

# Generalized Schrödinger Bridges in Learning and Control

Abhishek Halder

Department of Aerospace Engineering, Iowa State University  
Department of Applied Mathematics, University of California Santa Cruz

Joint work with students and collaborators

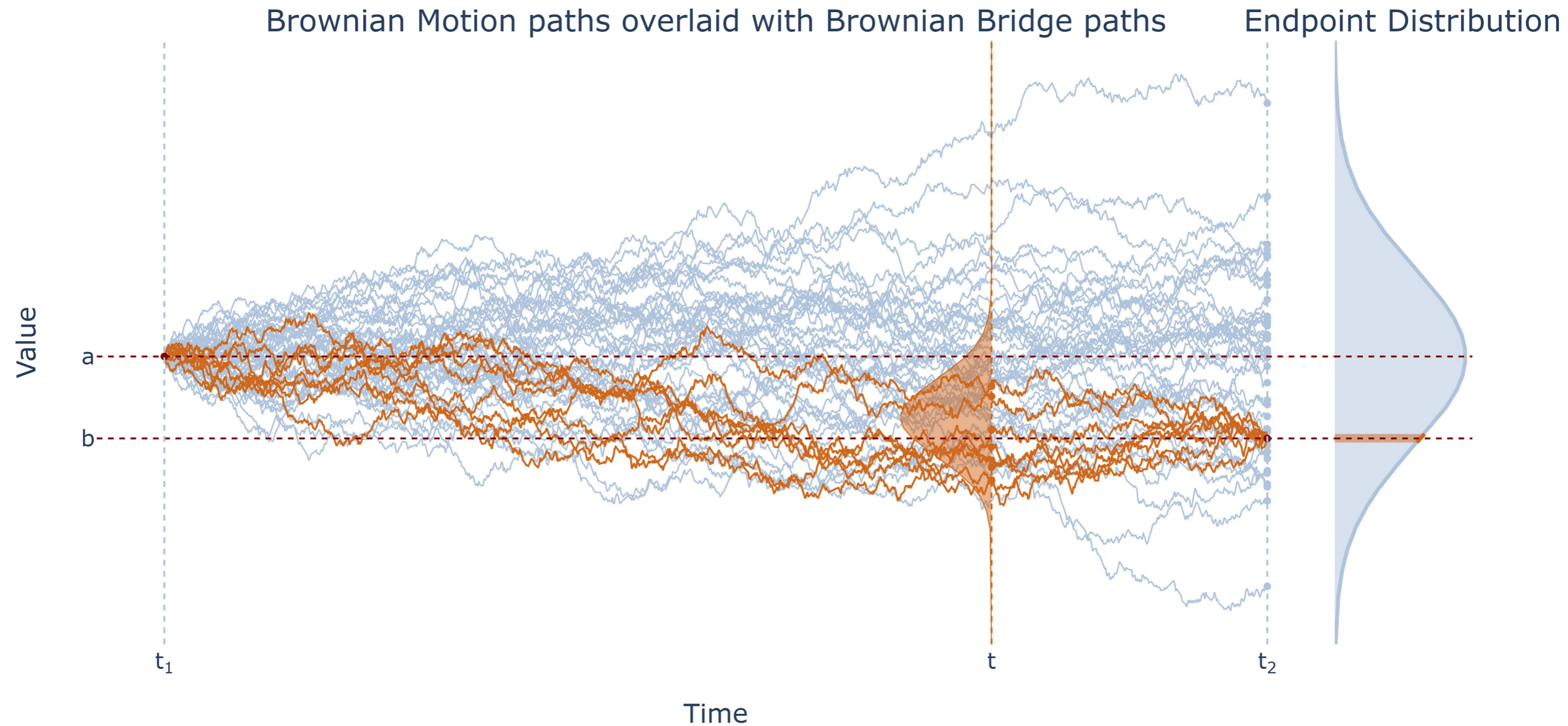


Computational and Applied Math Seminar  
Department of Mathematics, Iowa State University  
February 10, 2025



# What is a bridge

A stochastic process connecting two given states  $a, b$  in a given deadline  $[t_1, t_2]$

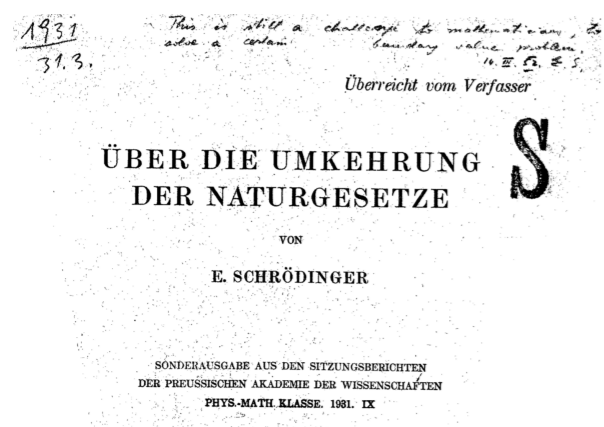


Source: <https://medium.com/@christopher.tabori/between-certainty-and-chance-tracing-the-probability-distribution-of-paths-of-brownian-bridges-b1f97eba638d>



# What is a Schrödinger bridge

Prior physics = Brownian motion



[1931]

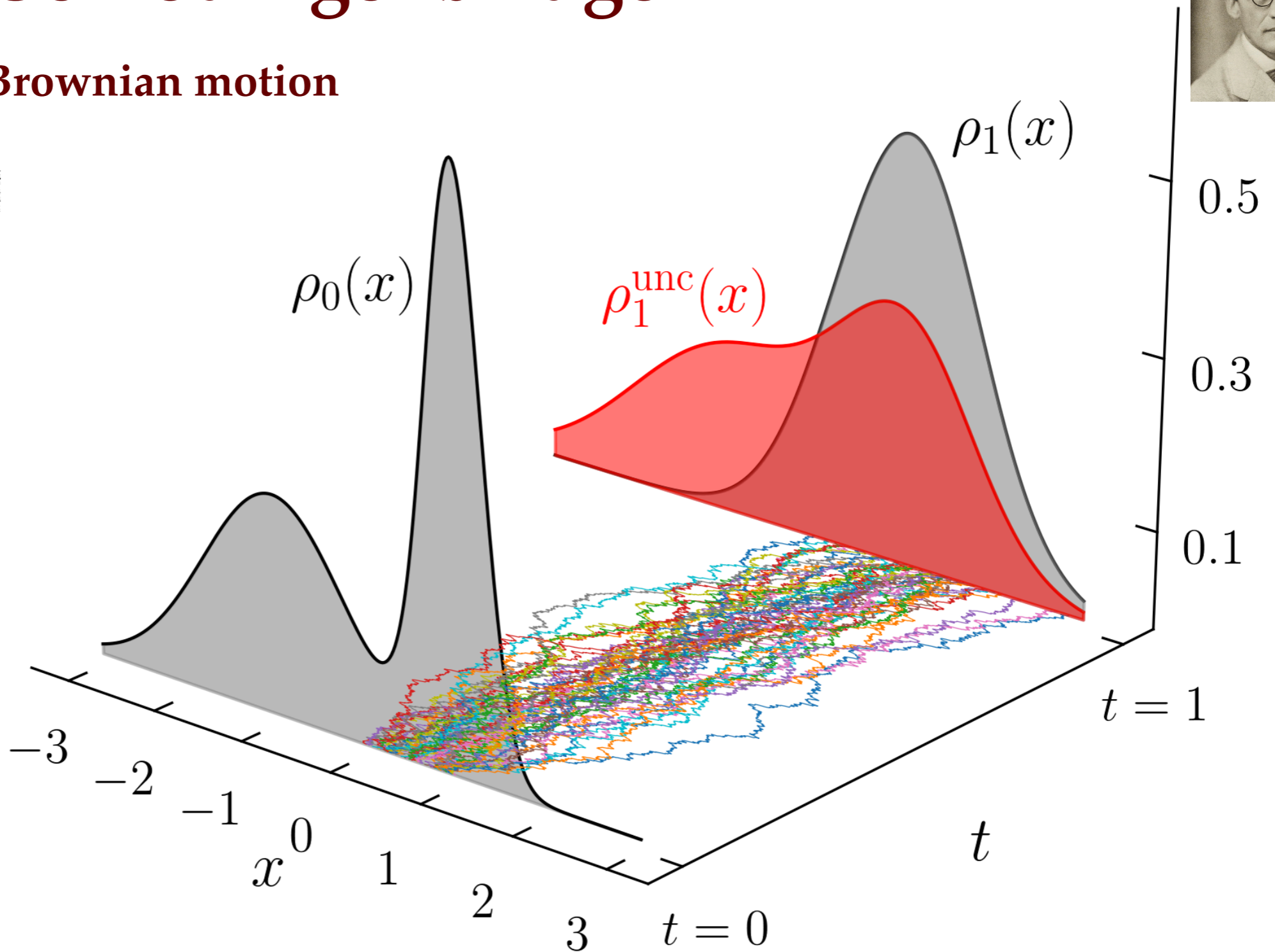
Sur la théorie relativiste de l'électron  
et l'interprétation de la mécanique quantique

