PDF

$$\rho^{+} := \rho\left(x, t \mid z(s), 0 \leq s \leq t\right)$$



Trajectory flow:

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}(\mathbf{X}, t)\,\mathrm{d}t + \mathbf{g}(\mathbf{X}, t)\,\mathrm{d}\mathbf{w}(t), \quad \mathrm{d}\mathbf{w}(t) \sim \mathcal{N}(0, \mathbf{Q}\mathrm{d}t)$$

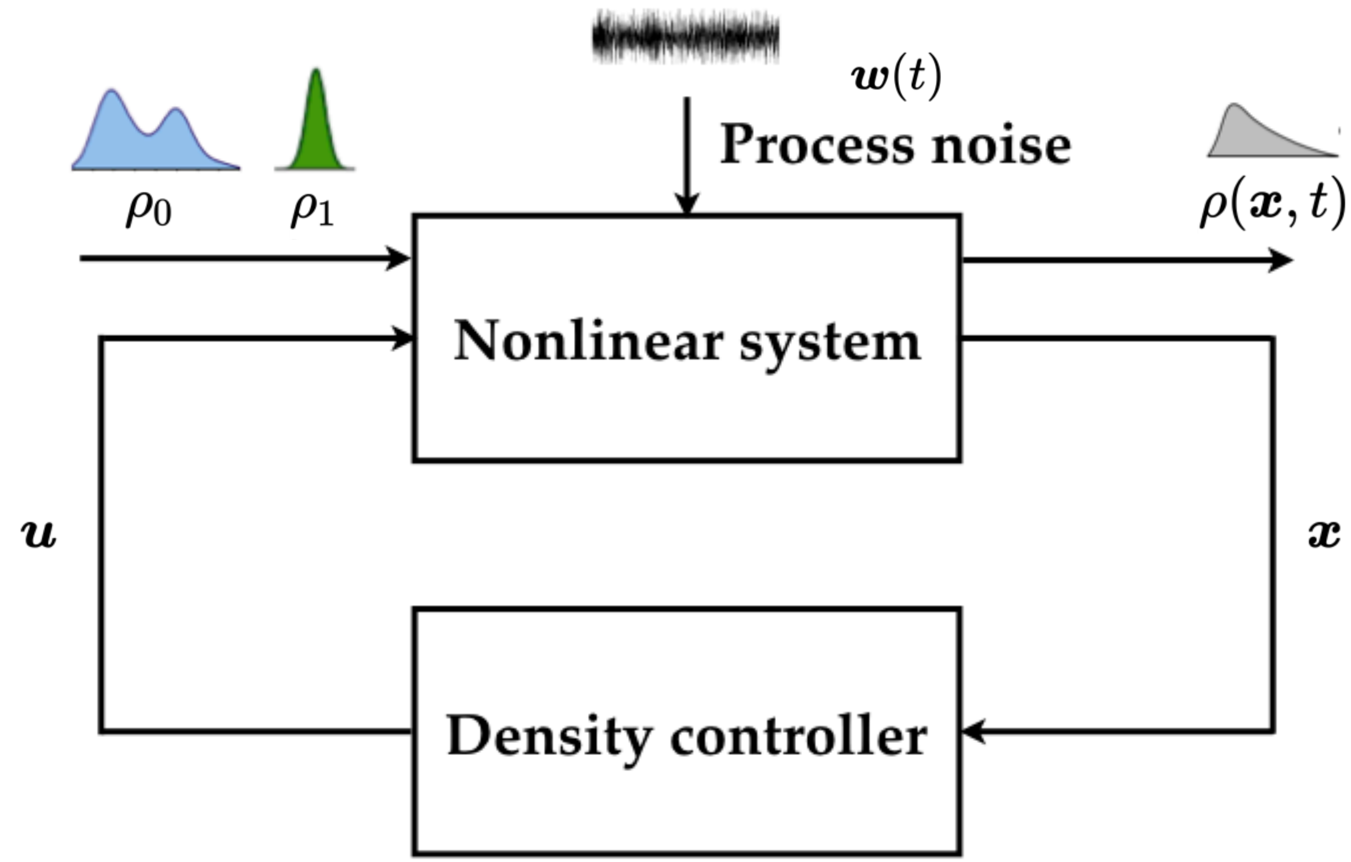
$$\mathrm{d}\mathbf{Z}(t) = \mathbf{h}(\mathbf{X}, t)\,\mathrm{d}t + \mathrm{d}\mathbf{v}(t), \quad \mathrm{d}\mathbf{v}(t) \sim \mathcal{N}(0, \mathbf{R}\mathrm{d}t)$$

Density flow:

$$\mathrm{d}\rho^{+} = \left[\mathcal{L}_{\mathrm{FP}}\mathrm{d}t + \left(\mathbf{h}(\mathbf{x}, t) - \mathbb{E}_{\rho^{+}}\{\mathbf{h}(\mathbf{x}, t)\}\right)^{\top}\mathbf{R}^{-1}\left(\mathrm{d}\mathbf{z}(t) - \mathbb{E}_{\rho^{+}}\{\mathbf{h}(\mathbf{x}, t)\}\mathrm{d}t\right)\right]\rho^{+}$$

Control Problem

Steer joint state PDF via feedback control over finite time horizon



$$\underset{u \in \mathcal{U}}{\text{minimize}} \quad \mathbb{E}\left[\int_0^1 \|\boldsymbol{u}\|_2^2 \, \mathrm{d}t\right]$$

subject to

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)\, \mathrm{d}t + \boldsymbol{g}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{w},$$

$$\boldsymbol{x}(t=0) \sim \rho_0, \quad \boldsymbol{x}(t=1) \sim \rho_1$$

Neural Network Learning Problem

Consider fully connected NN

Think "layers" as interacting population of neurons

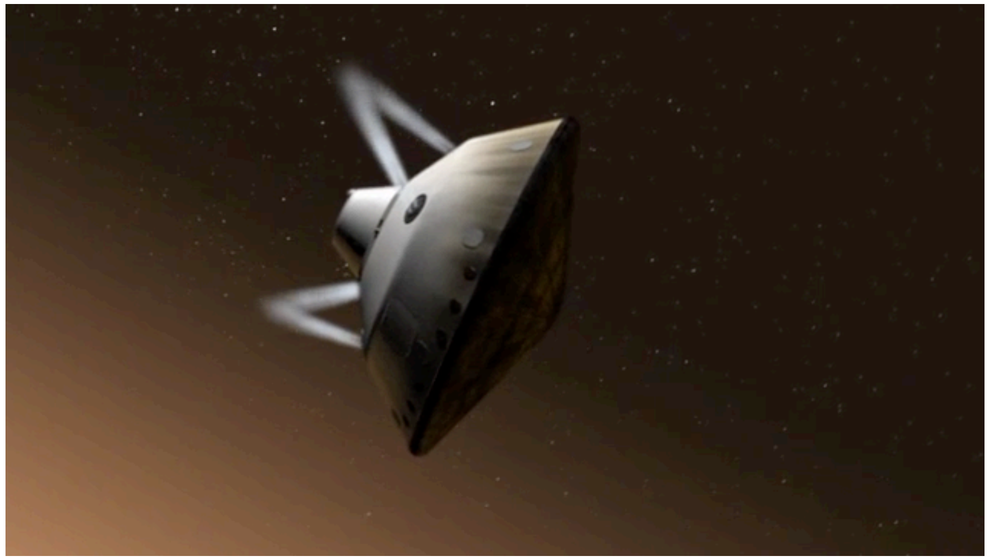
Mean field learning problem: $$\inf_{\rho\in\mathcal{P}_2(\mathbb{R}^p)} R\left(\int \Phi(\boldsymbol{x},\boldsymbol{\theta})\rho(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}\right)$$

PDF dynamics:

$$\frac{\partial\rho}{\partial t} = -\nabla^W R\left(\int\Phi\rho\right) = \nabla\cdot\left(\rho\nabla\frac{\delta}{\delta\rho}R\left(\int\Phi\rho\right)\right)$$

PDFs in Mars Entry-Descent-Landing

Prediction problem



Predict heating rate uncertainty

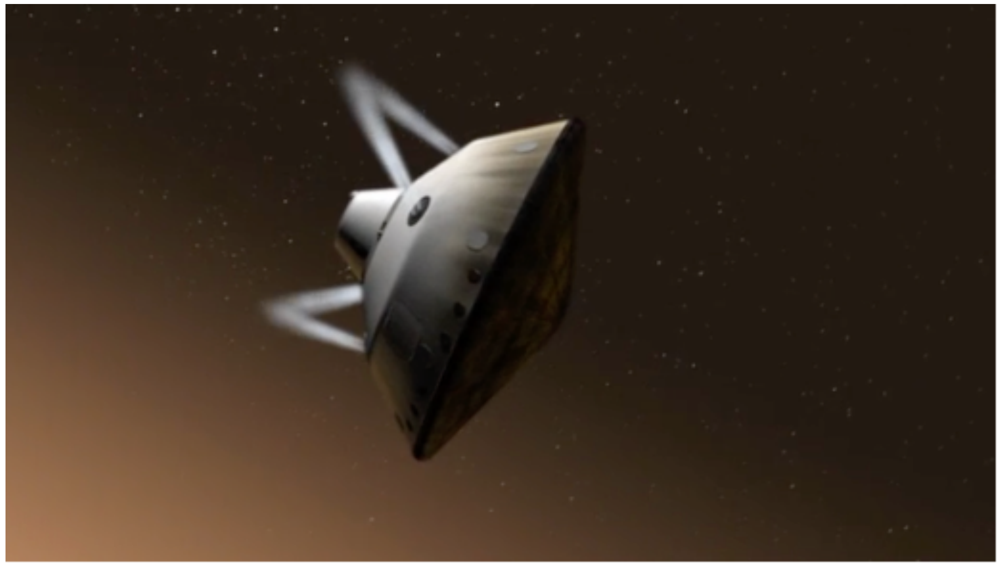
Filtering problem

Control problem

Learning problem

PDFs in Mars Entry-Descent-Landing

Prediction problem



Predict heating rate uncertainty

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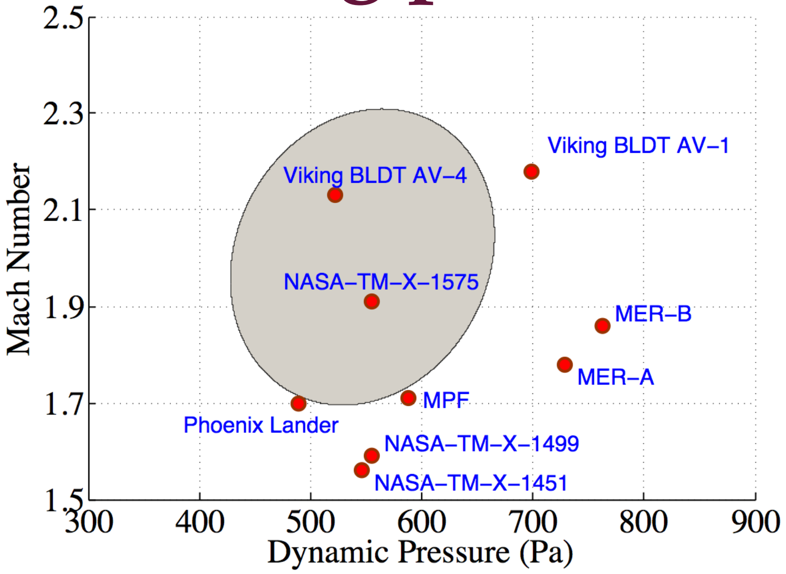


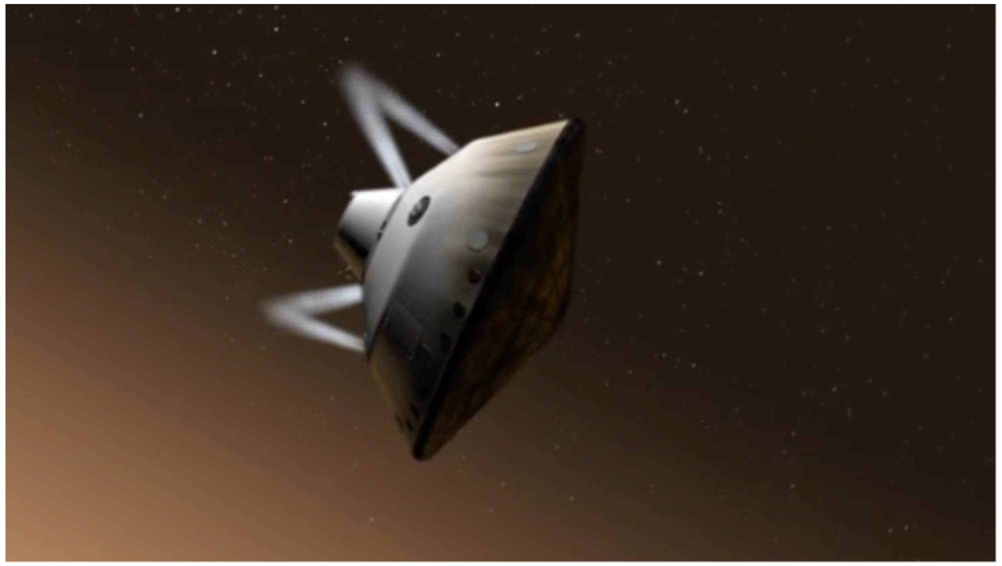
Estimate state to deploy parachute

Control problem

Learning problem

PDFs in Mars Entry-Descent-Landing

Prediction problem



Predict heating rate uncertainty

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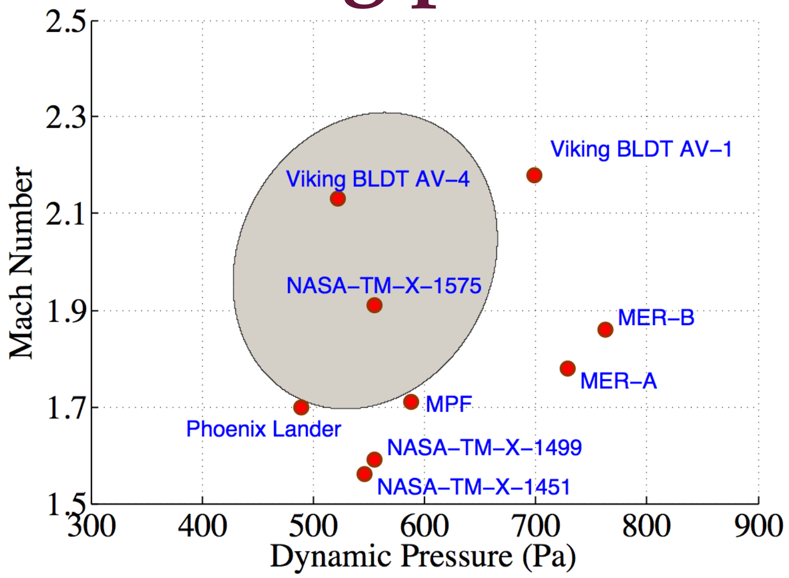

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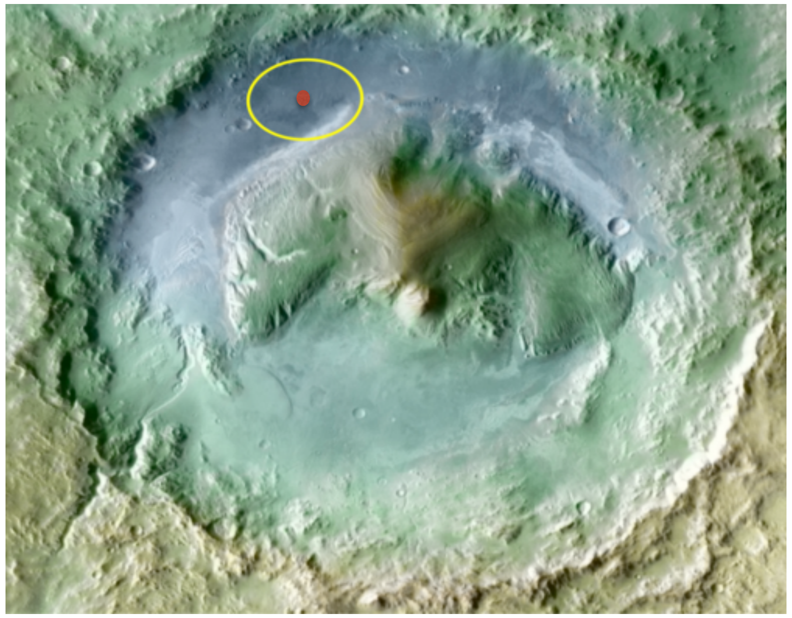


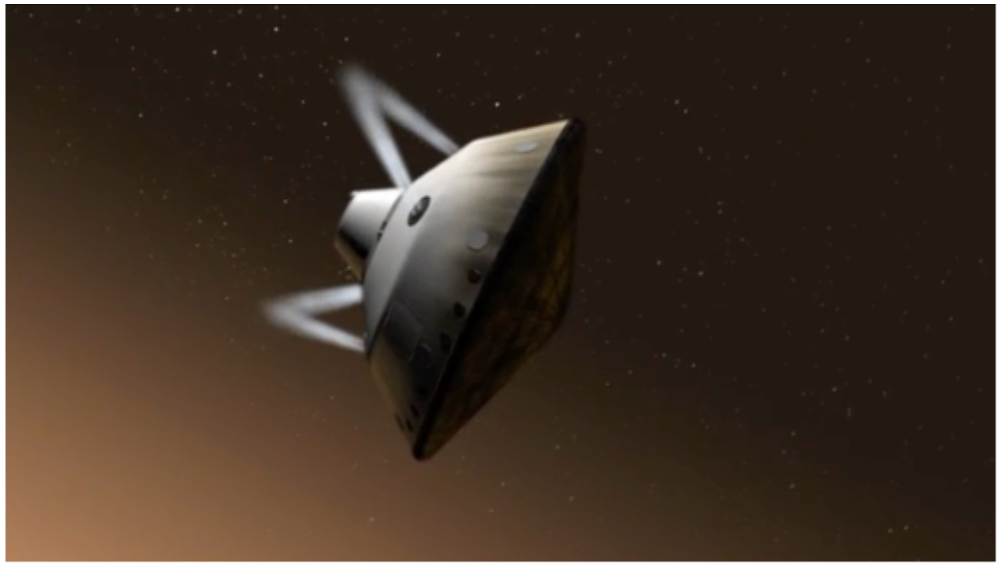
Gale Crater (4.49S, 137.42E)