PAR  
E. SCHRÖDINGER

I. — Introduction

J'ai l'intention d'exposer dans ces conférences diverses idées concernant la mécanique quantique et l'interprétation qu'on en donne généralement à l'heure actuelle ; je parlerai principalement de la théorie quantique relativiste du mouvement de l'électron. Autant que nous pouvons nous en rendre compte aujourd'hui, il semble à peu près sûr que la mécanique quantique de l'électron, sous sa forme idéale, que nous ne possédons pas encore, doit former un jour la base de toute la physique. A cet intérêt tout à fait général, s'ajoute, ici à Paris, un intérêt particulier : vous savez tous que les bases de la théorie moderne de l'électron ont été posées à Paris par votre célèbre compatriote Louis de BROGLIE.

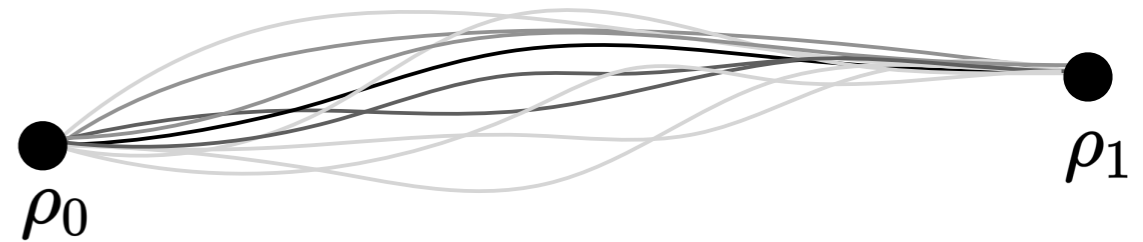
[1932]



Find the most likely explanation of observation vs prior physics mismatch

# What is a Schrödinger bridge

Path space  $\Omega := C([t_0, t_1]; \mathbb{R}^n)$



Denote the collection of all probability measures on  $\Omega$  as  $\mathcal{M}(\Omega)$

$\Pi_{01} := \{\mathbb{M} \in \mathcal{M}(\Omega) \mid \mathbb{M} \text{ has marginal } \rho_i \text{ d}\mathbf{x} \text{ at time } t_i \forall i \in \{0, 1\}, \rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^n)\}$

Schrödinger bridge =  $\arg \inf_{\mathbb{P} \in \Pi_{01}} D_{\text{KL}}(\mathbb{P} \parallel \mathbb{W})$

Generated by Itô diffusion

Wiener measure

$$d\mathbf{x} = \mathbf{u}(t, \mathbf{x})dt + d\mathbf{w}(t)$$

**Most parsimonious correction of prior physics**

Constrained maximum likelihood problem on measure-valued paths



# What is a Schrödinger bridge

Schrödinger bridge as large deviation principle: **Sanov's theorem [1957]**

$$\lim_{N \uparrow \infty} \log(\text{empirical prob}_N \text{ under } \mathbb{W} \in \Pi_{01}) \asymp - \inf_{\mathbb{P} \in \Pi_{01}} D_{\text{KL}}(\mathbb{P} \parallel \mathbb{W})$$

**KL div as rate function**

Schrödinger bridge as stochastic optimal control: **[1990s]**

$$\begin{aligned} & \underset{u \in \mathcal{U}}{\text{minimize}} \mathbb{E} \left[ \int_{t_0}^{t_1} \frac{1}{2} \|\mathbf{u}(t, \mathbf{x}_t^u)\|_2^2 dt \right] \\ & \text{subject to} \\ & d\mathbf{x}_t^u = \mathbf{u}(t, \mathbf{x}_t^u) dt + d\mathbf{w}_t \\ & \mathbf{x}_t^u(t = t_0) \sim \rho_0, \quad \mathbf{x}_t^u(t = t_1) \sim \rho_1 \end{aligned}$$

# What is a Schrödinger bridge

Schrödinger bridge as large deviation principle: **Sanov's theorem [1957]**

$$\lim_{N \uparrow \infty} \log(\text{empirical prob}_N \text{ under } \mathbb{W} \in \Pi_{01}) \asymp - \inf_{\mathbb{P} \in \Pi_{01}} D_{\text{KL}}(\mathbb{P} \parallel \mathbb{W})$$

**KL div as rate function**

Schrödinger bridge as stochastic optimal control: **[1990s]**

$$\text{minimize}_{u \in \mathcal{U}} \mathbb{E} \left[ \int_{t_0}^{t_1} \frac{1}{2} \|\mathbf{u}(t, \mathbf{x}_t^u)\|_2^2 dt \right]$$

subject to

$$d\mathbf{x}_t^u = \mathbf{u}(t, \mathbf{x}_t^u) dt + d\mathbf{w}_t$$

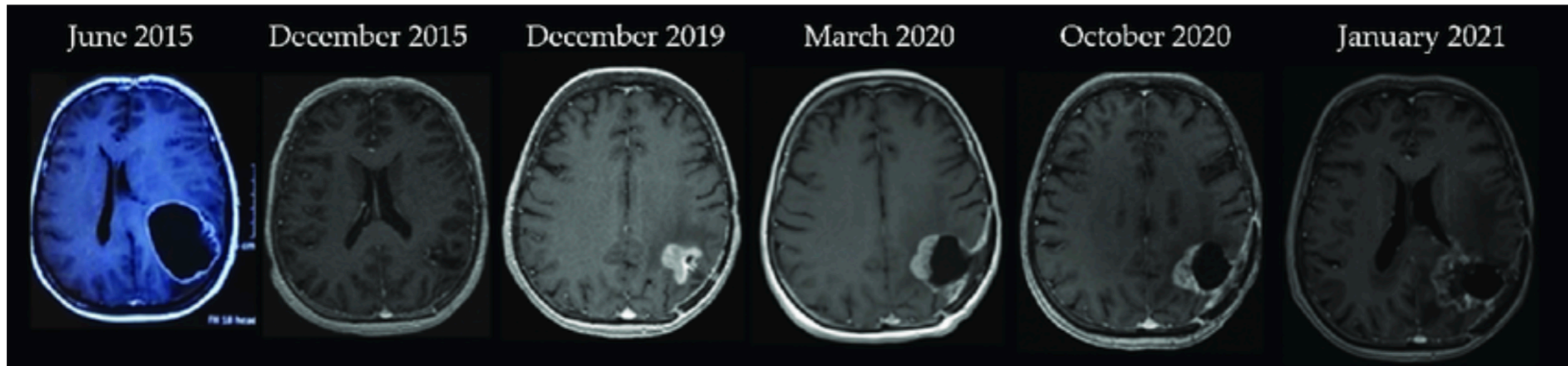
$$\mathbf{x}_t^u(t = t_0) \sim \rho_0, \quad \mathbf{x}_t^u(t = t_1) \sim \rho_1$$