Steer state PDF to achieve desired landing footprint accuracy

Learning problem

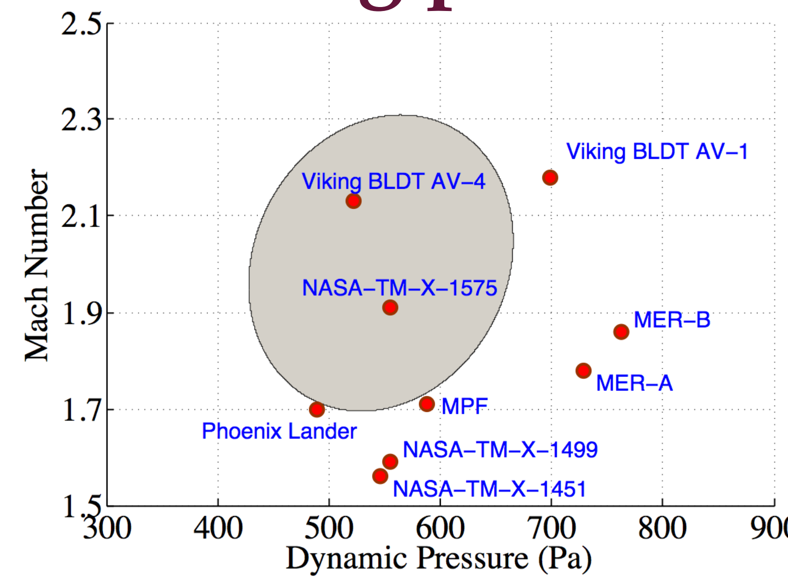
PDFs in Mars Entry-Descent-Landing

Prediction problem



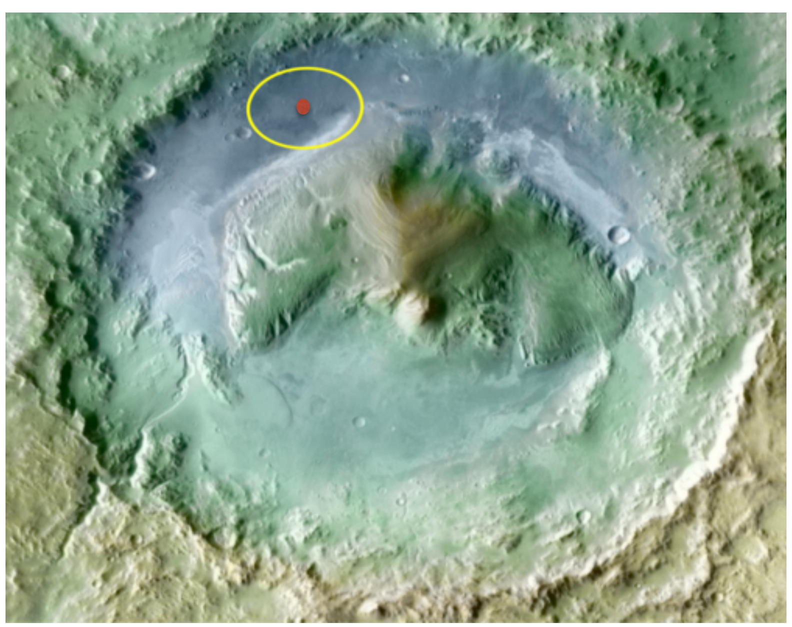
Predict heating rate uncertainty

Filtering problem



Estimate state to deploy parachute

Control problem



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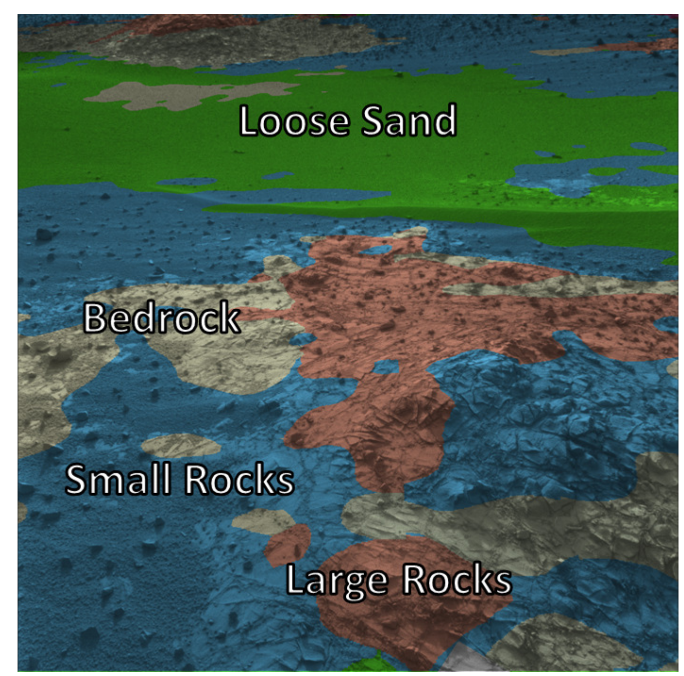


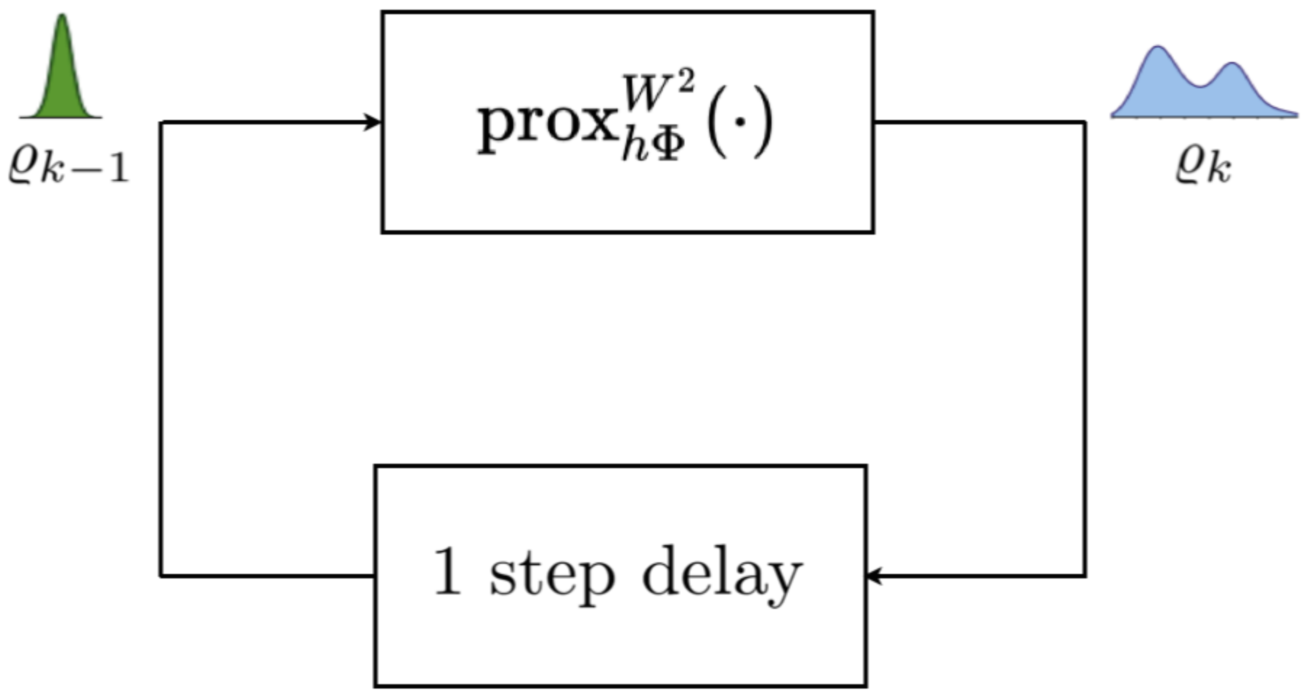
Learn surface feature from data

Solving prediction problem as generalized gradient flow

What's New?

Main idea: Solve $\dfrac{\partial \rho}{\partial t} = \mathcal{L}_{\mathrm{FP}}\rho, \; \rho(x, t=0) = \rho_0$ **as gradient flow in** $\mathcal{P}_2(\mathcal{X})$

Infinite dimensional variational recursion:



Proximal operator: $\varrho_k = \mathrm{prox}_{h\Phi}^{W^2}(\varrho_{k-1}) := \arg\inf_{\varrho \in \mathcal{P}_2(\mathcal{X})} \left\{ \frac{1}{2} W^2(\varrho, \varrho_{k-1}) + h\Phi(\varrho) \right\}$

Optimal transport cost: $W^2(\varrho, \varrho_{k-1}) := \inf_{\pi \in \Pi(\varrho, \varrho_{k-1})} \int_{\mathcal{X} \times \mathcal{X}} c(x, y)\, \mathrm{d}\pi(x, y)$

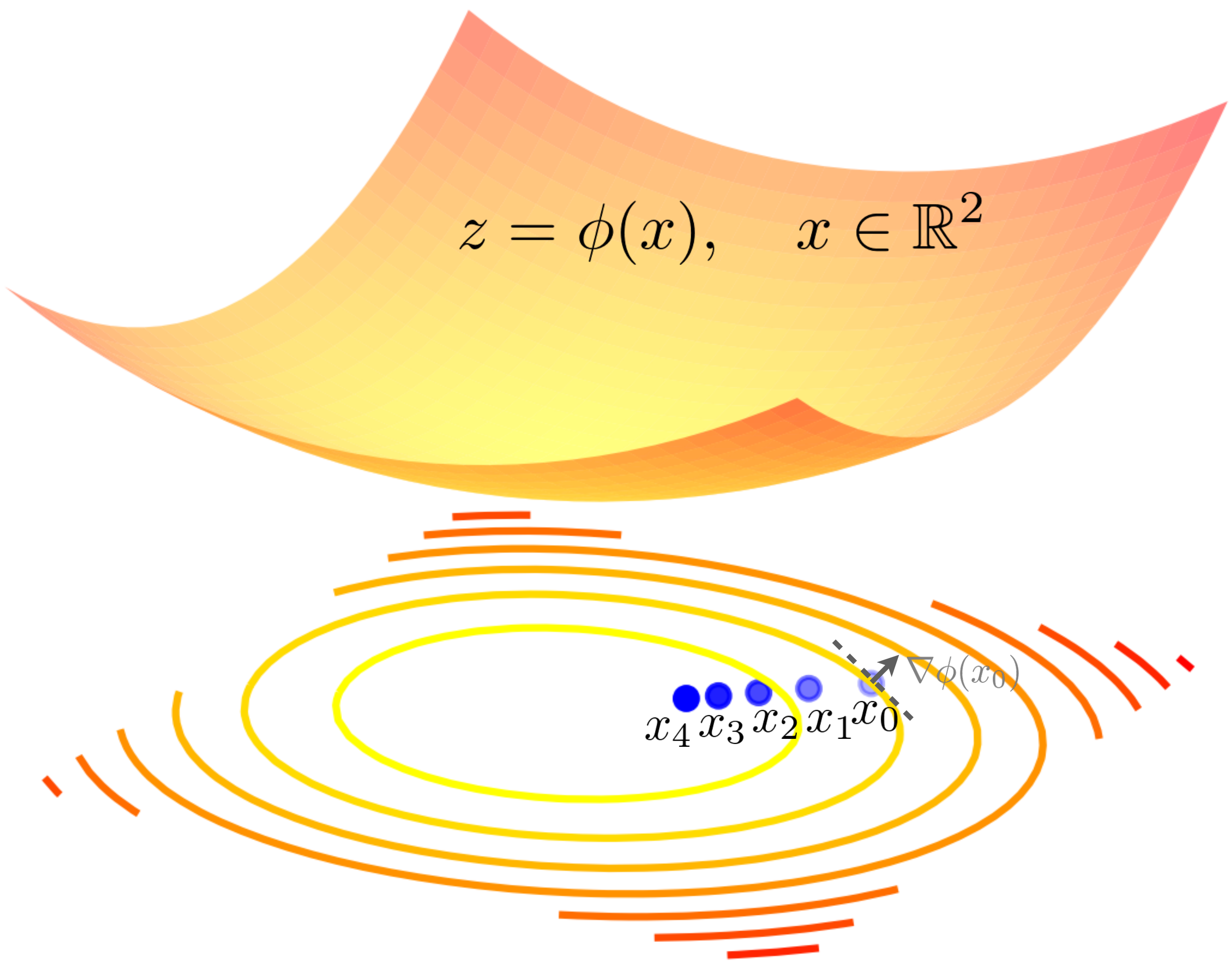
Free energy functional: $\Phi(\varrho) := \int_{\mathcal{X}} \psi \varrho \, \mathrm{d}x + \beta^{-1} \int_{\mathcal{X}} \varrho \log \varrho \, \mathrm{d}x$

Geometric Meaning of Gradient Flow

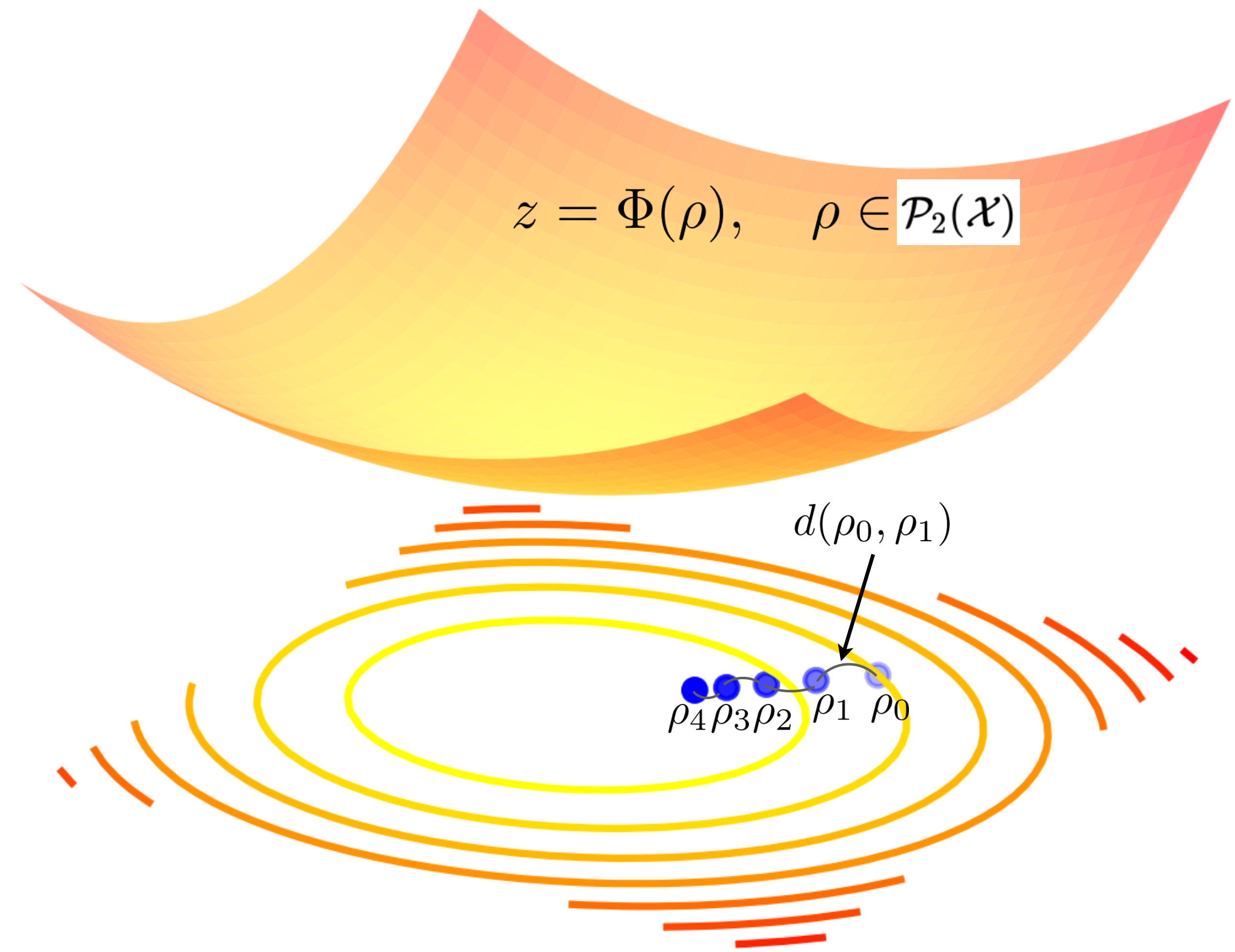
Gradient Flow in $\mathcal{X}$	**Gradient Flow in $\mathcal{P}_2(\mathcal{X})$**
$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -\nabla\varphi(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$	$\frac{\partial\rho}{\partial t} = -\nabla^{W}\Phi(\rho), \quad \rho(\boldsymbol{x},0) = \rho_0$
Recursion:	**Recursion:**
$\boldsymbol{x}_k = \boldsymbol{x}_{k-1} - h\nabla\varphi(\boldsymbol{x}_k) = \underset{\boldsymbol{x}\in\mathcal{X}}{\arg\min}\left\{\frac{1}{2}\lVert\boldsymbol{x}-\boldsymbol{x}_{k-1}\rVert_2^2 + h\varphi(\boldsymbol{x})\right\} =: \mathrm{prox}_{h\varphi}^{\lVert\cdot\rVert_2}(\boldsymbol{x}_{k-1})$	$\rho_k = \rho(\cdot, t = kh) = \underset{\rho\in\mathcal{P}_2(\mathcal{X})}{\arg\min}\left\{\frac{1}{2}W^2(\rho,\rho_{k-1}) + h\Phi(\rho)\right\} =: \mathrm{prox}_{h\Phi}^{W^2}(\rho_{k-1})$
Convergence:	**Convergence:**
$\boldsymbol{x}_k \to \boldsymbol{x}(t = kh) \quad \text{as} \quad h\downarrow 0$	$\rho_k \to \rho(\cdot, t = kh) \quad \text{as} \quad h\downarrow 0$
φ as Lyapunov function:	**Φ as Lyapunov functional:**
$\frac{\mathrm{d}}{\mathrm{d}t}\varphi = -\lVert\nabla\varphi\rVert_2^2 \leq 0$	$\frac{\mathrm{d}}{\mathrm{d}t}\Phi = -\mathbb{E}_\rho\left[\left\lVert\nabla\frac{\delta\Phi}{\delta\rho}\right\rVert_2^2\right] \leq 0$