**0** **Benamou-Brenier OMT [1999]**

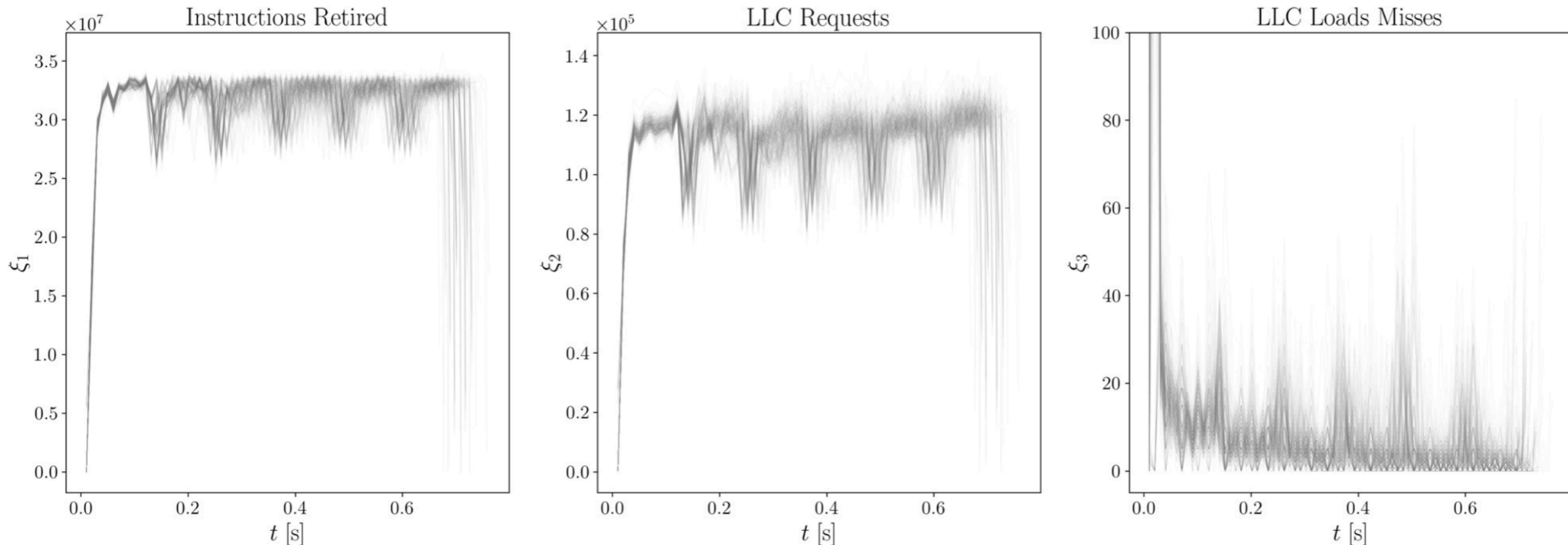


# Resurgence of Schrödinger bridge in ML/AI

Learn most likely progression of medical condition



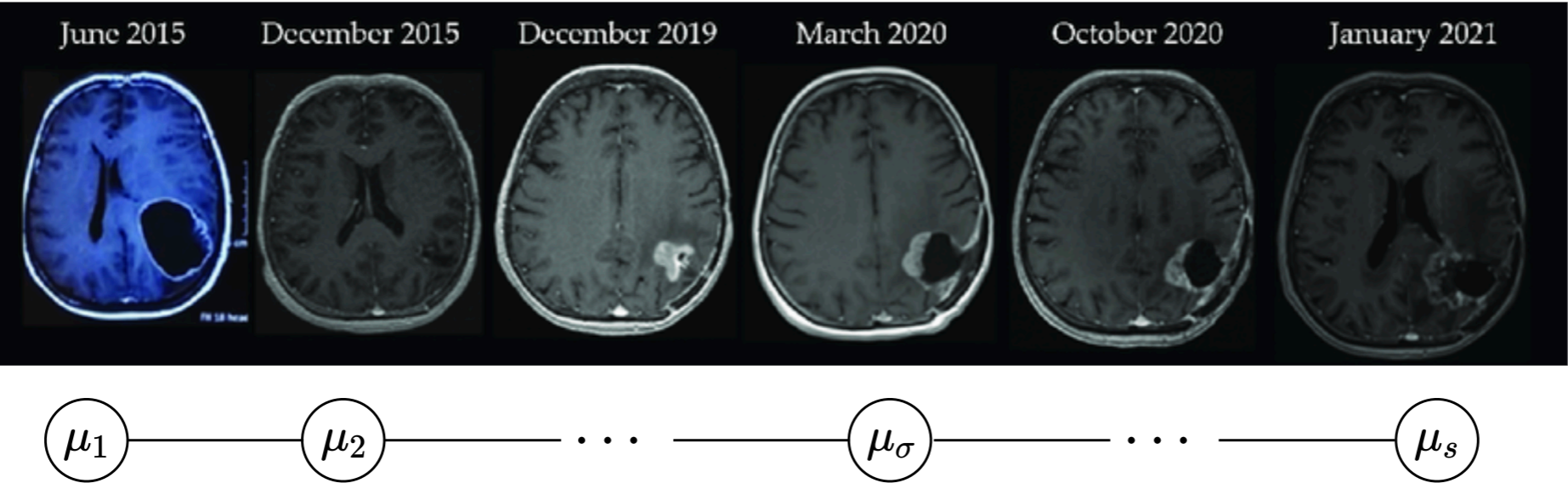
Learn joint stochastic time-varying hardware resource availability



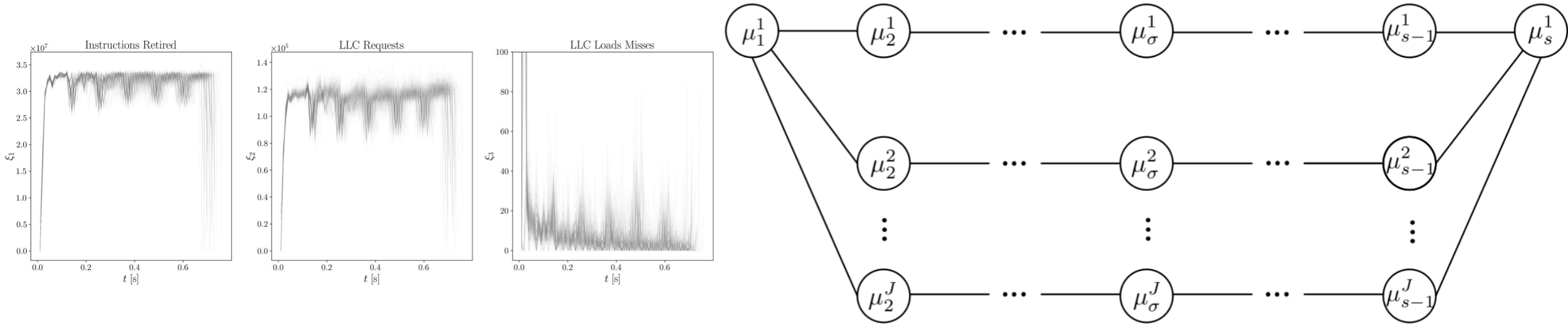
G.A. Bondar, R. Gifford, L.T.X. Phan, and A.H., ACC 2024,  
*arXiv:2310.00604*  
*arXiv:2405.12463*

# Connections with graphical models

Learn most likely progression of medical condition



Learn joint stochastic time-varying hardware resource availability



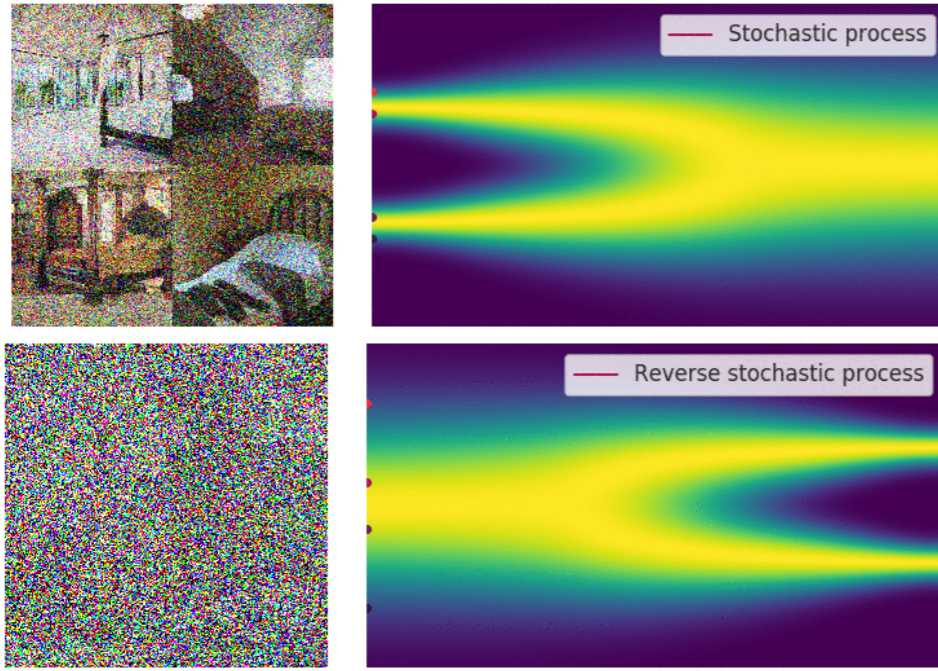
G.A. Bondar, R. Gifford, L.T.X. Phan, and A.H., ACC 2024,  
*arXiv:2310.00604*  
*arXiv:2405.12463*