Geometric Meaning of Gradient Flow

Gradient Flow in $\mathcal{X}$

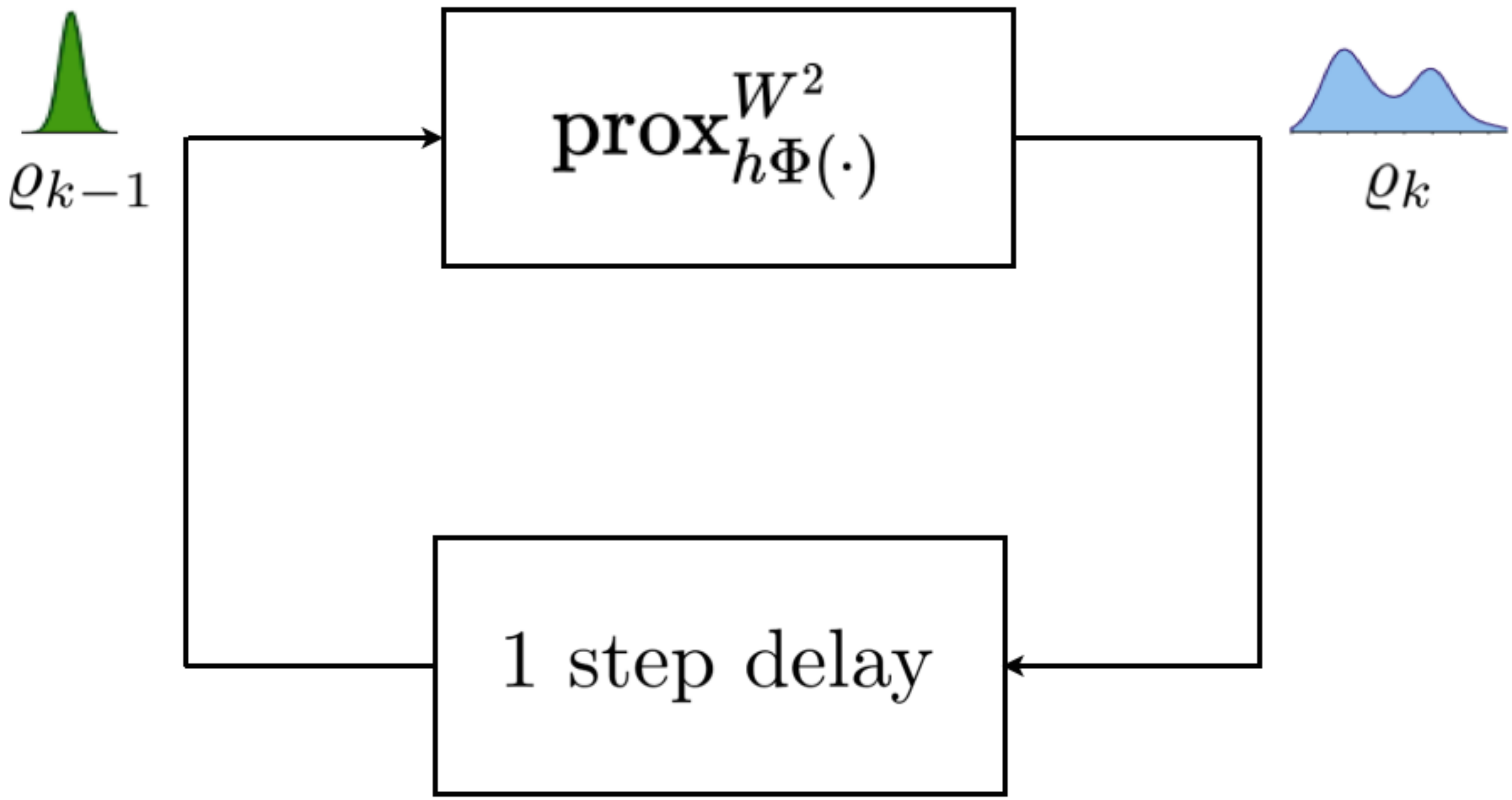


Gradient Flow in $\mathcal{P}_2(\mathcal{X})$



Algorithm: Gradient Ascent on the Dual Space

Uncertainty propagation via point clouds



No spatial discretization or function approximation

Algorithm: Gradient Ascent on the Dual Space

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla \psi \rho) + \beta^{-1} \Delta \rho$$

$\Updownarrow$ **Proximal Recursion**

$$\rho_k = \rho(\boldsymbol{x}, t = kh) = \underset{\rho \in \mathcal{P}_2(\mathbb{R}^n)}{\arg\inf} \left\{ \frac{1}{2} W^2(\rho, \rho_{k-1}) + h\, \Phi(\rho) \right\}$$

$\Downarrow$ **Discrete Primal Formulation**

$$\boldsymbol{\varrho}_k = \underset{\boldsymbol{\varrho}}{\arg\min} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + h \, \langle \boldsymbol{\psi}_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \rangle \right\}$$

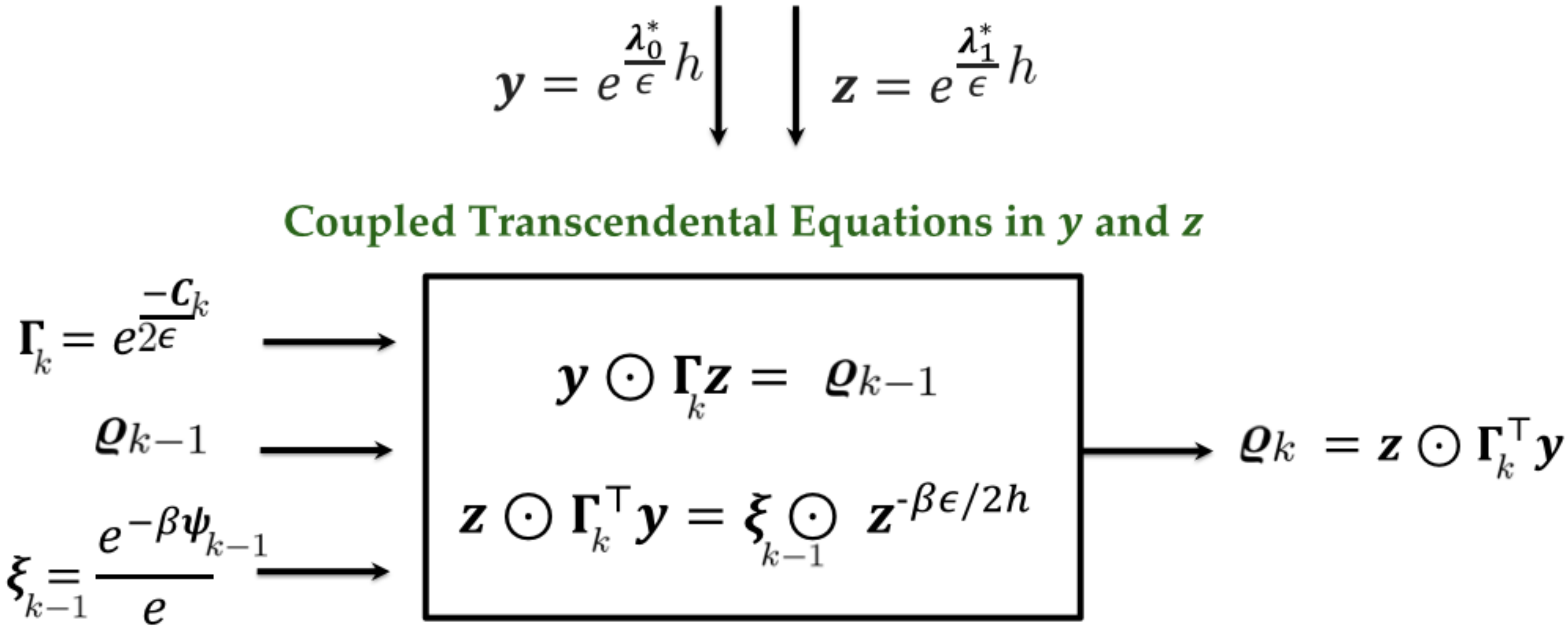
$\Downarrow$ **Entropic Regularization**

$$\boldsymbol{\varrho}_k = \underset{\boldsymbol{\varrho}}{\arg\min} \left\{ \min_{\boldsymbol{M} \in \Pi(\boldsymbol{\varrho}_{k-1}, \boldsymbol{\varrho})} \frac{1}{2} \langle \boldsymbol{C}_k, \boldsymbol{M} \rangle + \epsilon H(\boldsymbol{M}) + h \, \langle \boldsymbol{\psi}_{k-1} + \beta^{-1} \log \boldsymbol{\varrho}, \boldsymbol{\varrho} \rangle \right\}$$

$\Updownarrow$ **Dualization**

$$\boldsymbol{\lambda}_0^{\text{opt}}, \boldsymbol{\lambda}_1^{\text{opt}} = \underset{\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1 \geq 0}{\arg\max} \Big\{ \langle \boldsymbol{\lambda}_0, \boldsymbol{\varrho}_{k-1} \rangle - F^{\star}(-\boldsymbol{\lambda}_1) - \frac{\epsilon}{h} \left(\exp(\boldsymbol{\lambda}_0^{\top} h/\epsilon) \exp(-\boldsymbol{C}_k/2\epsilon) \exp(\boldsymbol{\lambda}_1 h/\epsilon) \right) \Big\}$$

Recursion on the Cone



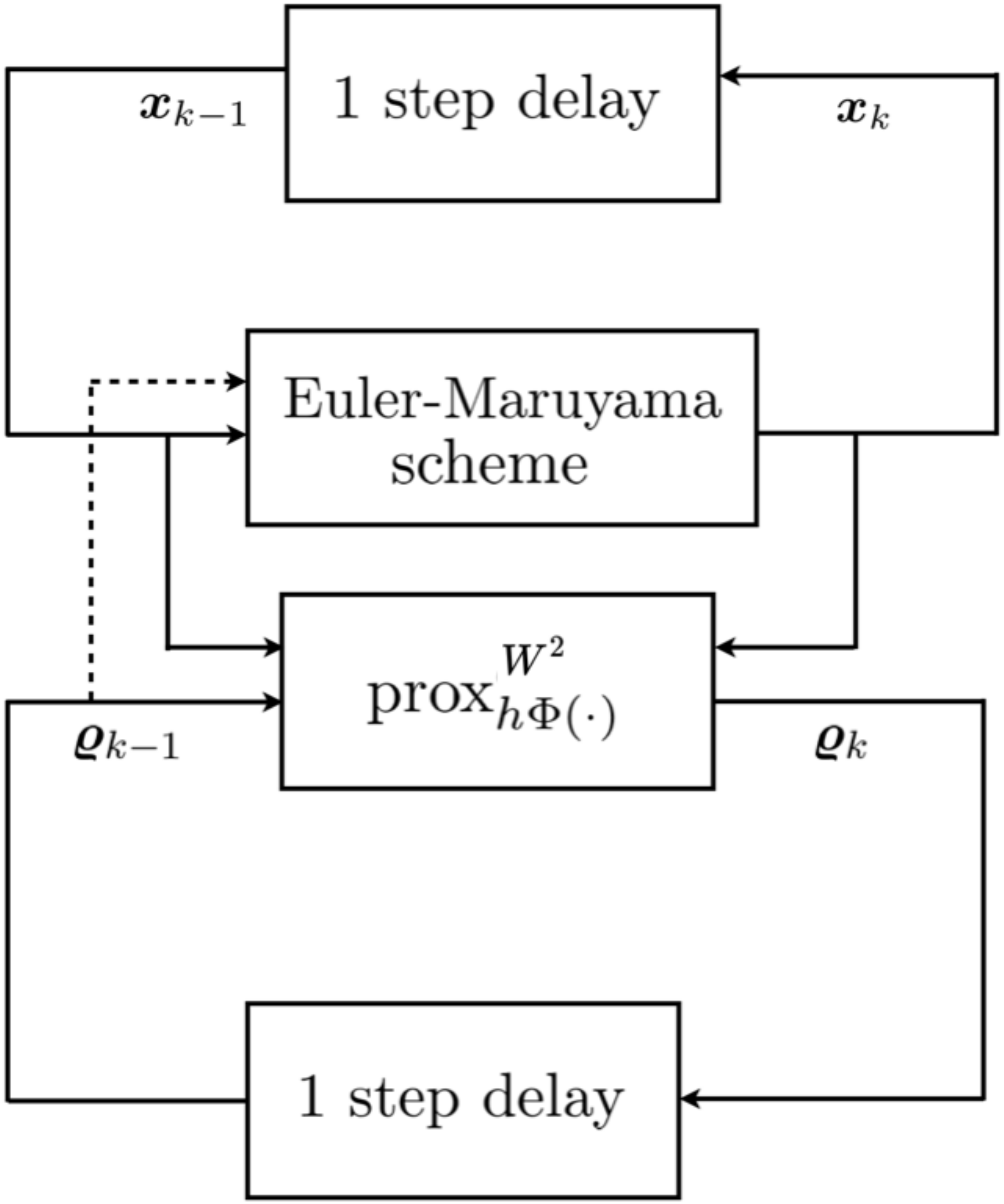
Theorem: Consider the recursion on the cone $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0}$

$$\boldsymbol{y} \odot (\boldsymbol{\Gamma}_k \boldsymbol{z}) = \boldsymbol{\varrho}_{k-1}, \quad \boldsymbol{z} \odot \left(\boldsymbol{\Gamma}_k{}^\top \boldsymbol{y}\right) = \boldsymbol{\xi}_{k-1} \odot \boldsymbol{z}^{-\frac{\beta\epsilon}{h}},$$

Then the solution $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ gives the proximal update $\boldsymbol{\varrho}_k = \boldsymbol{z}^* \odot (\boldsymbol{\Gamma}_k{}^\top \boldsymbol{y}^*)$

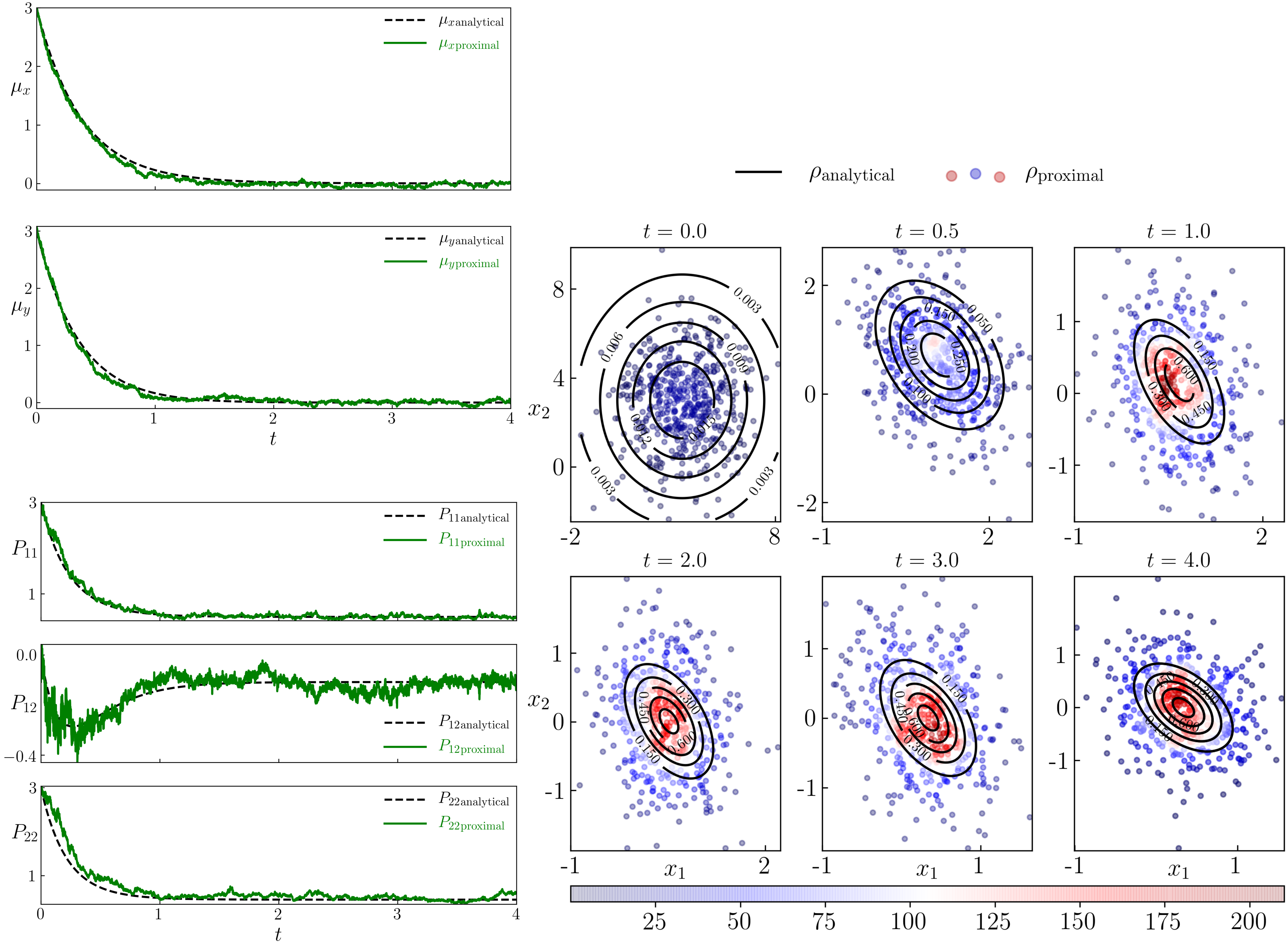
— K.F. Caluya and A.H., Gradient flow algorithms for density propagation in stochastic systems, *IEEE TAC* 2020.

Algorithmic Setup

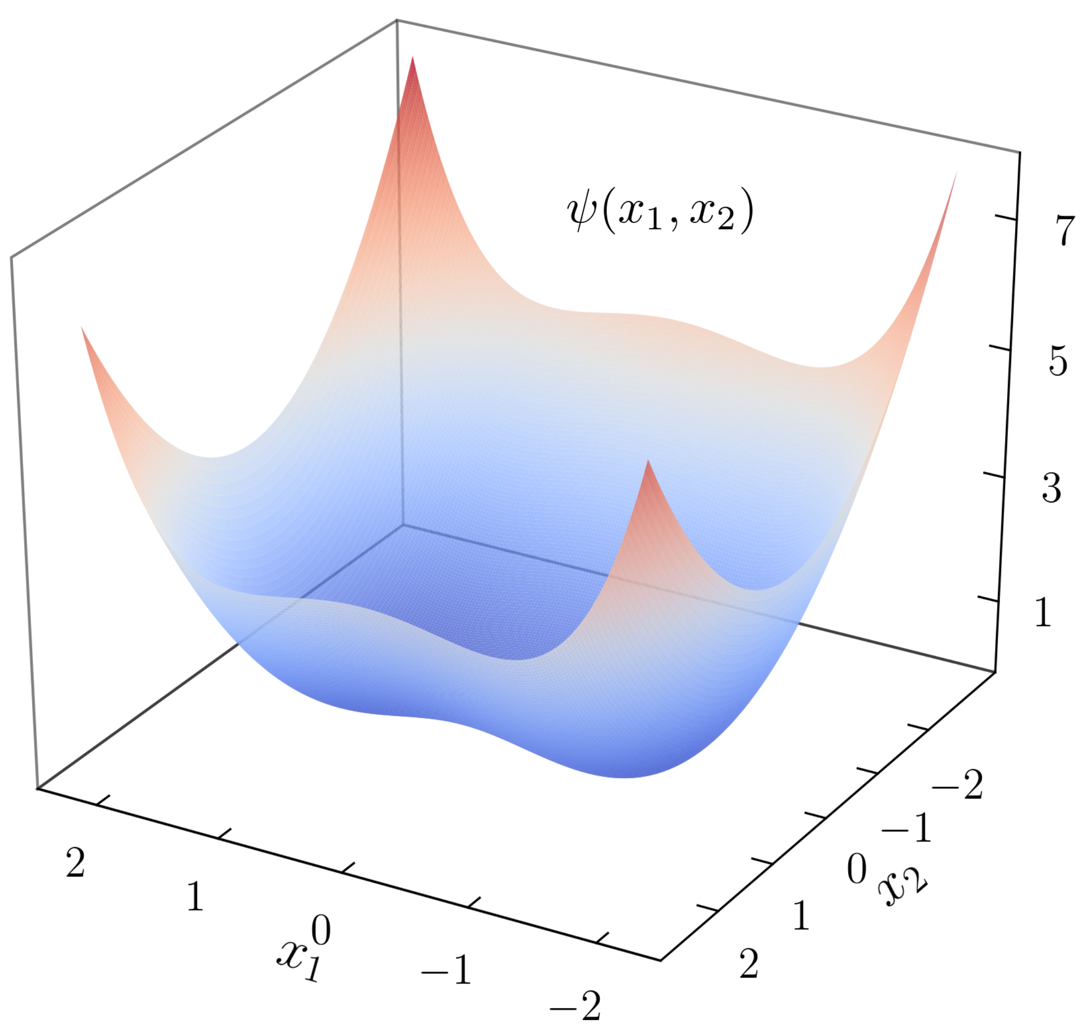


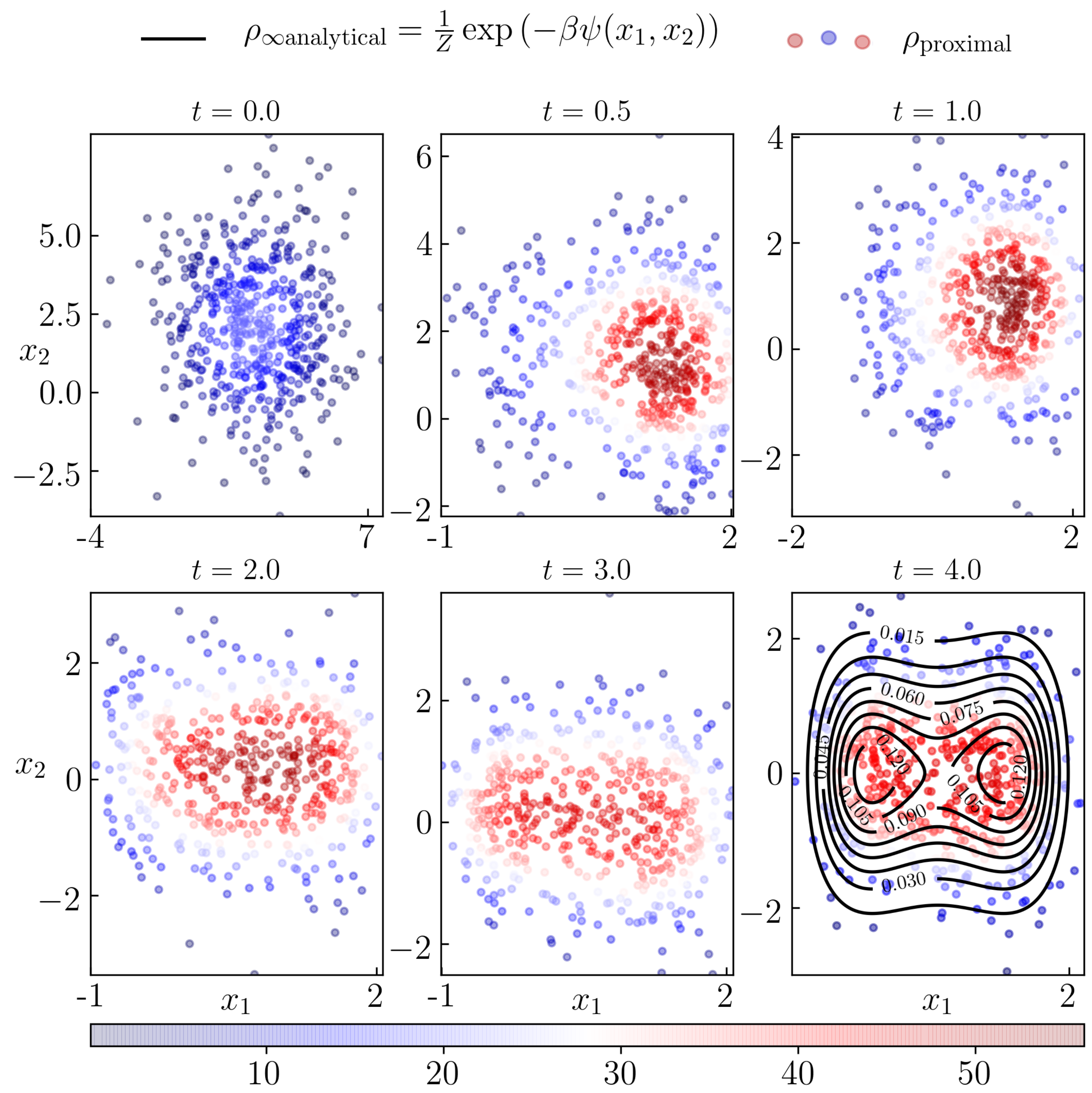
Theorem: Block co-ordinate iteration of $(\mathbf{y}, \mathbf{z})$ recursion is contractive on $\mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0}$.

Proximal Prediction: 2D Linear Gaussian

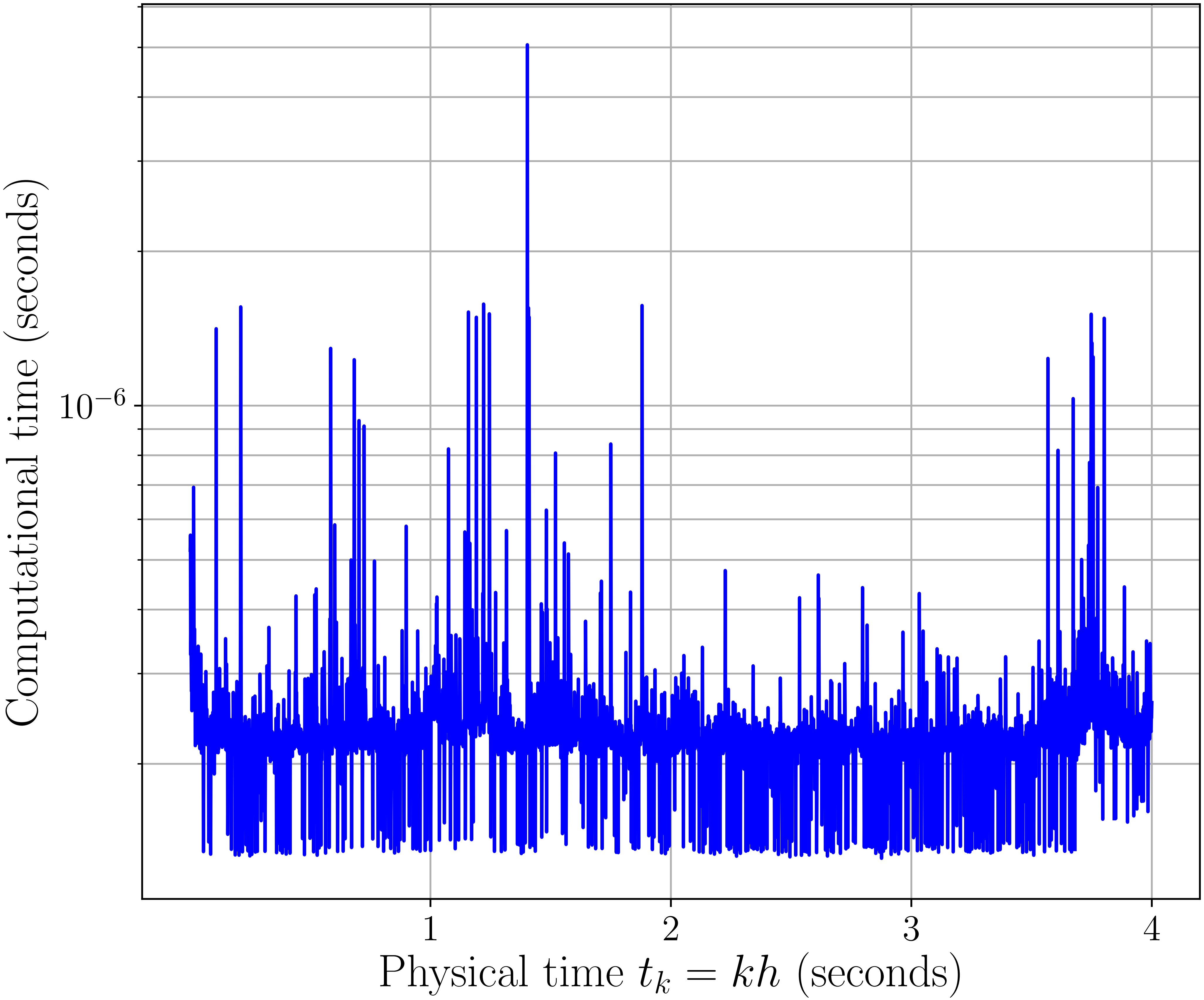


Proximal Prediction: Nonlinear Non-Gaussian





Computational Time: Nonlinear Non-Gaussian



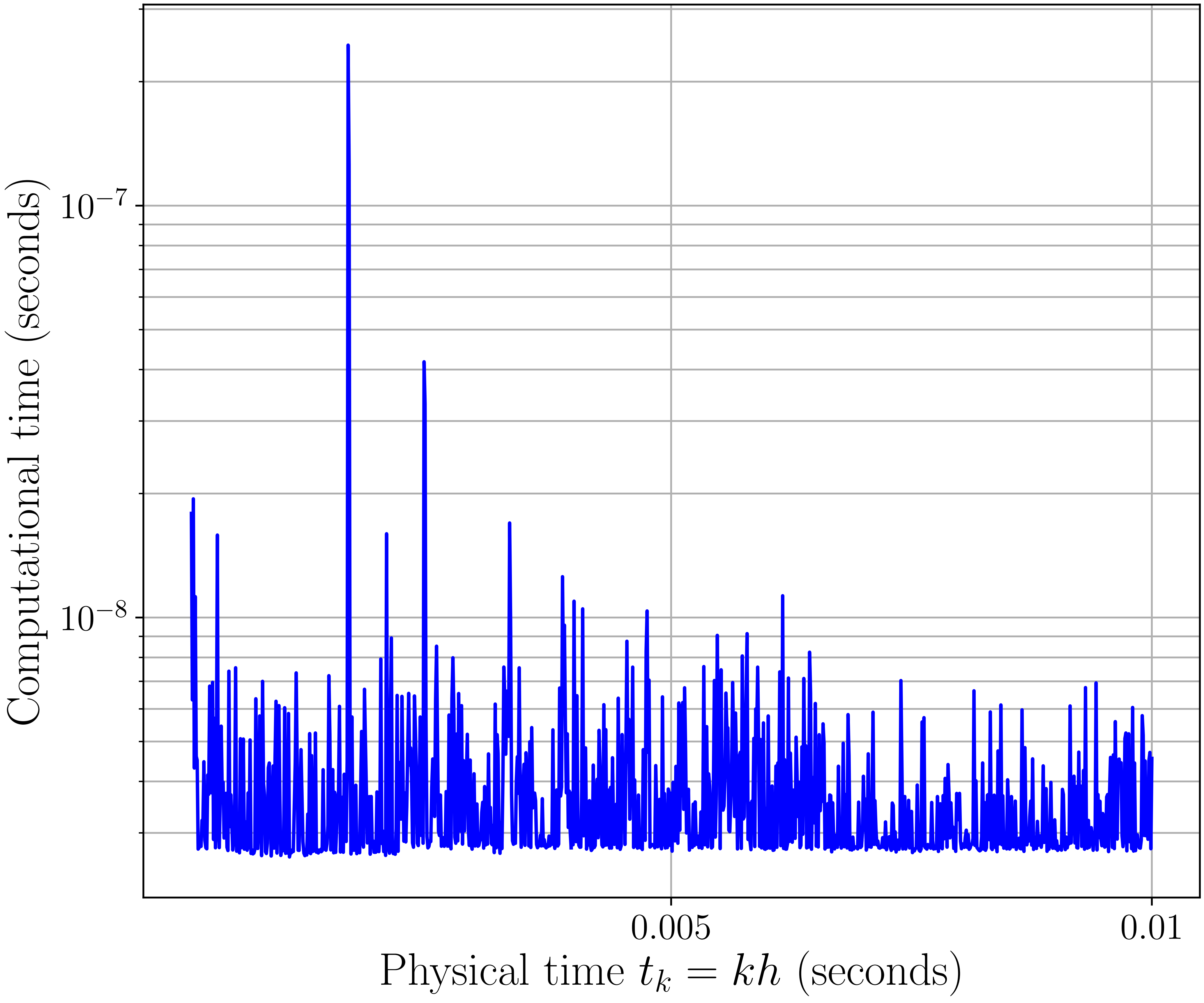
Proximal Prediction: Satellite in Geocentric Orbit

Here, $\mathcal{X} \equiv \mathbb{R}^6$

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \\ \mathrm{d}v_x \\ \mathrm{d}v_y \\ \mathrm{d}v_z \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ -\dfrac{\mu x}{r^3} + (f_x)_{\text{pert}} - \gamma v_x \\ -\dfrac{\mu y}{r^3} + (f_y)_{\text{pert}} - \gamma v_y \\ -\dfrac{\mu z}{r^3} + (f_z)_{\text{pert}} - \gamma v_z \end{pmatrix} \mathrm{d}t + \sqrt{2\beta^{-1}\gamma} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{d}w_1 \\ \mathrm{d}w_2 \\ \mathrm{d}w_3 \end{pmatrix},$$

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}_{\text{pert}} = \begin{pmatrix} s\theta\, c\phi & c\theta\, c\phi & -s\phi \\ s\theta\, s\phi & c\theta\, s\phi & c\phi \\ c\theta & -s\theta & 0 \end{pmatrix} \begin{pmatrix} \dfrac{k}{2r^4}\left(3(s\theta)^2 - 1\right) \\ -\dfrac{k}{r^5} s\theta\, c\theta \\ 0 \end{pmatrix}, k := 3J_2 R_{\mathrm{E}}^2, \mu = \text{constant}$$

Computational Time: Satellite in Geocentric Orbit



Extensions: Nonlocal Interactions

PDF dependent sample path dynamics:

$$\mathrm{d}\mathbf{x} = -\left(\nabla U\left(\mathbf{x}\right) + \nabla\rho * V\right)\mathrm{d}t + \sqrt{2\beta^{-1}}\,\mathrm{d}\mathbf{w}$$

Mckean-Vlasov-Fokker-Planck-Kolmogorov integro PDE:

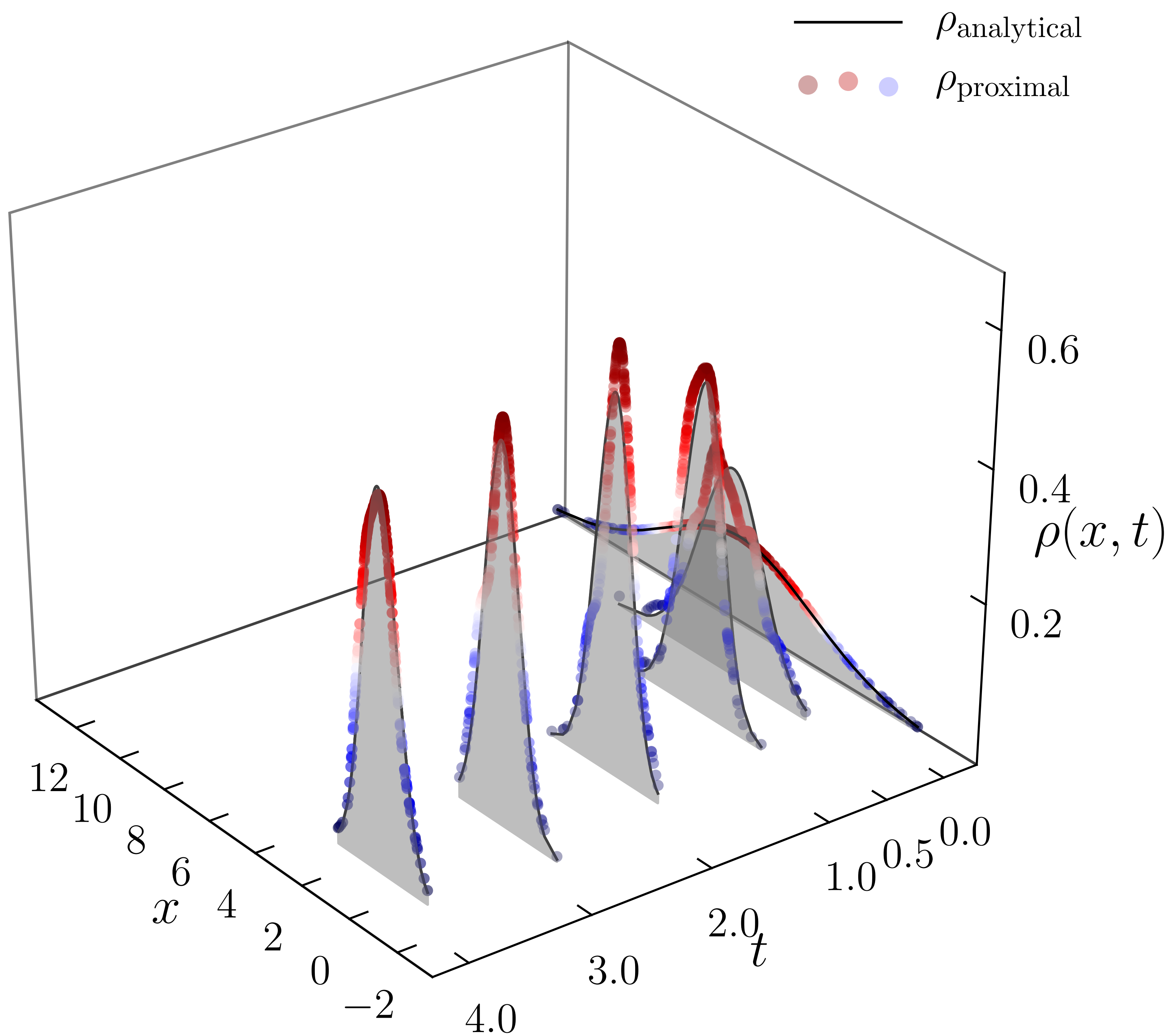
$$\frac{\partial\rho}{\partial t} = \nabla\cdot\left(\rho\nabla\left(U + \rho * V\right)\right) + \beta^{-1}\Delta\rho$$