# Resurgence of Schrödinger bridge in ML/AI

## Diffusion models for generative AI

Source: <https://yang-song.net/blog/2021/score/>



UAI 2023

### Aligned Diffusion Schrödinger Bridges

Vignesh Ram Somnath<sup>\*1,2</sup>  
Maria Rodriguez Martinez<sup>2</sup>

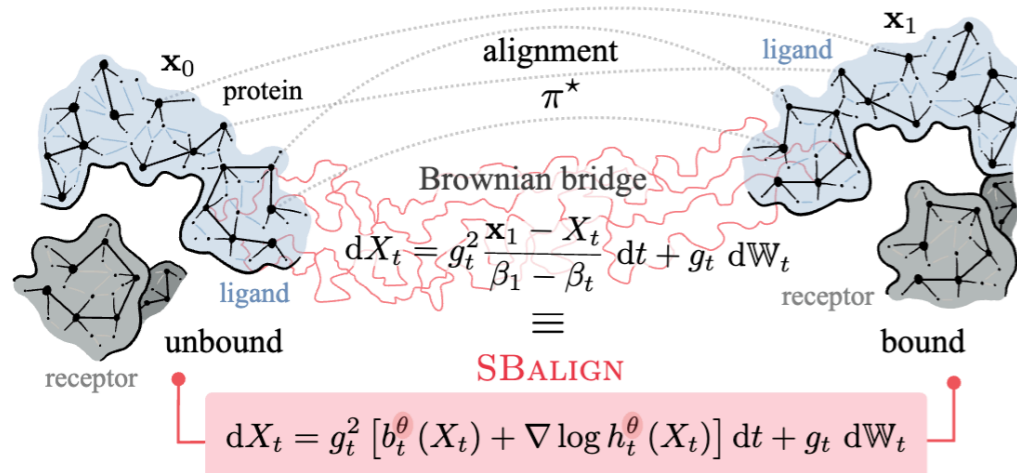
Matteo Pariset<sup>\*1,3</sup>  
Andreas Krause<sup>1</sup>

Ya-Ping Hsieh<sup>1</sup>  
Charlotte Bunne<sup>1</sup>

<sup>1</sup>Department of Computer Science, ETH Zürich

<sup>2</sup>IBM Research Zürich

<sup>3</sup>Department of Computer Science, EPFL



NeurIPS 2021

### Diffusion Schrödinger Bridge with Applications to Score-Based Generative Modeling

Valentin De Bortoli  
Department of Statistics,  
University of Oxford, UK

James Thornton  
Department of Statistics,  
University of Oxford, UK

Jeremy Heng  
ESSEC Business School,  
Singapore

Arnaud Doucet  
Department of Statistics,  
University of Oxford, UK

NeurIPS 2024

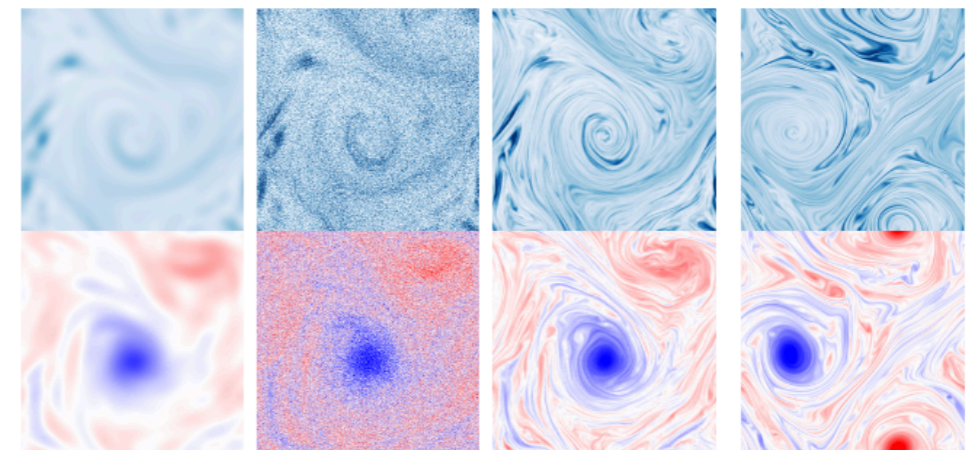
### Diffusion Schrödinger Bridge Matching

Yuyang Shi<sup>\*</sup>  
University of Oxford

Valentin De Bortoli<sup>\*</sup>  
ENS ULM

Andrew Campbell  
University of Oxford

Arnaud Doucet  
University of Oxford



Low res

High res

# **This talk: generalized Schrödinger bridges**

**# 1. general controlled dynamics**

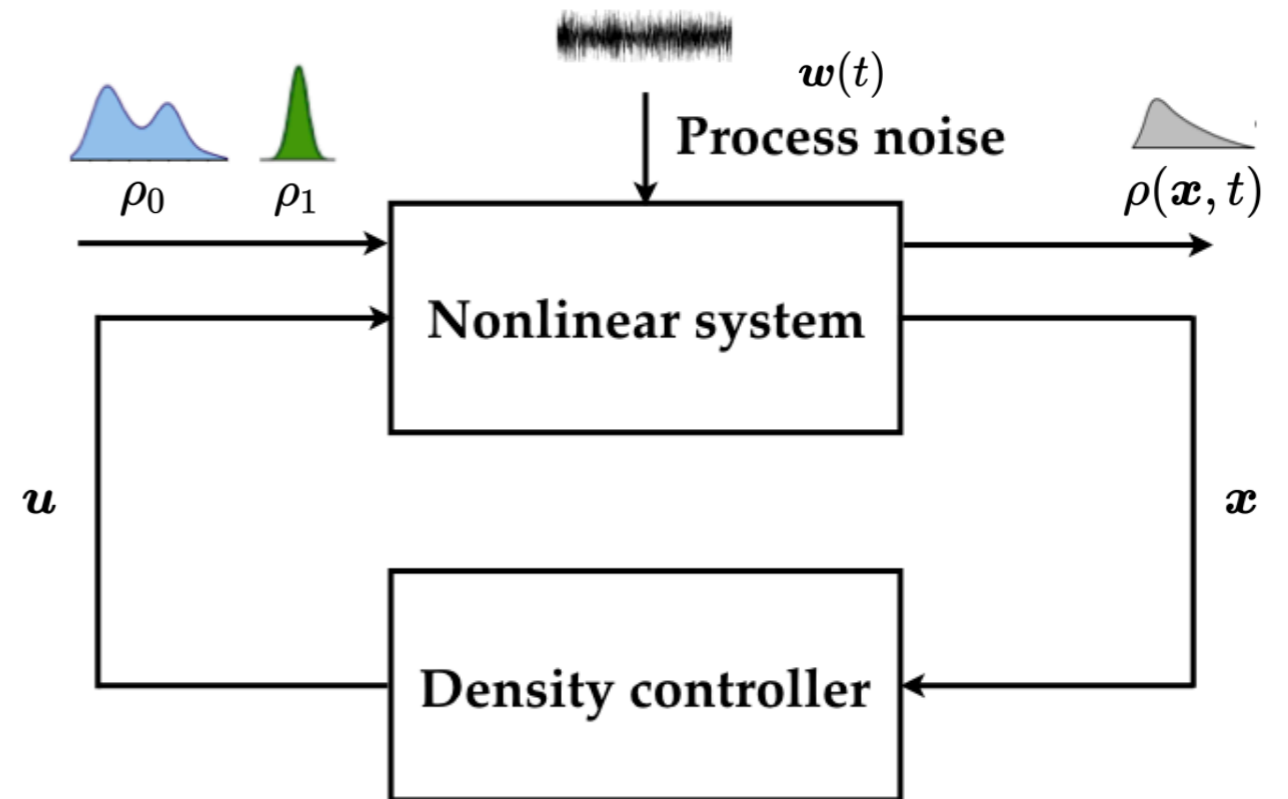
**# 2. extra sample path constraints**

**# 3. additive state cost**



# Generalization # 1: more general controlled dyn.

Steer joint state PDF via feedback control over finite time horizon



Common scenario:  $G \equiv B$

$$\text{minimize}_{u \in \mathcal{U}} \quad \mathbb{E} \left[ \int_0^1 \left( \frac{1}{2} \|u(t, x_t^u)\|_2^2 + q(t, x_t^u) \right) dt \right]$$