Free energy:

$$F(\rho) := \mathbb{E}_{\rho}\left[U + \beta^{-1}\rho\log\rho + \rho * V\right]$$

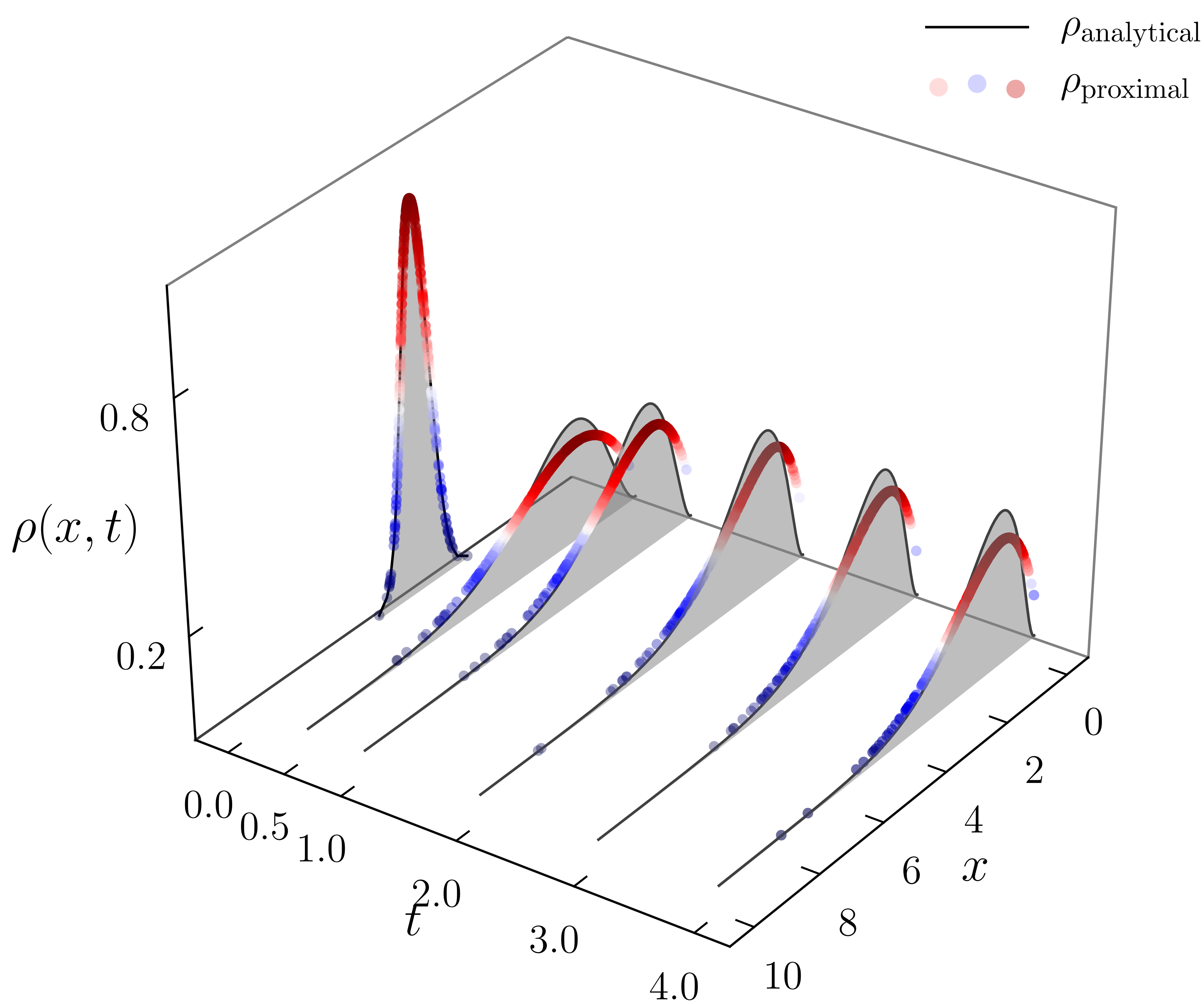
Extensions: Nonlocal Interactions

$$U(\cdot) = V(\cdot) = \| \cdot \|_2^2$$



Extensions: Multiplicative Noise

Cox-Ingersoll-Ross: $\mathrm{d}x = a\,(\theta - x)\;\mathrm{d}t + b\sqrt{x}\,\mathrm{d}w,\ 2a > b^2, \theta > 0$



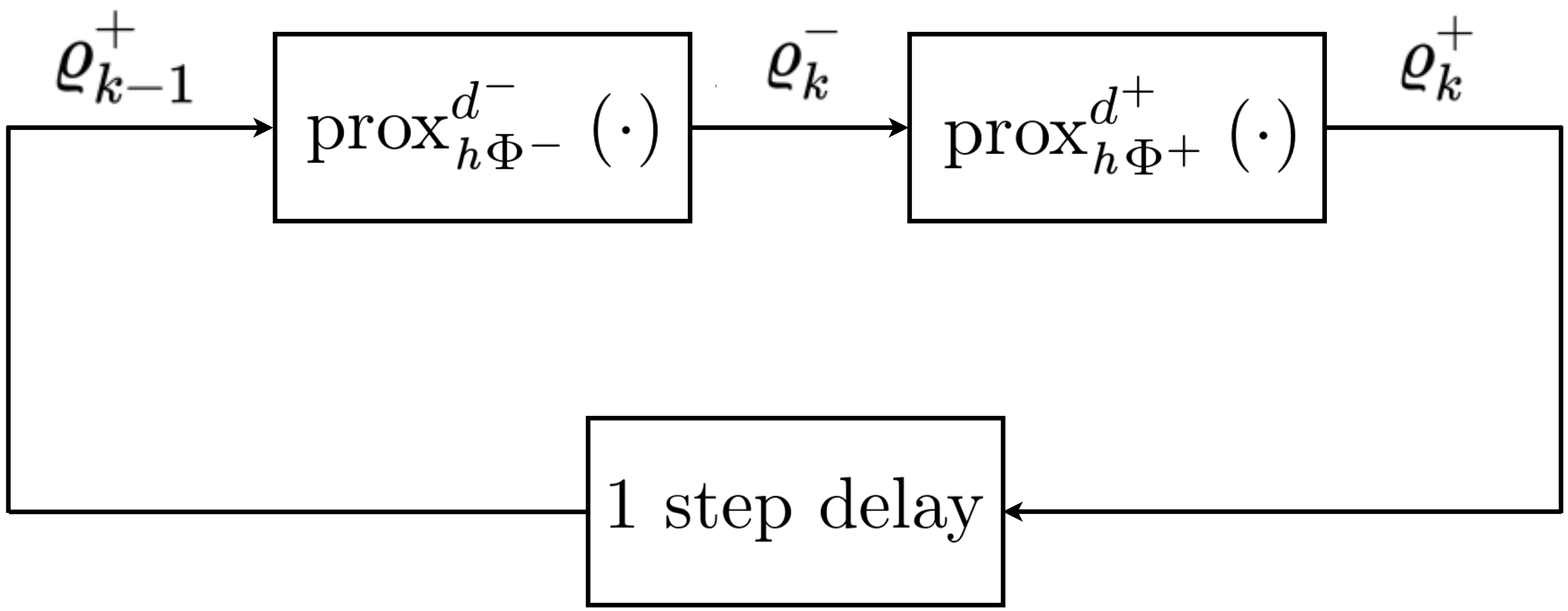
Solving filtering as generalized gradient flow

What's New?

Main idea: Solve the Kushner-Stratonovich SPDE

$\mathrm{d}\rho^+ = \left[\mathcal{L}_{\mathrm{FP}}\mathrm{d}t + \mathcal{L}\left(\mathrm{d}z, \mathrm{d}t, \rho^+\right)\right]\rho^+,\ \rho(x, t=0) = \rho_0$ as gradient flow in $\mathcal{P}_2(\mathcal{X})$

Recursion of {deterministic ∘ stochastic} proximal operators:



Convergence: $\varrho_k^+(h) \to \rho^+(x, t=kh) \quad \text{as} \quad h \downarrow 0$

For prior, as before: $d^- \equiv W^2, \quad \Phi^- \equiv \mathbb{E}_{\varrho}\left[\psi + \beta^{-1}\log\varrho\right]$

For posterior: $d^+ \equiv d_{\mathrm{FR}}^2$ or $D_{\mathrm{KL}}, \quad \Phi^+ \equiv \dfrac{1}{2}\mathbb{E}_{\varrho^+}\left[(y_k - h(x))^\top R^{-1}(y_k - h(x))\right]$

Explicit Recovery of the Kalman-Bucy Filter

Model:

$$\mathrm{d}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)\mathrm{d}t + \mathbf{B}\mathrm{d}\mathbf{w}(t), \quad \mathrm{d}\mathbf{w}(t) \sim \mathcal{N}(0, \mathbf{Q}\mathrm{d}t)$$

$$\mathrm{d}\mathbf{z}(t) = \mathbf{C}\mathbf{x}(t)\mathrm{d}t + \mathrm{d}\mathbf{v}(t), \quad \mathrm{d}\mathbf{v}(t) \sim \mathcal{N}(0, \mathbf{R}\mathrm{d}t)$$

Given $\mathbf{x}(0) \sim \mathcal{N}(\mu_0, \mathbf{P}_0)$, want to recover:

$\mathbf{P}^{+}\mathbf{C}\mathbf{R}^{-1}$

$$\mathrm{d}\mu^{+}(t) = \mathbf{A}\mu^{+}(t)\mathrm{d}t + \mathbf{K}(t)\left(\mathrm{d}\mathbf{z}(t) - \mathbf{C}\mu^{+}(t)\mathrm{d}t\right),$$

$$\dot{\mathbf{P}}^{+}(t) = \mathbf{A}\mathbf{P}^{+}(t) + \mathbf{P}^{+}(t)\mathbf{A}^{\top} + \mathbf{B}\mathbf{Q}\mathbf{B}^{\top} - \mathbf{K}(t)\mathbf{R}\mathbf{K}(t)^{\top}.$$

— A.H. and T.T. Georgiou, Gradient Flows in Uncertainty Propagation and Filtering of Linear Gaussian Systems, *CDC 2017*.

— A.H. and T.T. Georgiou, Gradient Flows in Filtering and Fisher-Rao Geometry, *ACC 2018*.

Explicit Recovery of the Wonham Filter

Model:

$$x(t) \sim \text{Markov}(Q),$$
$$\mathrm{d}z(t) = h(x(t))\,\mathrm{d}t \;+\; \sigma_v(t)\mathrm{d}v(t)$$

J.SIAM CONTROL
Ser. A, Vol. 2, No. 3
Printed in U.S.A., 1965

SOME APPLICATIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS TO OPTIMAL NONLINEAR FILTERING*

W. M. WONHAM†

State space: $\Omega := \{a_1, \ldots, a_m\}$

Posterior $\pi^+(t) := \{\pi_1^+(t), \ldots, \pi_m^+(t)\}$ **solves the nonlinear SDE:**

$$\mathrm{d}\pi^+(t) = \pi^+(t)Q\,\mathrm{d}t \;+\; \frac{1}{(\sigma_v(t))^2}\pi^+(t)\Big(H - \widehat{h}(t)I\Big)\Big(\mathrm{d}z(t) - \widehat{h}(t)\mathrm{d}t\Big),$$

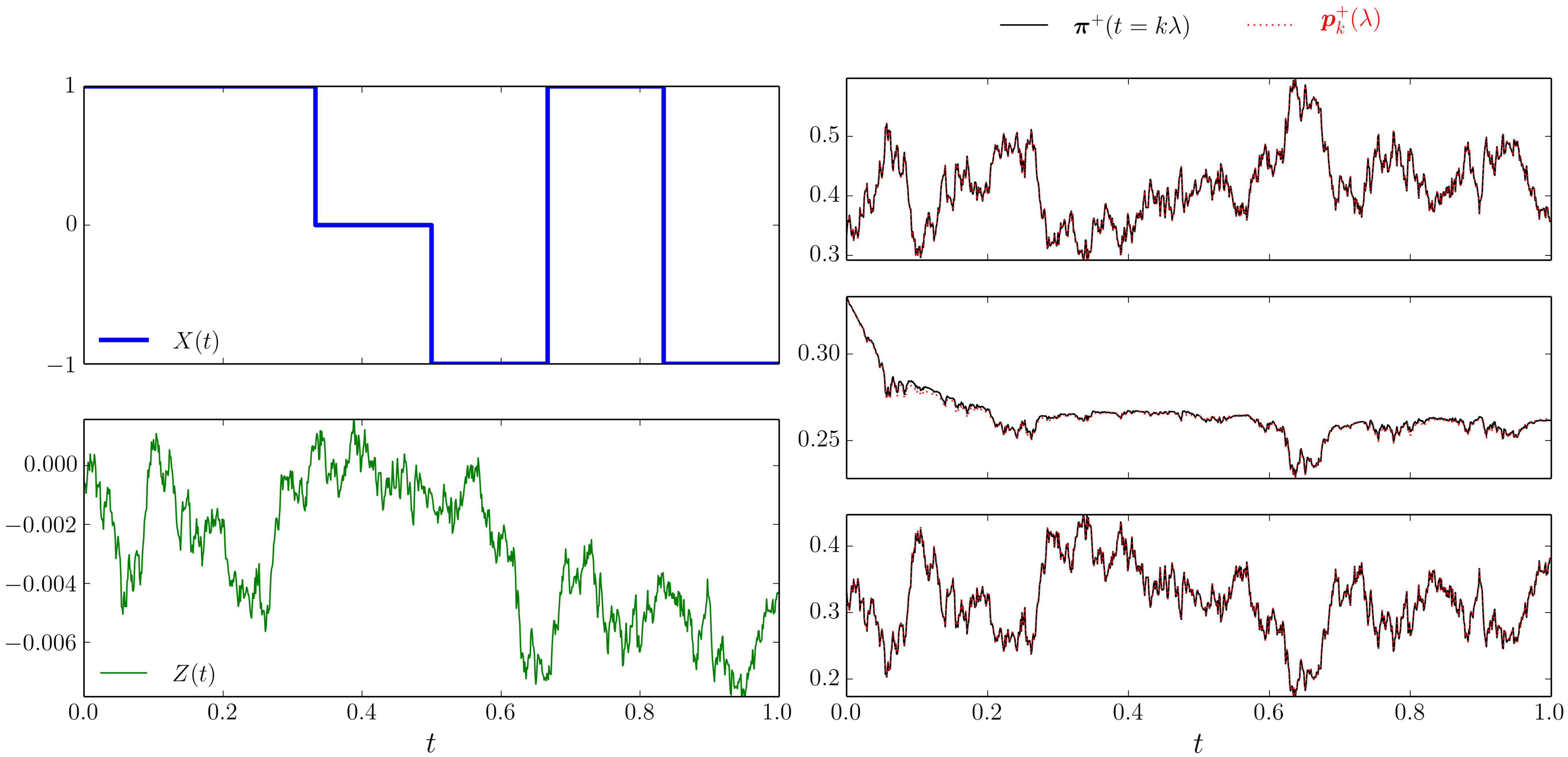
where $H := \mathrm{diag}(h(a_1), \ldots, h(a_m)), \quad \widehat{h}(t) := \sum_{i=1}^{m} h(a_i)\pi_i^+(t),$

Initial condition: $\pi^+(t=0) = \pi_0,$

By defn. $\pi^+(t) = \mathbb{P}(x(t) = a_i \mid z(s), 0 \leq s \leq t)$

— A.H. and T.T. Georgiou, Proximal Recursion for the Wonham Filter, *CDC 2019*.

Numerical Results for the Wonham Filter



— A.H. and T.T. Georgiou, Proximal Recursion for the Wonham Filter, *CDC 2019*.

Solving density control as generalized gradient flow

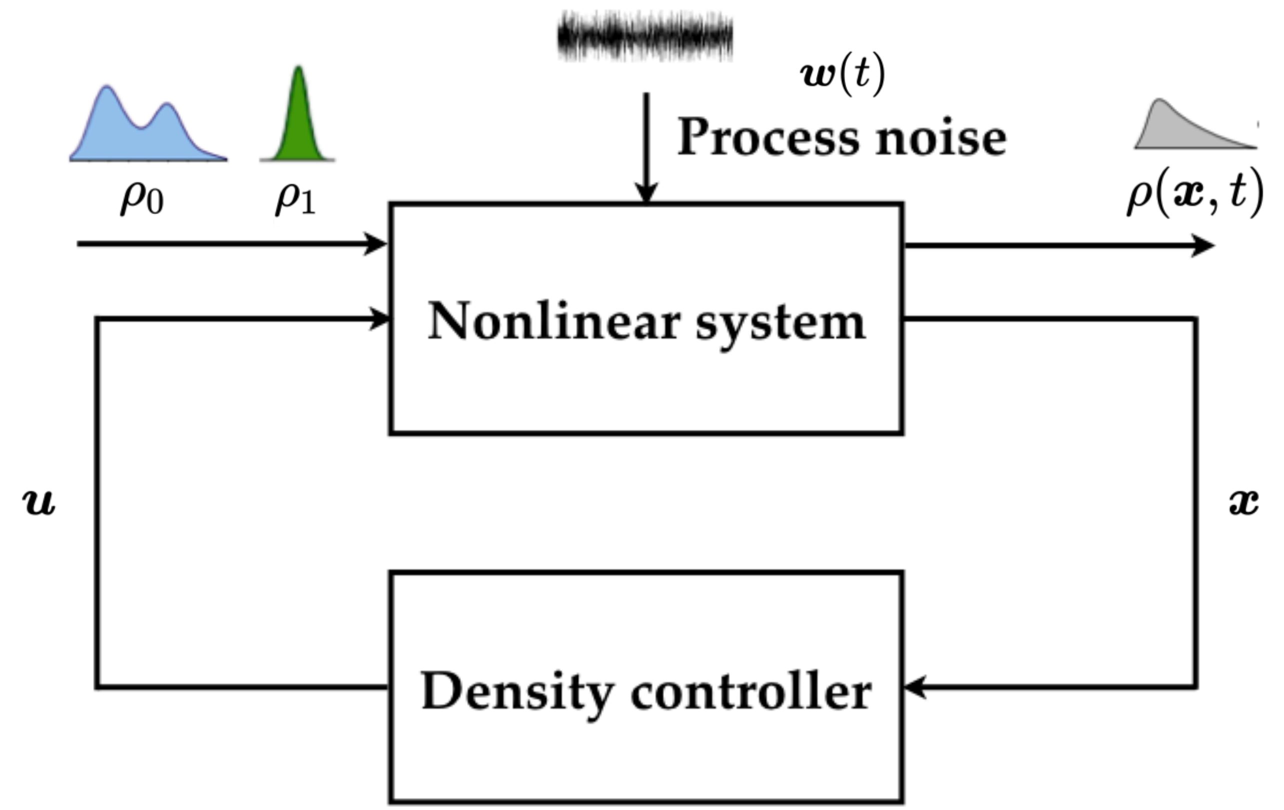
Finite Horizon Feedback Density Control

$$\underset{u \in \mathcal{U}}{\text{minimize}} \quad \mathbb{E}\left[\int_0^1 \|\boldsymbol{u}(\boldsymbol{x},t)\|_2^2 \,\mathrm{d}t\right]$$

subject to

$$\mathrm{d}\boldsymbol{x} = \left\{\boldsymbol{f}(\boldsymbol{x},t) + \boldsymbol{B}(t)\boldsymbol{u}(\boldsymbol{x},t)\right\}\mathrm{d}t + \sqrt{2\epsilon}\boldsymbol{B}(t)\,\mathrm{d}\boldsymbol{w},$$

$$\boldsymbol{x}(t=0) \sim \rho_0, \quad \boldsymbol{x}(t=1) \sim \rho_1$$



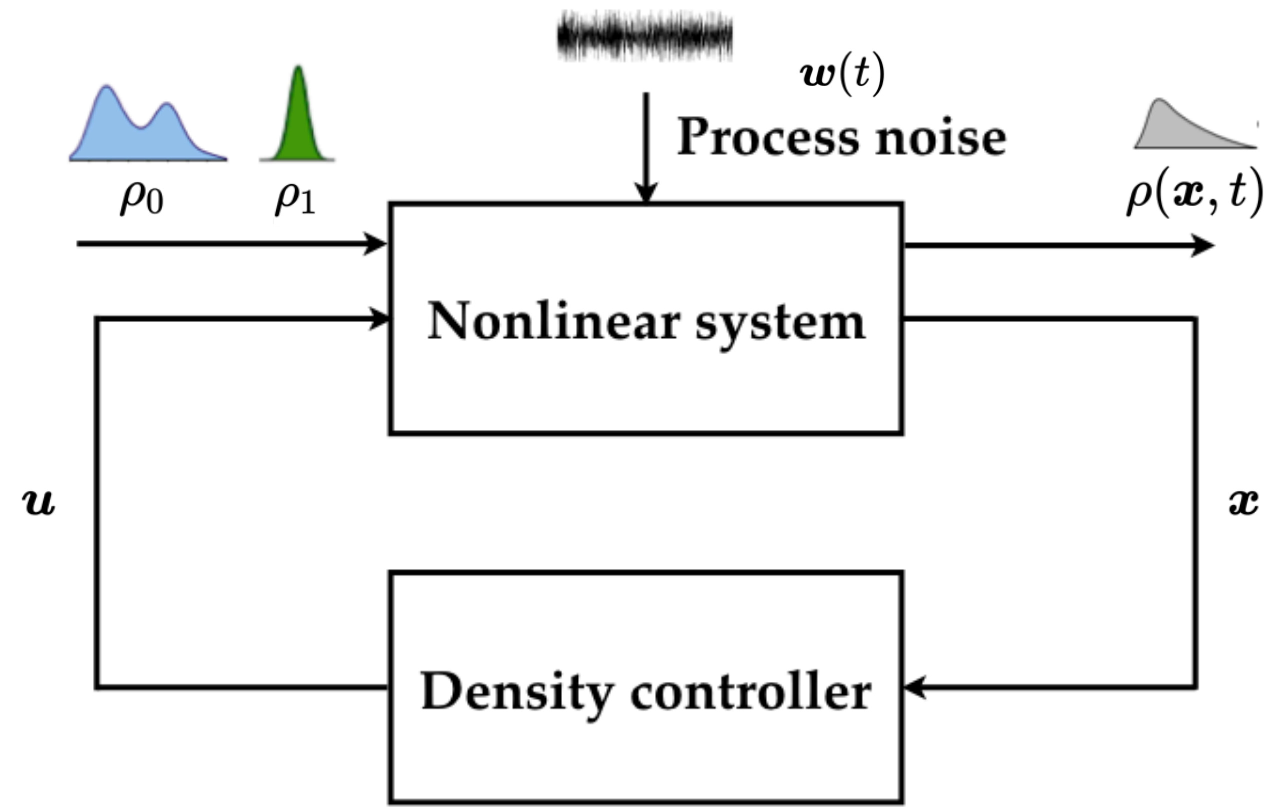
Finite Horizon Feedback Density Control

$$\underset{u \in \mathcal{U}}{\text{minimize}} \quad \mathbb{E}\left[\int_0^1 \|\boldsymbol{u}(\boldsymbol{x},t)\|_2^2 \, \mathrm{d}t\right]$$

subject to

$$\mathrm{d}\boldsymbol{x} = \left\{\boldsymbol{f}(\boldsymbol{x},t) + \boldsymbol{B}(t)\boldsymbol{u}(\boldsymbol{x},t)\right\}\mathrm{d}t + \sqrt{2\epsilon}\boldsymbol{B}(t)\,\mathrm{d}\boldsymbol{w},$$

$$\boldsymbol{x}(t=0) \sim \rho_0, \quad \boldsymbol{x}(t=1) \sim \rho_1$$



Necessary conditions for optimality: coupled nonlinear PDEs (FPK + HJB)

$$\frac{\partial \rho^{\text{opt}}}{\partial t} + \nabla \cdot \left(\rho^{\text{opt}}\left(\boldsymbol{f} + \boldsymbol{B}(t)^\top \nabla\psi\right)\right) = \epsilon \mathbf{1}^\top \left(\boldsymbol{D}(t) \odot \text{Hess}\left(\rho^{\text{opt}}\right)\right)\mathbf{1},$$

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\|\boldsymbol{B}(t)^\top \nabla\psi\|_2^2 + \langle \nabla\psi, \boldsymbol{f}\rangle = -\epsilon\langle \boldsymbol{D}(t), \text{Hess}(\psi)\rangle$$

Boundary conditions:

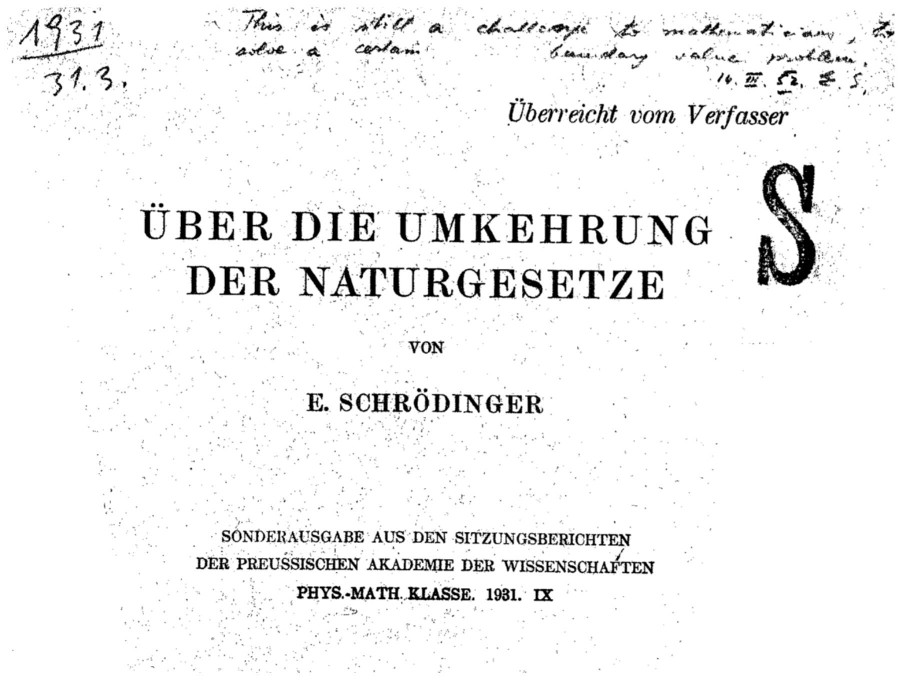
$$\rho^{\text{opt}}(\boldsymbol{x},0) = \rho_0(\boldsymbol{x}), \quad \rho^{\text{opt}}(\boldsymbol{x},1) = \rho_1(\boldsymbol{x})$$

Optimal control:

$$\boldsymbol{u}^{\text{opt}}(\boldsymbol{x},t) = \boldsymbol{B}(t)^\top \nabla\psi$$

Feedback Synthesis via the Schrödinger System

Schrödinger's (until recently) forgotten papers:



1931
31.3.

This is still a challenge to mathematicians, to solve a certain boundary value problem. 14.III.53. E.S.

Überreicht vom Verfasser

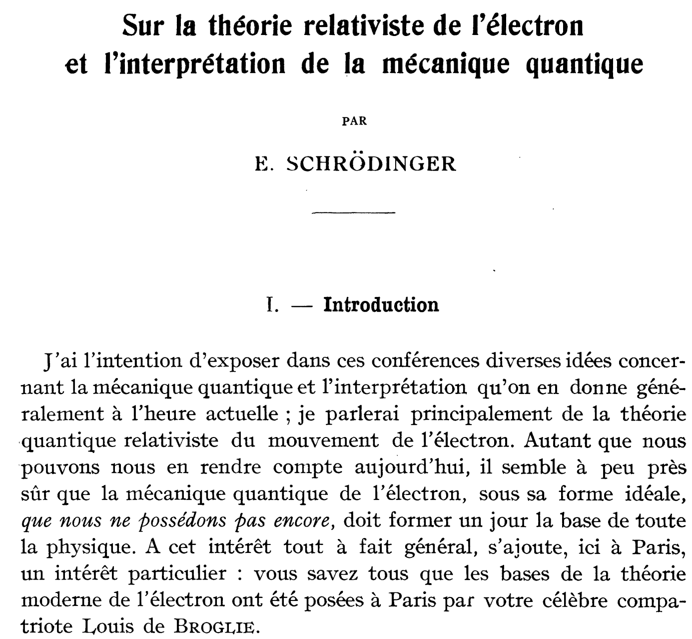
S

ÜBER DIE UMKEHRUNG DER NATURGESETZE

VON

E. SCHRÖDINGER

SONDERAUSGABE AUS DEN SITZUNGSBERICHTEN DER PREUSSISCHEN AKADEMIE DER WISSENSCHAFTEN PHYS.-MATH. KLASSE. 1931. IX



Sur la théorie relativiste de l'électron et l'interprétation de la mécanique quantique

PAR

E. SCHRÖDINGER

I. — **Introduction**

J'ai l'intention d'exposer dans ces conférences diverses idées concernant la mécanique quantique et l'interprétation qu'on en donne généralement à l'heure actuelle ; je parlerai principalement de la théorie quantique relativiste du mouvement de l'électron. Autant que nous pouvons nous en rendre compte aujourd'hui, il semble à peu près sûr que la mécanique quantique de l'électron, sous sa forme idéale, *que nous ne possédons pas encore*, doit former un jour la base de toute la physique. A cet intérêt tout à fait général, s'ajoute, ici à Paris, un intérêt particulier : vous savez tous que les bases de la théorie moderne de l'électron ont été posées à Paris par votre célèbre compatriote Louis de Broglie.

Hopf-Cole transform: $(\rho^{\text{opt}}, \psi) \mapsto (\varphi, \hat{\varphi})$

$$\varphi(x,t) = \exp\left(\frac{\psi(x,t)}{2\epsilon}\right),$$

$$\hat{\varphi}(x,t) = \rho^{\text{opt}}(x,t)\exp\left(-\frac{\psi(x,t)}{2\epsilon}\right),$$

Optimal controlled joint state PDF: $\rho^{\text{opt}}(x,t) = \hat{\varphi}(x,t)\varphi(x,t)$

Optimal control: $\boldsymbol{u}^{\text{opt}}(x,t) = 2\epsilon \boldsymbol{B}(t)^{\top}\nabla \log\varphi(x,t)$

Feedback Synthesis via the Schrödinger System

2 coupled nonlinear PDEs $\longrightarrow$ boundary-coupled linear PDEs!!

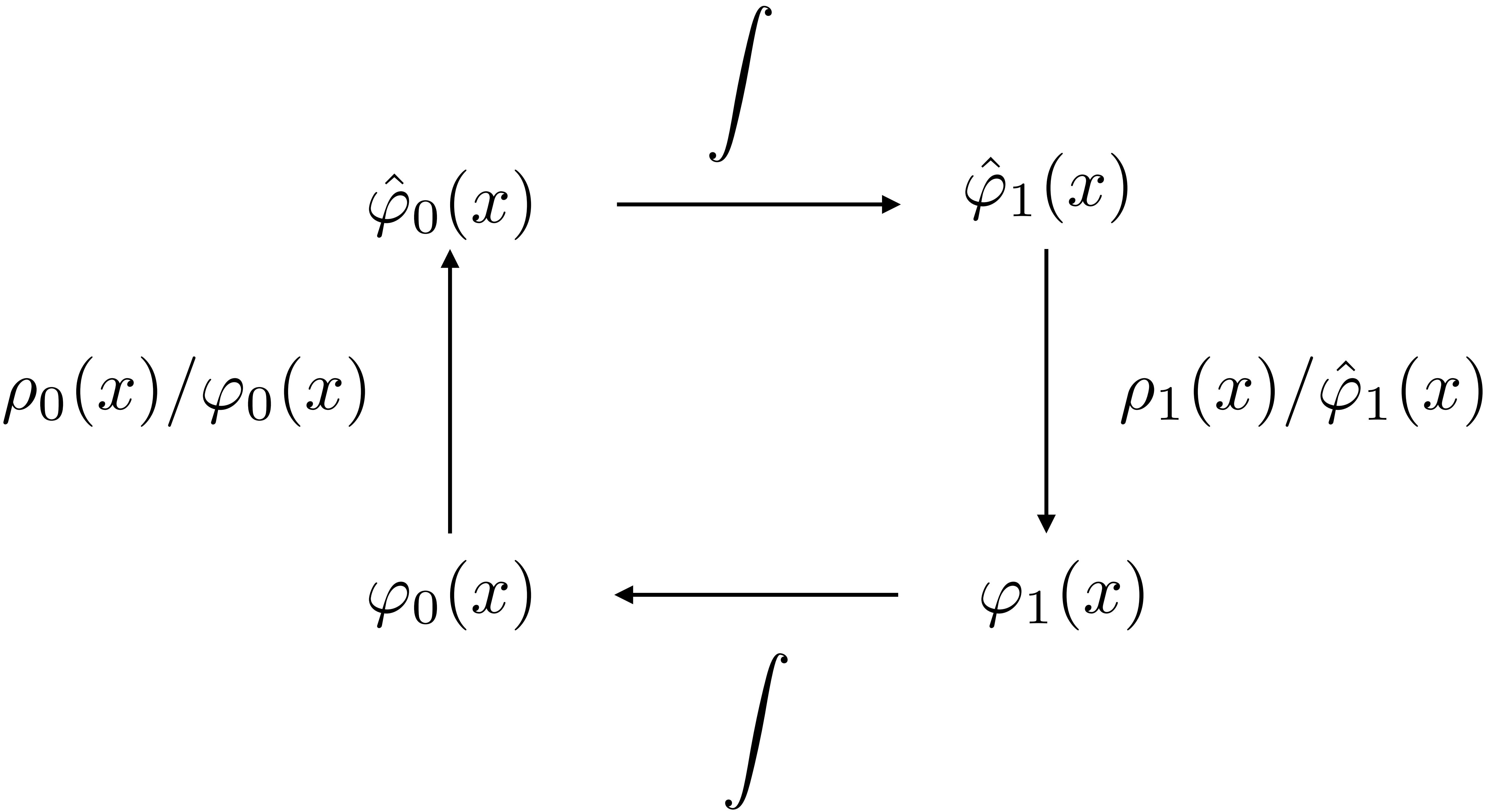
$$\underbrace{\frac{\partial \hat{\varphi}}{\partial t} = -\nabla \cdot (\hat{\varphi} \boldsymbol{f}) + \epsilon \mathbf{1}^{\top} (\boldsymbol{D}(t) \odot \mathrm{Hess}\,(\hat{\varphi}))\mathbf{1},}_{\text{forward Kolmogorov PDE}} \quad \varphi_0 \hat{\varphi}_0 = \rho_0,$$

$$\underbrace{\frac{\partial \varphi}{\partial t} = -\langle \nabla \varphi, \boldsymbol{f} \rangle - \epsilon \langle \boldsymbol{D}(t), \mathrm{Hess}\,(\varphi) \rangle,}_{\text{backward Kolmogorov PDE}} \quad \varphi_1 \hat{\varphi}_1 = \rho_1.$$