subject to

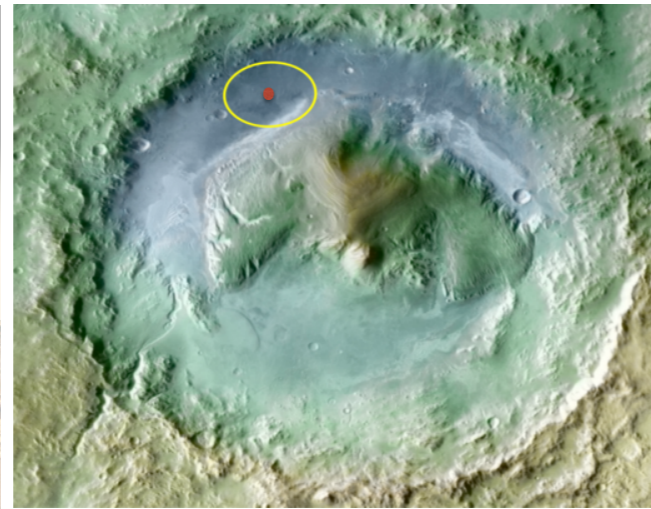
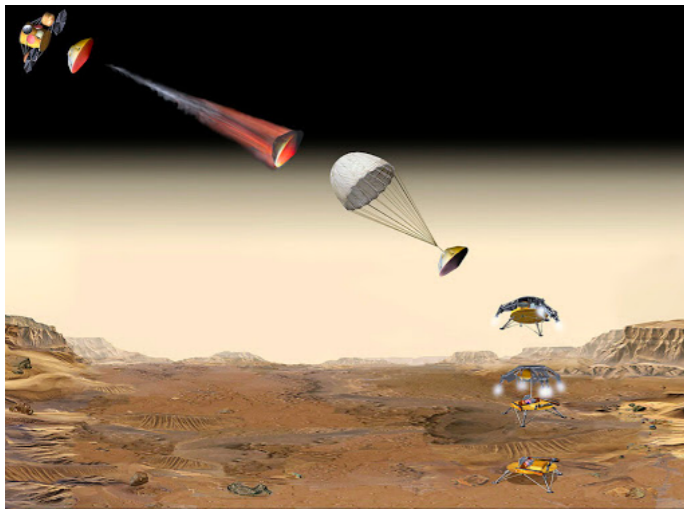
$$dx_t^u = \{f(t, x_t^u) + B(t, x_t^u)u\}dt + \sqrt{2}G(t, x_t^u)d\omega_t$$

$$x_0^u := x_t^u(t=0) \sim \rho_0, \quad x_1^u := x_t^u(t=1) \sim \rho_1$$

# Motivating applications

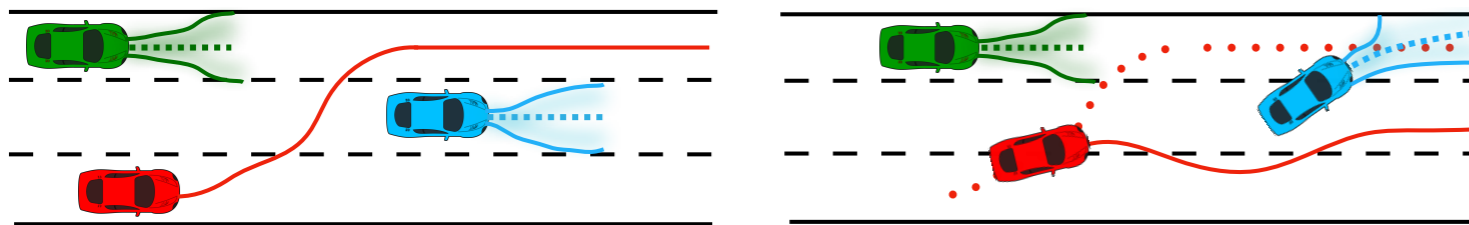
## Distribution ~ Probability

Spacecraft landing with desired statistical accuracy



Gale Crater (4.49S, 137.42E)

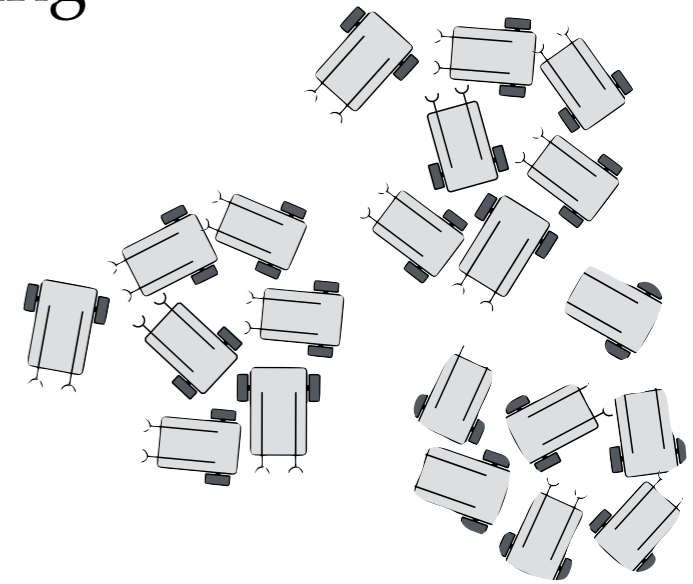
Risk management for automated driving in multi-lane highways



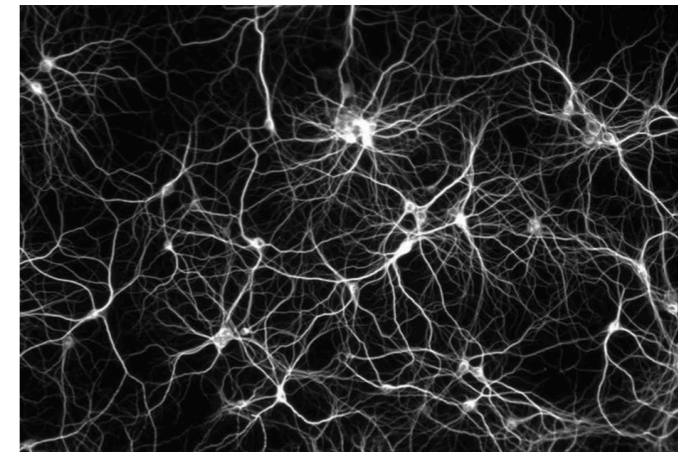
Control of uncertainties

## Distribution ~ Population

Dynamic shaping of swarms



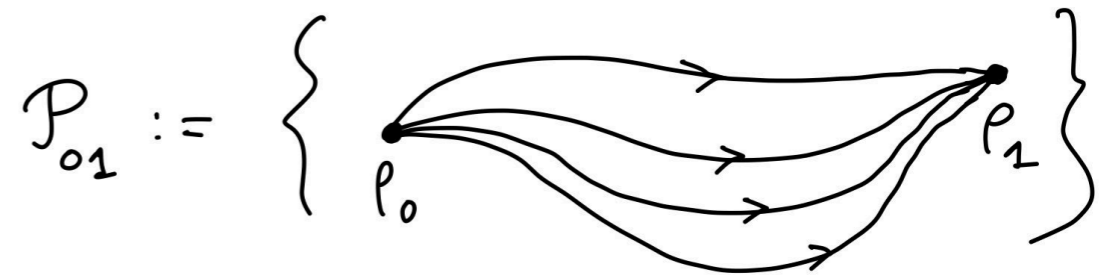
Feedback sync. and desync. of neuronal population



Control of ensemble

# Generalized Schrödinger bridge

Diffusion tensor:  $D := GG^\top$



Hessian operator w.r.t. state: Hess

$$\inf_{(\rho, \mathbf{u}) \in \mathcal{P}_{01} \times \mathcal{U}} \int_{\mathbb{R}^n} \int_0^1 \left( \frac{1}{2} \|\mathbf{u}(t, \mathbf{x}_t^u)\|_2^2 + q(t, \mathbf{x}_t^u) \right) \rho(t, \mathbf{x}_t^u) dt d\mathbf{x}_t^u$$

subject to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot ((\mathbf{f} + \mathbf{B}\mathbf{u})\rho) = \Delta_D \rho$$

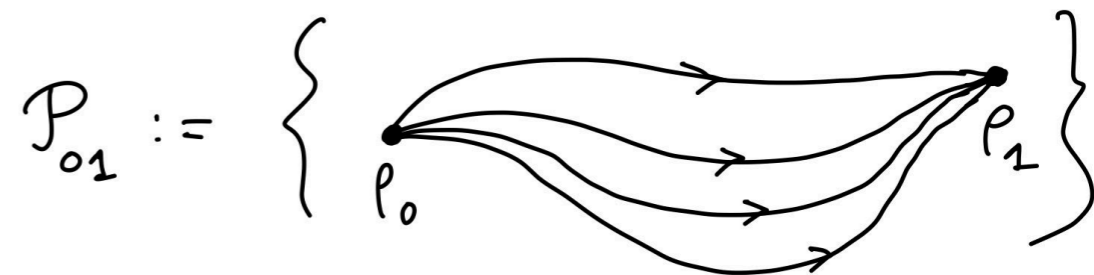
$$\rho(t=0, \mathbf{x}_0^u) = \rho_0, \quad \rho(t=1, \mathbf{x}_1^u) = \rho_1$$

Controlled Fokker-Planck or Kolmogorov's forward PDE



# Zero process noise $\rightsquigarrow$ Generalized OMT

Diffusion tensor:  $D := GG^\top$



Hessian operator w.r.t. state: Hess

$$\inf_{(\rho, \mathbf{u}) \in \mathcal{P}_{01} \times \mathcal{U}} \int_{\mathbb{R}^n} \int_0^1 \left( \frac{1}{2} \|\mathbf{u}(t, \mathbf{x}_t^u)\|_2^2 + q(t, \mathbf{x}_t^u) \right) \rho(t, \mathbf{x}_t^u) dt d\mathbf{x}_t^u$$

subject to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot ((\mathbf{f} + \mathbf{B}\mathbf{u})\rho) = \Delta_D \rho$$

$$\rho(t=0, \mathbf{x}_0^u) = \rho_0, \quad \rho(t=1, \mathbf{x}_1^u) = \rho_1$$

Controlled Liouville PDE

# Necessary Conditions of Optimality (Assuming $G \equiv B$ )

**Coupled nonlinear PDEs + linear boundary conditions**

**Controlled Fokker-Planck or Kolmogorov's forward PDE**

$$\frac{\partial \rho^{\text{opt}}}{\partial t} + \nabla \cdot ((f + D\nabla\psi) \rho^{\text{opt}}) = \Delta_D \rho^{\text{opt}}$$

**Hamilton-Jacobi-Bellman-like PDE**

$$\frac{\partial \psi}{\partial t} + \langle \nabla \psi, f \rangle + \langle D, \text{Hess}(\psi) \rangle + \frac{1}{2} \langle \nabla \psi, D \nabla \psi \rangle = q$$

**Boundary conditions:**

$$\rho^{\text{opt}}(\cdot, t = 0) = \rho_0, \quad \rho^{\text{opt}}(\cdot, t = 1) = \rho_1$$

**Optimal control:**  $u^{\text{opt}} = B^\top \nabla \psi$

# Feedback synthesis via the Schrödinger factors

Hopf-Cole a.k.a. Fleming's logarithmic transform:

$$(\rho^{\text{opt}}, \psi) \mapsto (\hat{\varphi}, \varphi) \text{ — Schrödinger factors} \quad \hat{\varphi}(\mathbf{x}, t) = \rho^{\text{opt}}(\mathbf{x}, t) \exp(-\psi(\mathbf{x}, t))$$

$$\varphi(\mathbf{x}, t) = \exp(\psi(\mathbf{x}, t))$$

2 coupled nonlinear PDEs  $\rightarrow$  boundary-coupled linear PDEs!!

Uncontrolled forward-backward advection-reaction-diffusion PDEs:

$$\begin{aligned} \frac{\partial \hat{\varphi}}{\partial t} &= \boxed{-\nabla \cdot (\hat{\varphi} \mathbf{f}) + \Delta_D \hat{\varphi} - q \hat{\varphi},} & \hat{\varphi}_0 \varphi_0 &= \rho_0 \\ \frac{\partial \varphi}{\partial t} &= \boxed{-\langle \nabla \varphi, \mathbf{f} \rangle - \Delta_D \hat{\varphi} + q \hat{\varphi},} & \hat{\varphi}_1 \varphi_1 &= \rho_1 \end{aligned}$$

Optimal controlled joint state PDF:  $\rho^{\text{opt}}(\mathbf{x}, t) = \hat{\varphi}(\mathbf{x}, t) \varphi(\mathbf{x}, t)$

Optimal control:  $\mathbf{u}^{\text{opt}}(\mathbf{x}, t) = 2\mathbf{B}^\top \nabla_x \log \varphi(\mathbf{x}, t)$



# What exactly are Schrödinger factors?

Consider Schrödinger's original case:  $f = \mathbf{0}, B = D = I$

**Classical:**  $\rho^{\text{opt}}(\mathbf{x}, t) = \varphi(\mathbf{x}, t)\hat{\varphi}(\mathbf{x}, t)$

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta - q\right)\varphi = 0 \quad [\text{Backward reaction-diffusion PDE}]$$

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta + q\right)\hat{\varphi} = 0 \quad [\text{Forward reaction-diffusion PDE}]$$

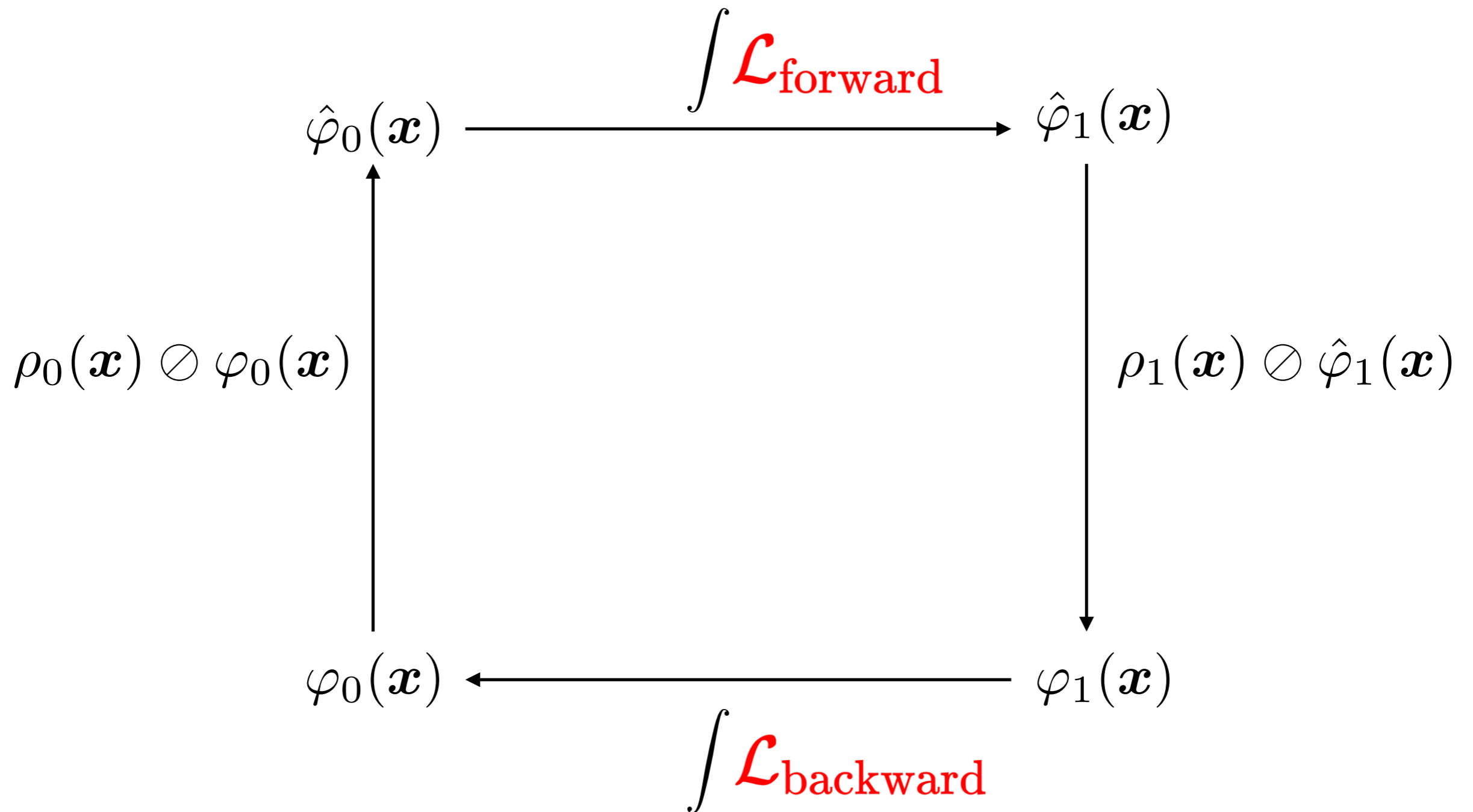
**Quantum:**  $\rho^{\text{opt}}(\mathbf{x}, t) = \Psi(\mathbf{x}, t)\hat{\Psi}(\mathbf{x}, t)$  [Born's relation]

wave function

$$\left(\sqrt{-1}\frac{\partial}{\partial t} + \frac{1}{2}\Delta - q\right)\Psi = 0 \quad [\text{Schrödinger PDE}]$$