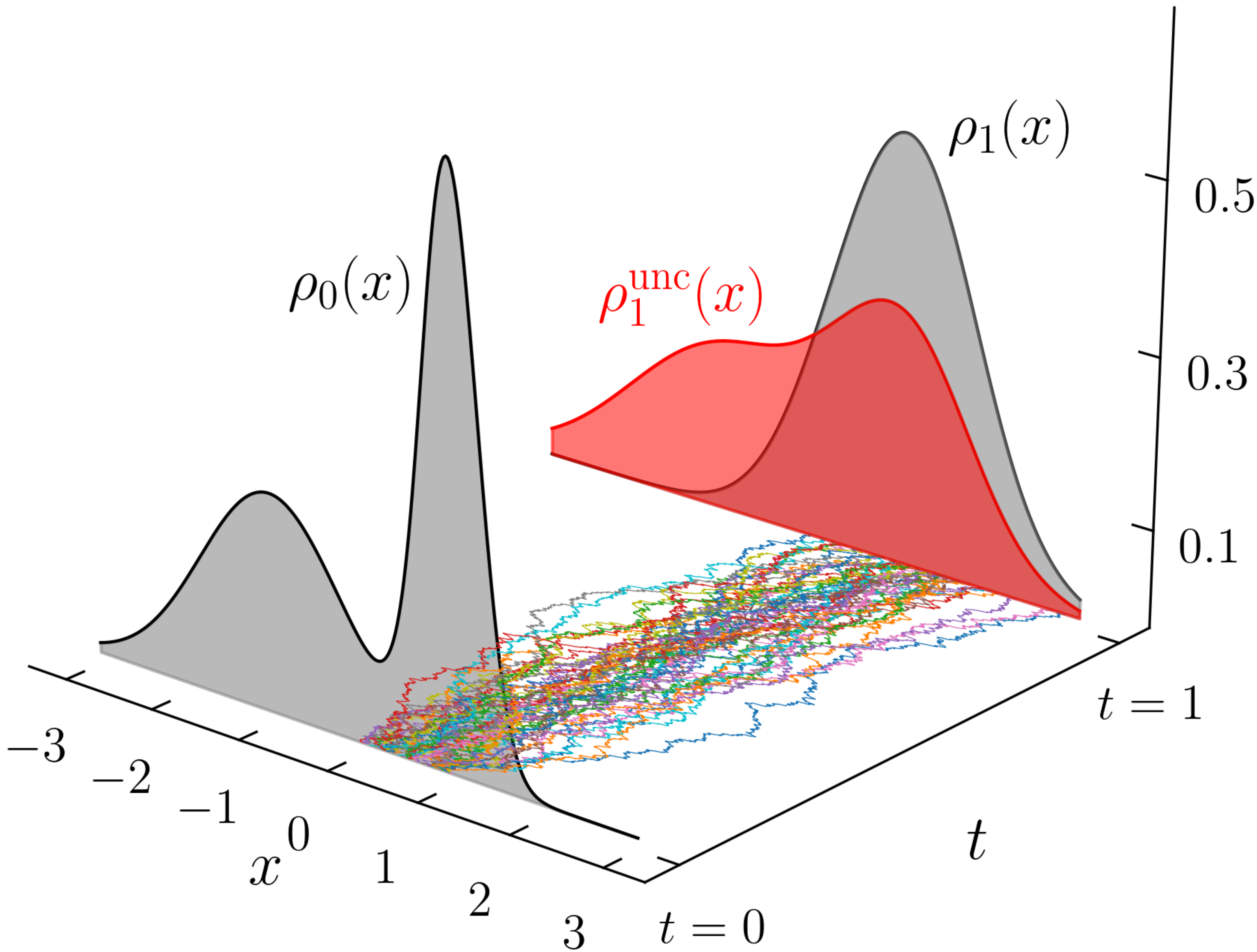
Wasserstein proximal algorithm $\longrightarrow$ fixed point recursion over $(\hat{\varphi}_0, \varphi_1)$

(Contractive in Hilbert metric)

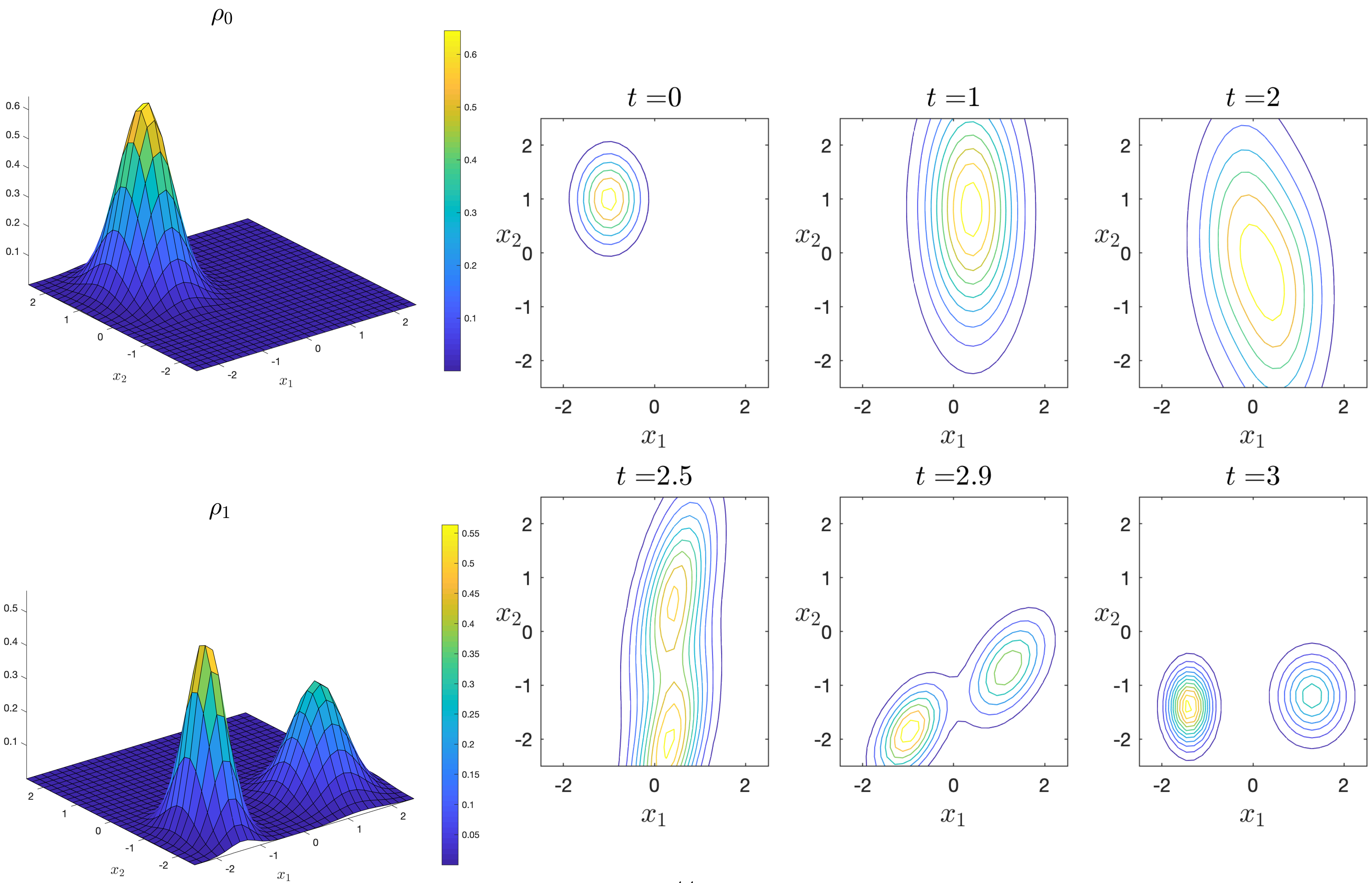
Fixed Point Recursion over $(\hat{\varphi}_0, \varphi_1)$



Feedback Density Control: Zero Prior Dynamics

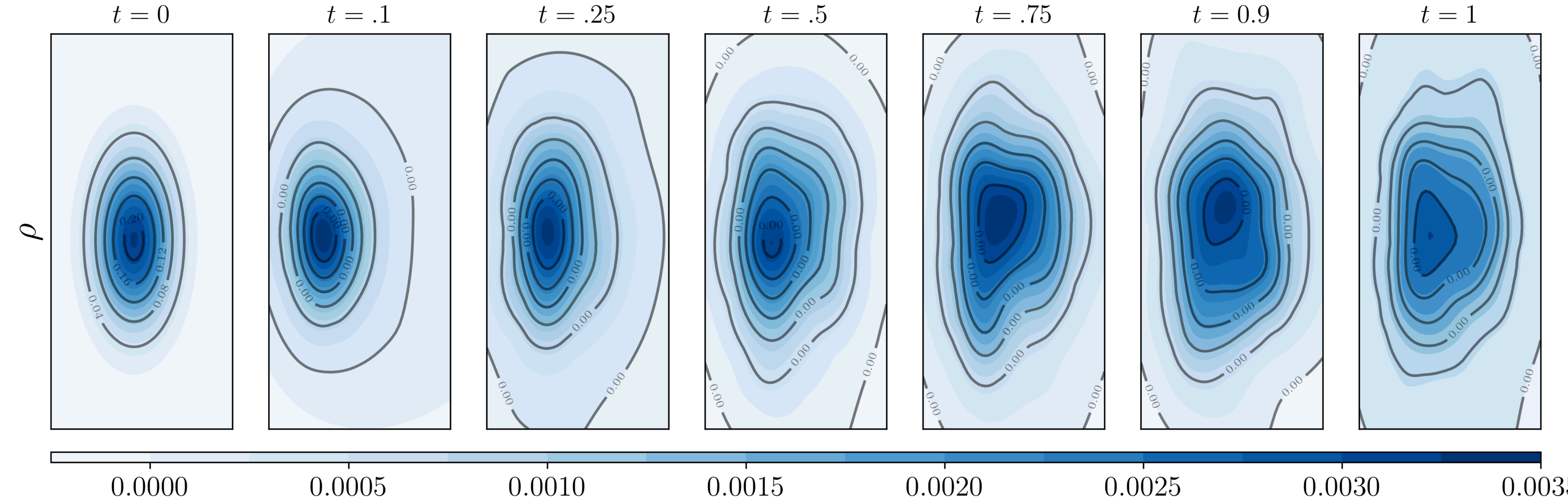


Feedback Density Control: LTI Prior Dynamics

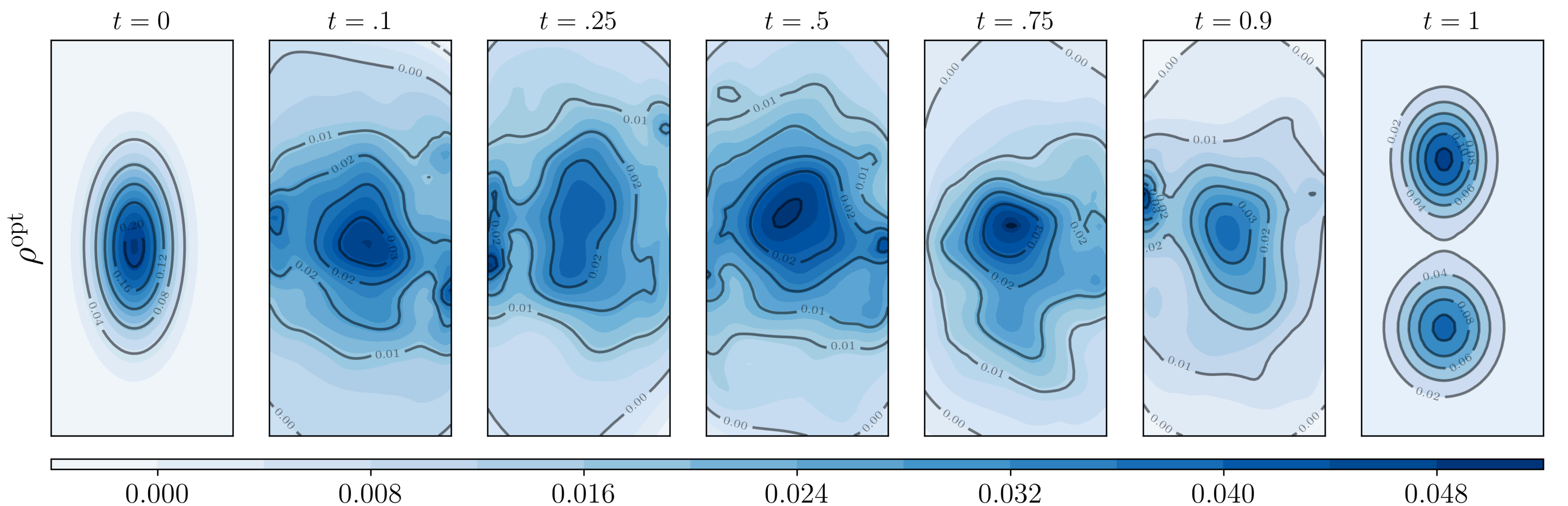


Feedback Density Control: Nonlinear Grad. Drift

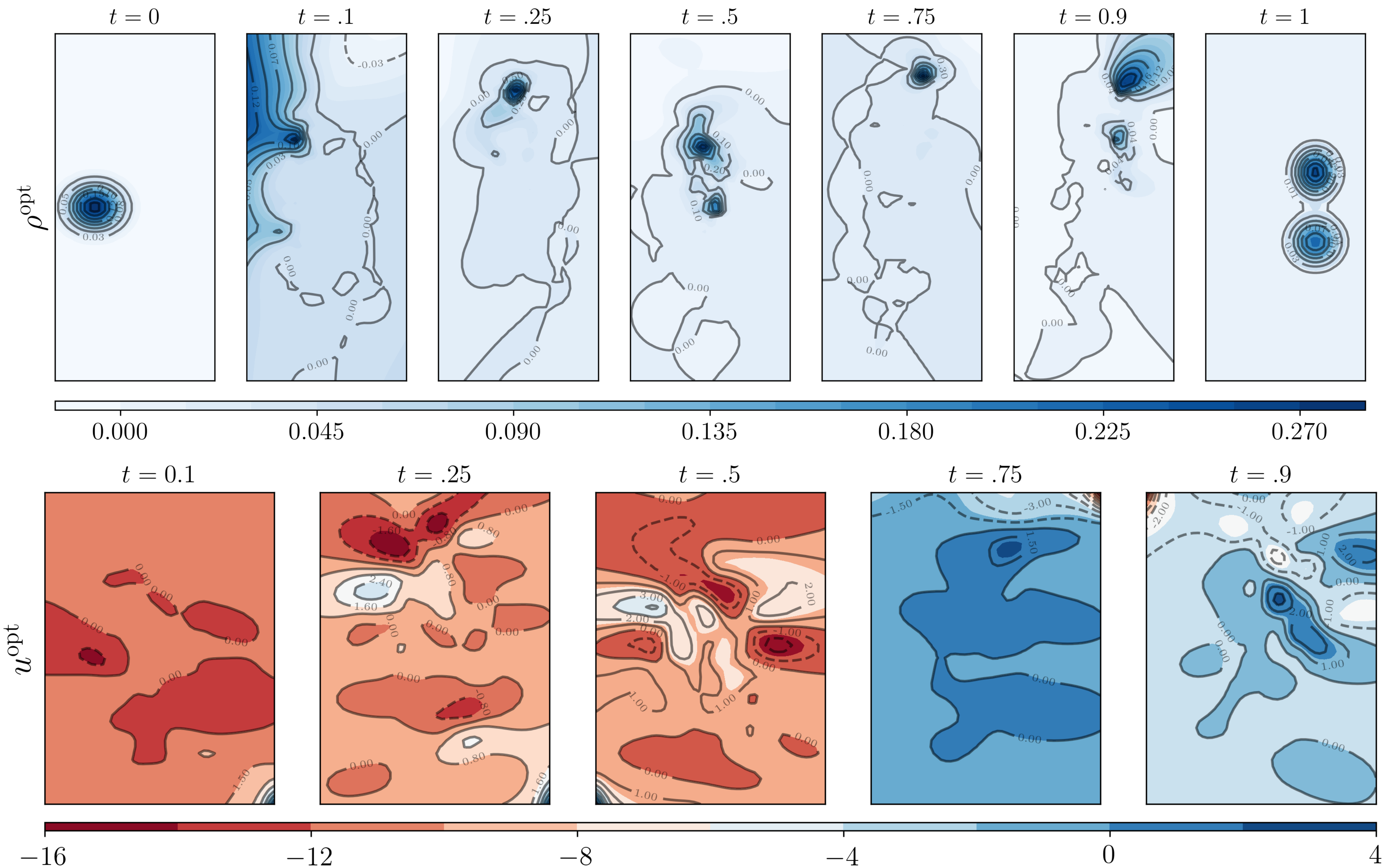
Uncontrolled joint PDF evolution:



Optimal controlled joint PDF evolution:

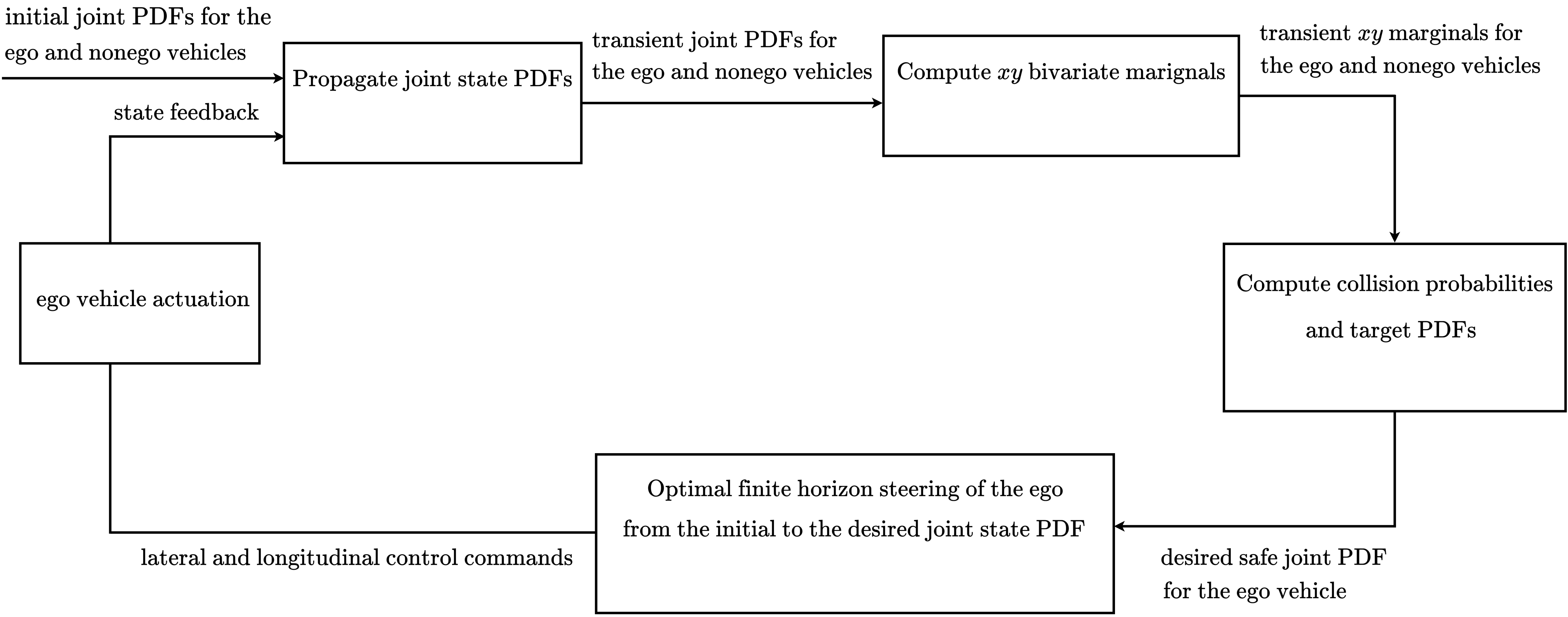


Feedback Density Control: Mixed Conservative-Dissipative Drift

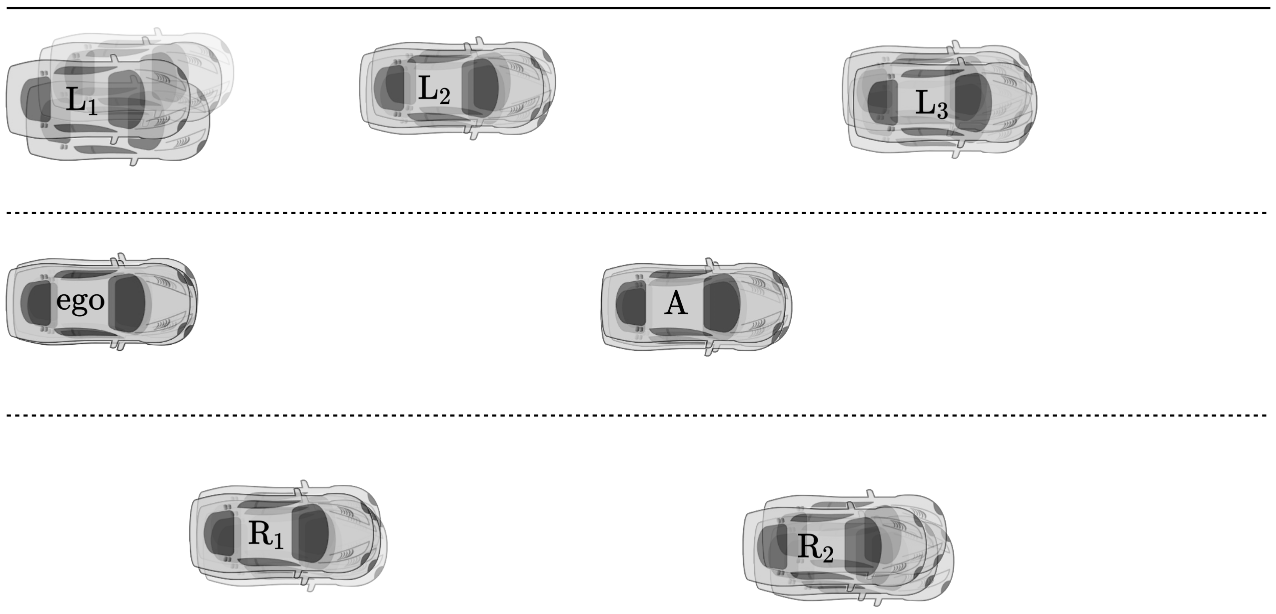


— K.F. Caluya and A.H., Wasserstein proximal algorithms for the Schrodinger bridge problem: density control with nonlinear drift, *IEEE TAC* 2021.