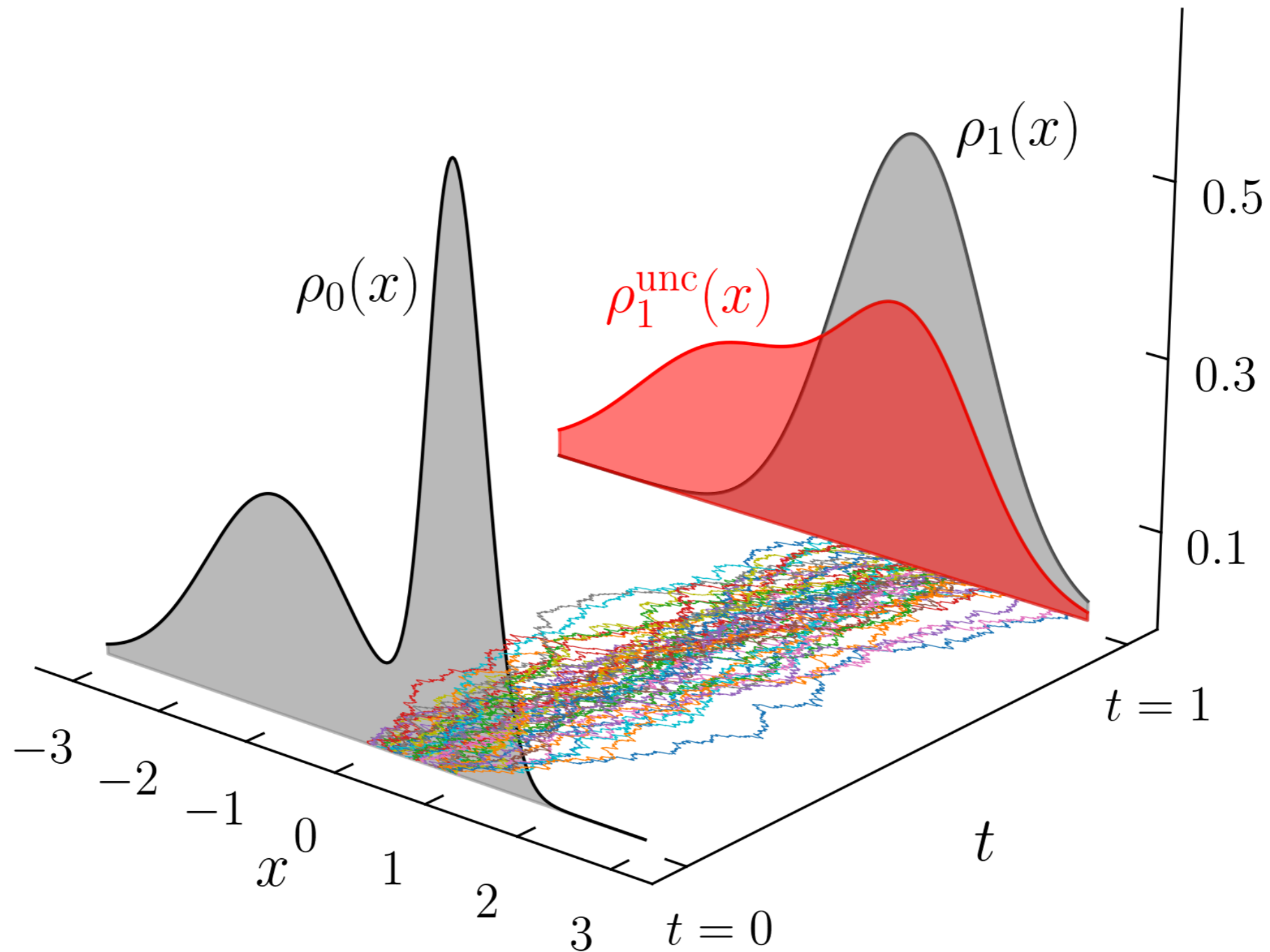
$$\left(-\sqrt{-1}\frac{\partial}{\partial t} - \frac{1}{2}\Delta + q\right)\hat{\Psi} = 0 \quad [\text{Adjoint Schrödinger PDE}]$$

# Fixed Point Recursion Over Pair $(\varphi_1, \hat{\varphi}_0)$



**This recursion is contractive in the Hilbert's projective metric!!**

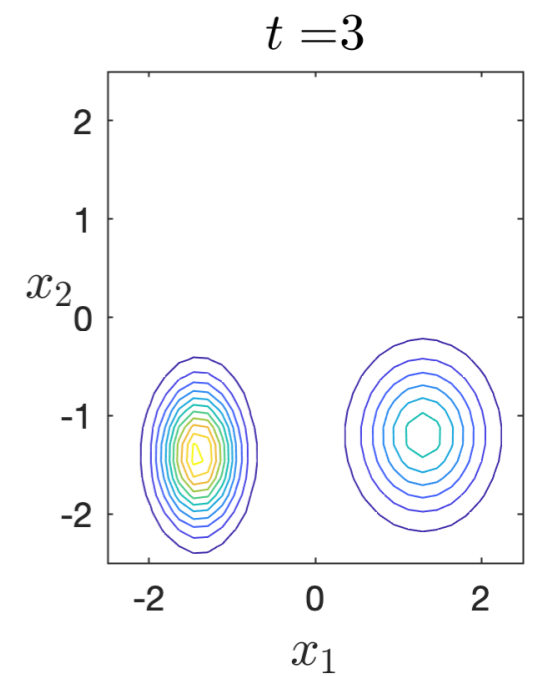
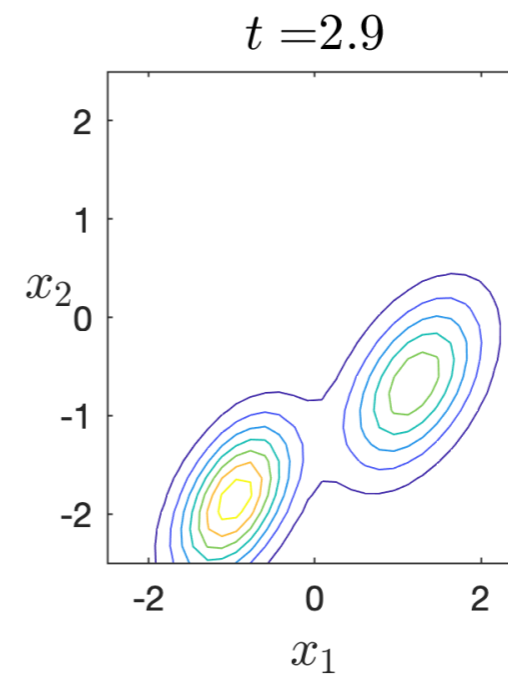
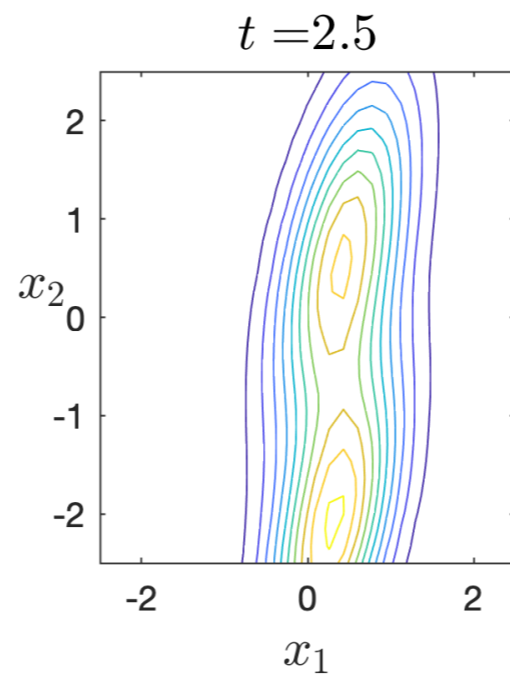
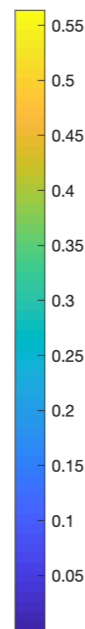
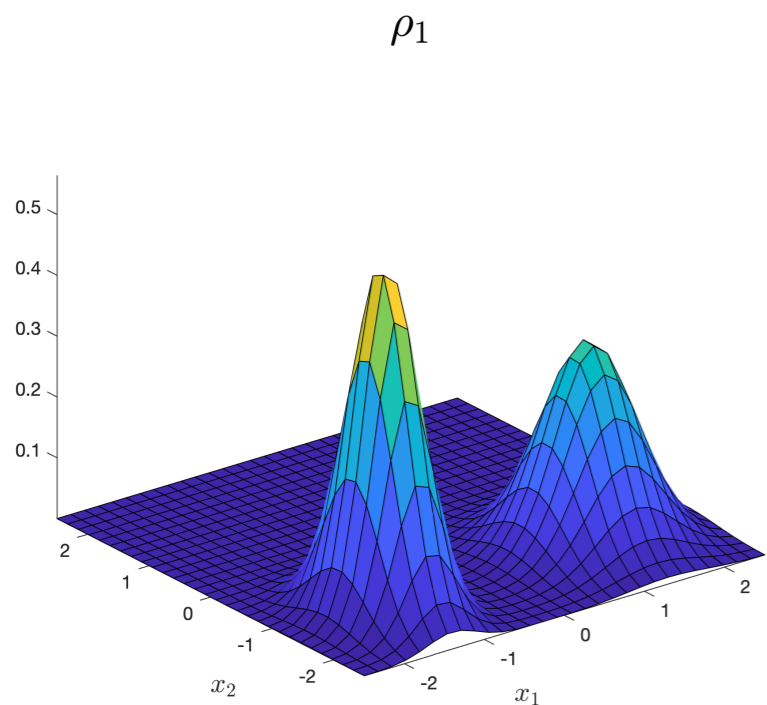
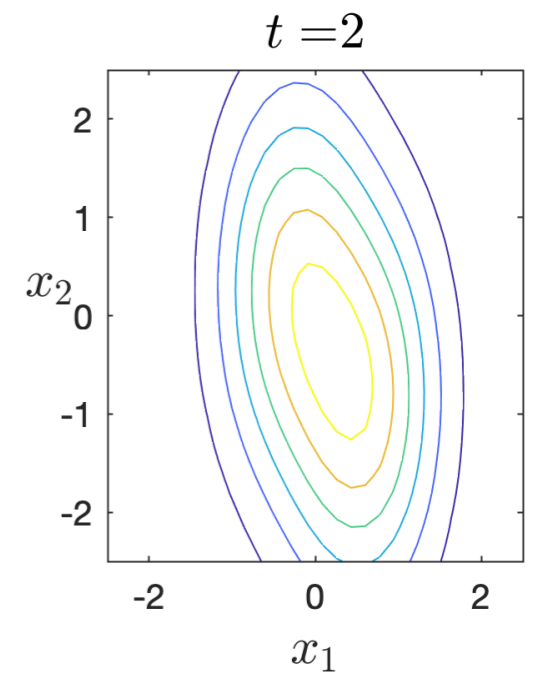
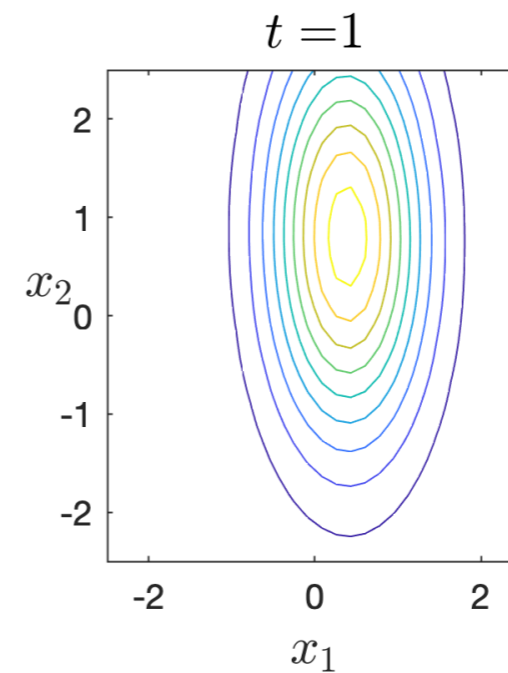
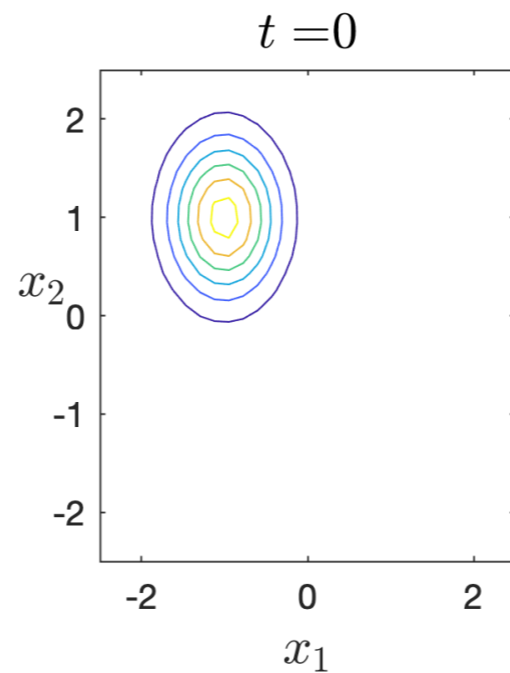
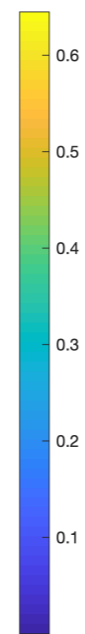
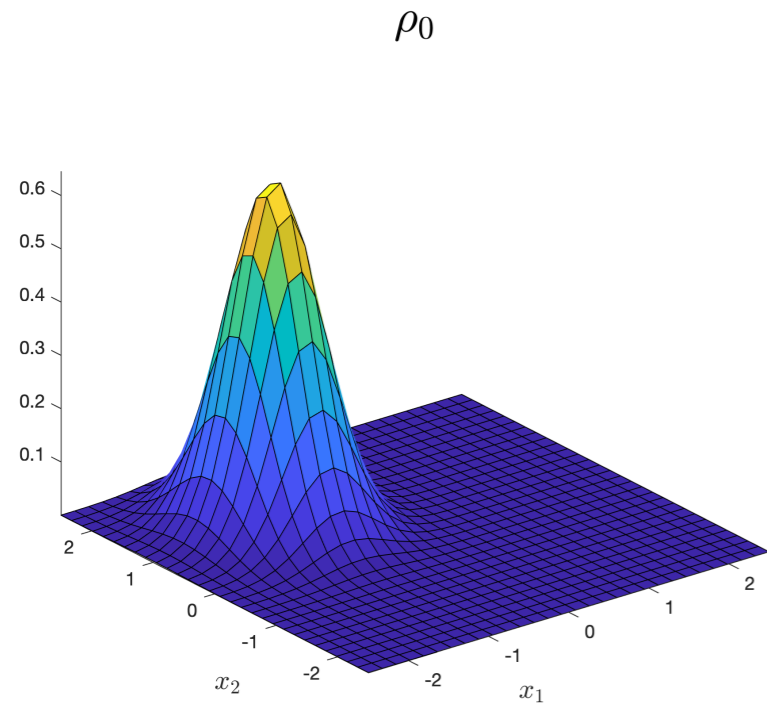
# Feedback Density Control: $f \equiv 0, B = G \equiv I, q \equiv 0$



Zero prior dynamics