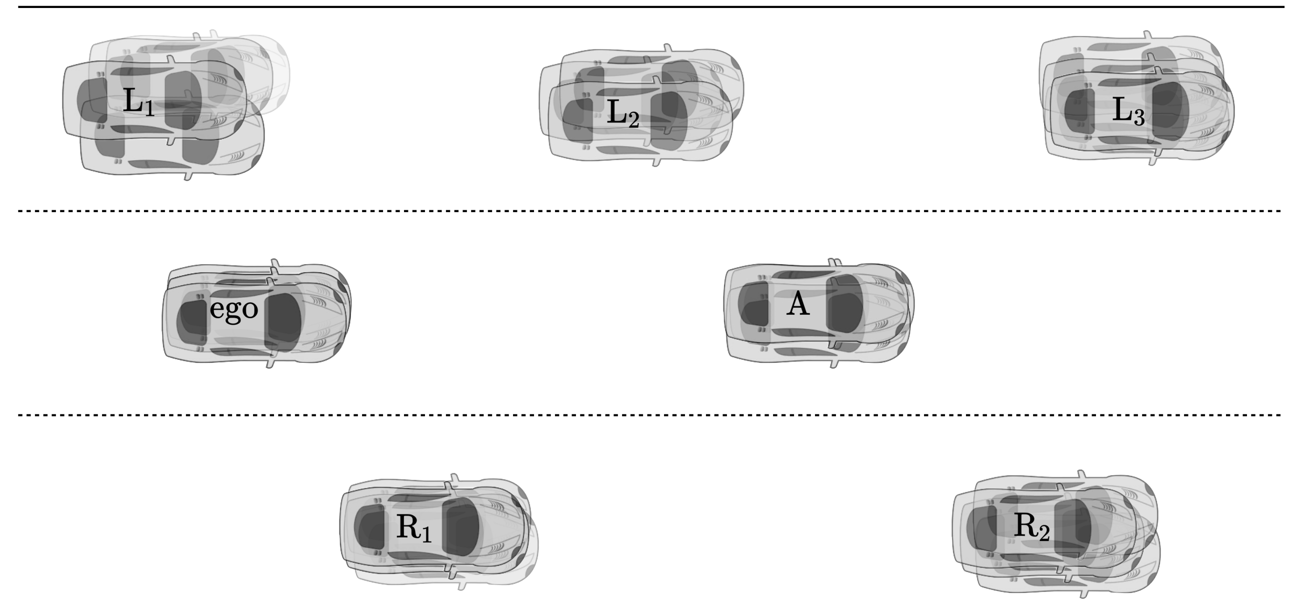
Application to Safe Automated Driving



Density Prediction for Safe Automated Driving

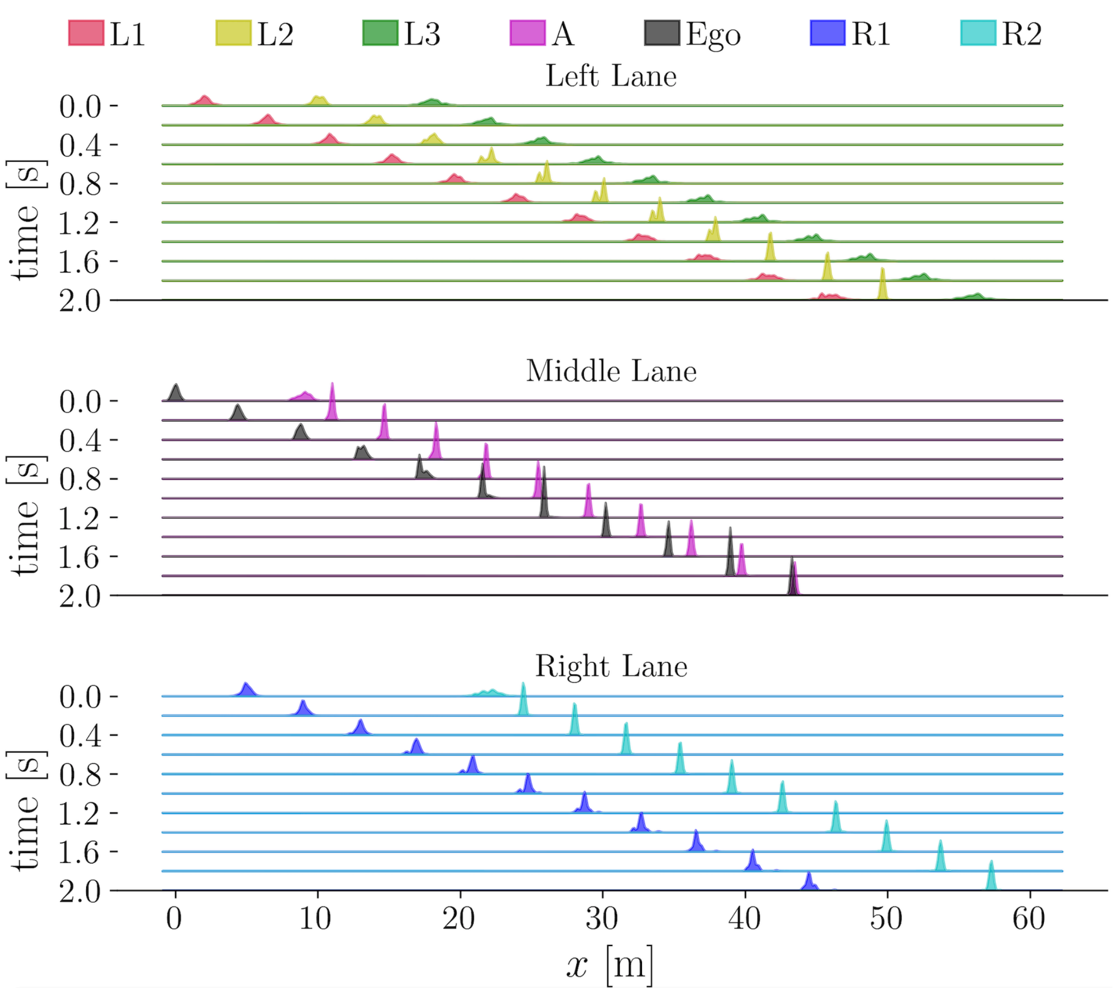


t_0

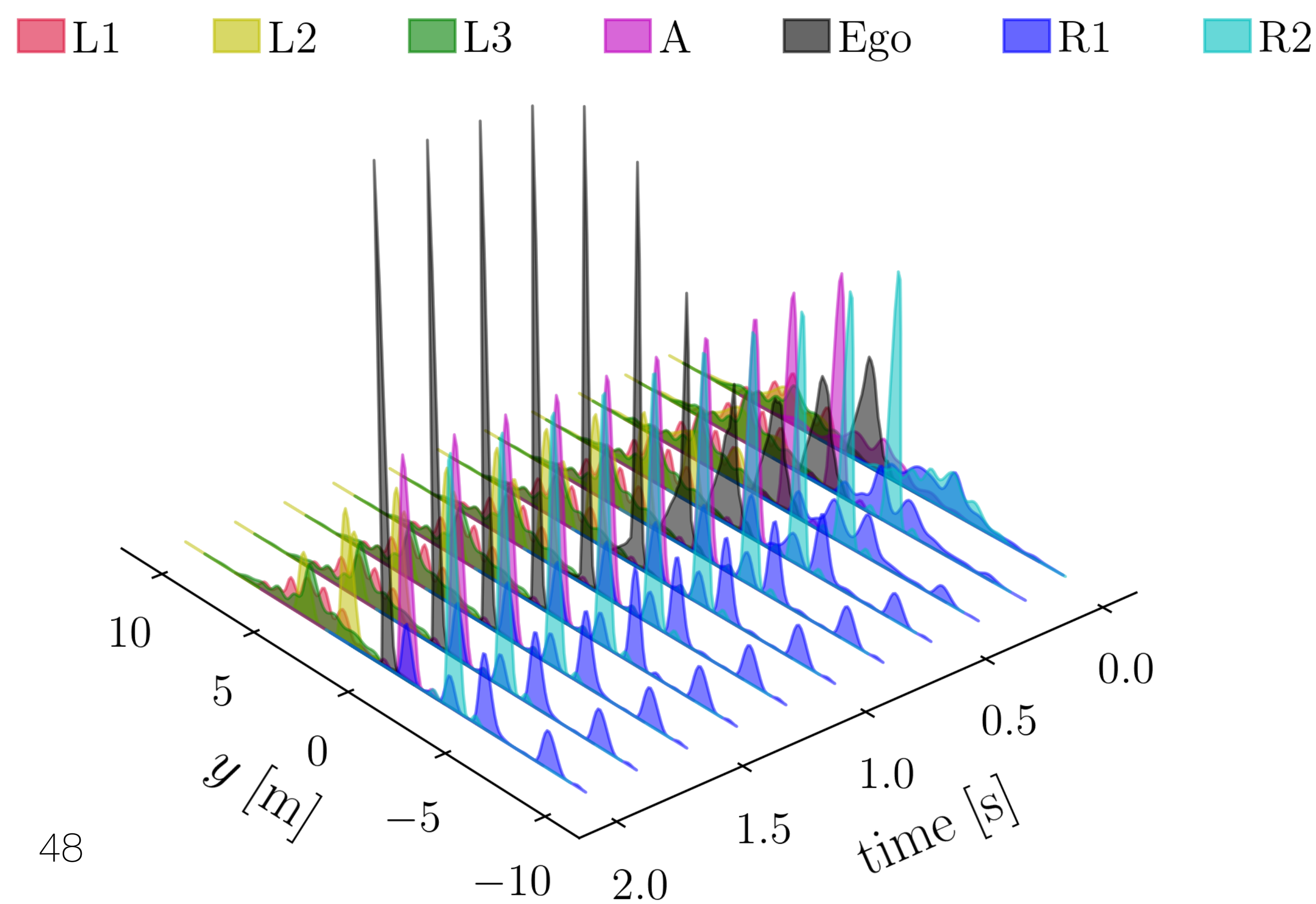


t_1

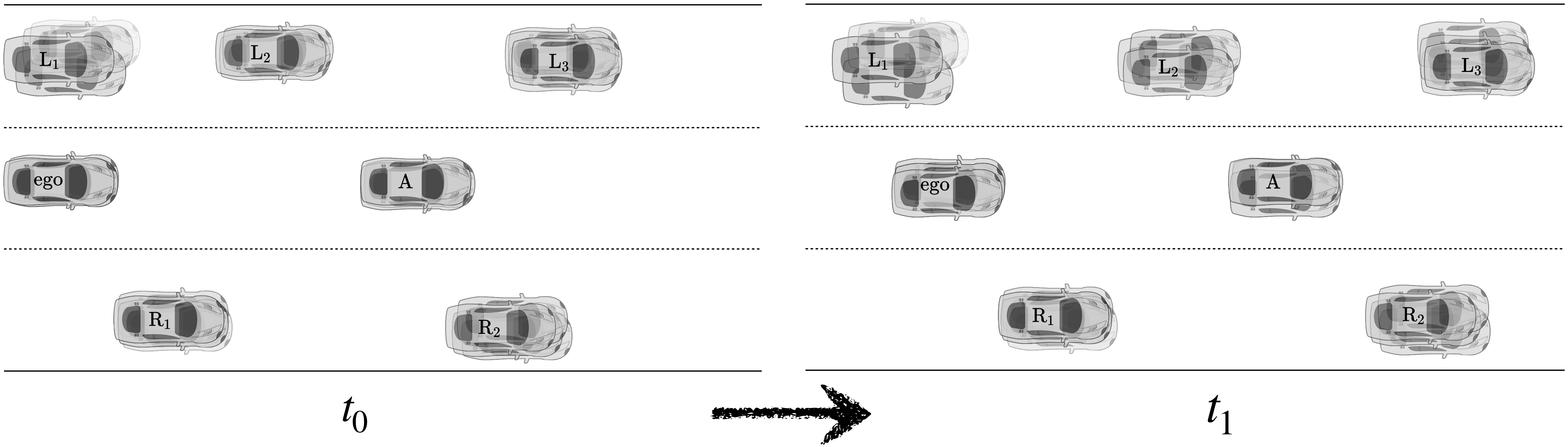
x marginals

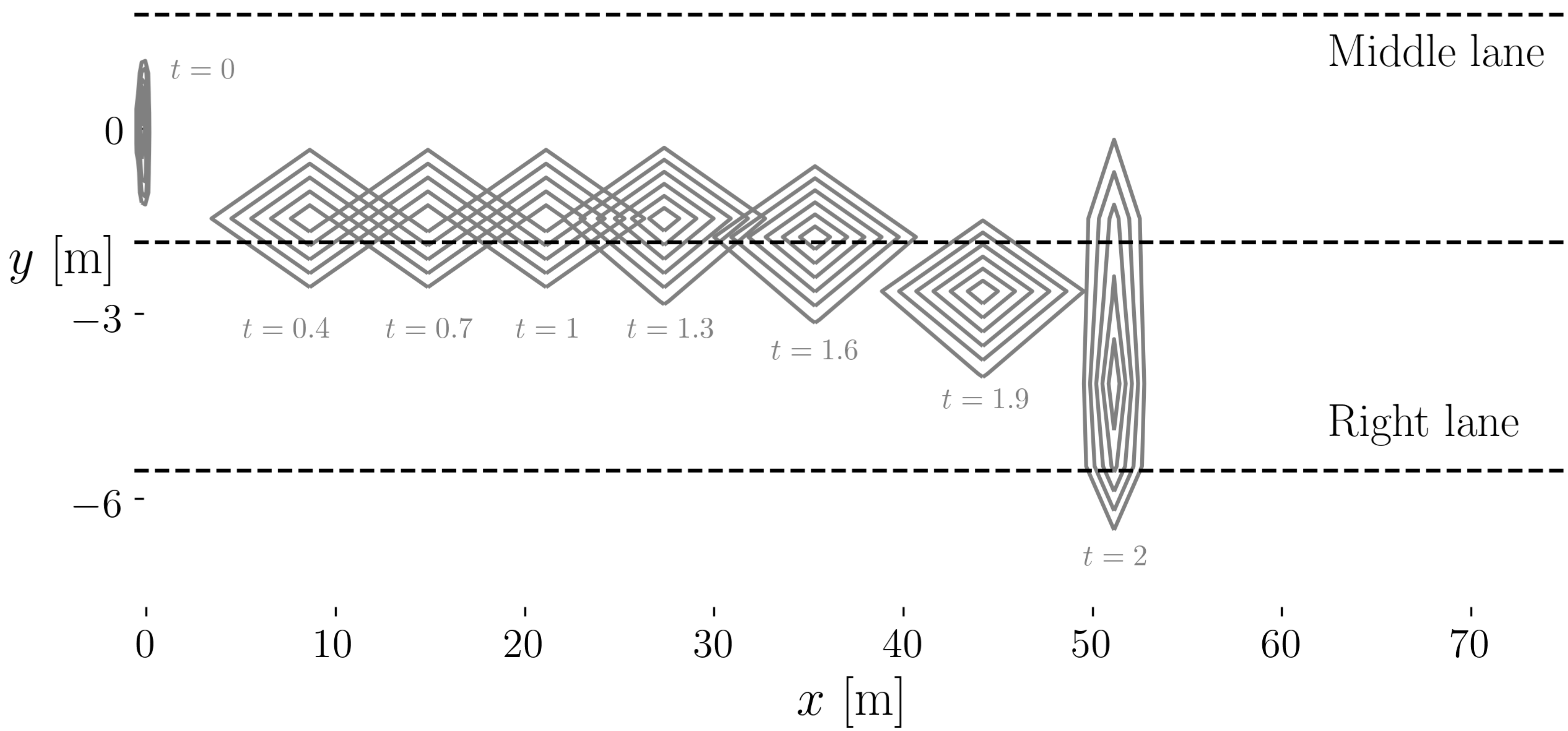


y marginals



Density Control for Safe Automated Driving





Application to Safe Automated Driving

S. Haddad, A.H., and B. Singh, Density-based stochastic reachability computation for occupancy prediction in automated driving, *IEEE Transactions on Control Systems Technology*, 2022.

S. Haddad, K.F. Caluya, A.H., and B. Singh, Prediction and optimal feedback steering of probability density functions for safe automated driving, *IEEE Control Systems Letters*, 2021.

Summary

Thank You

Support: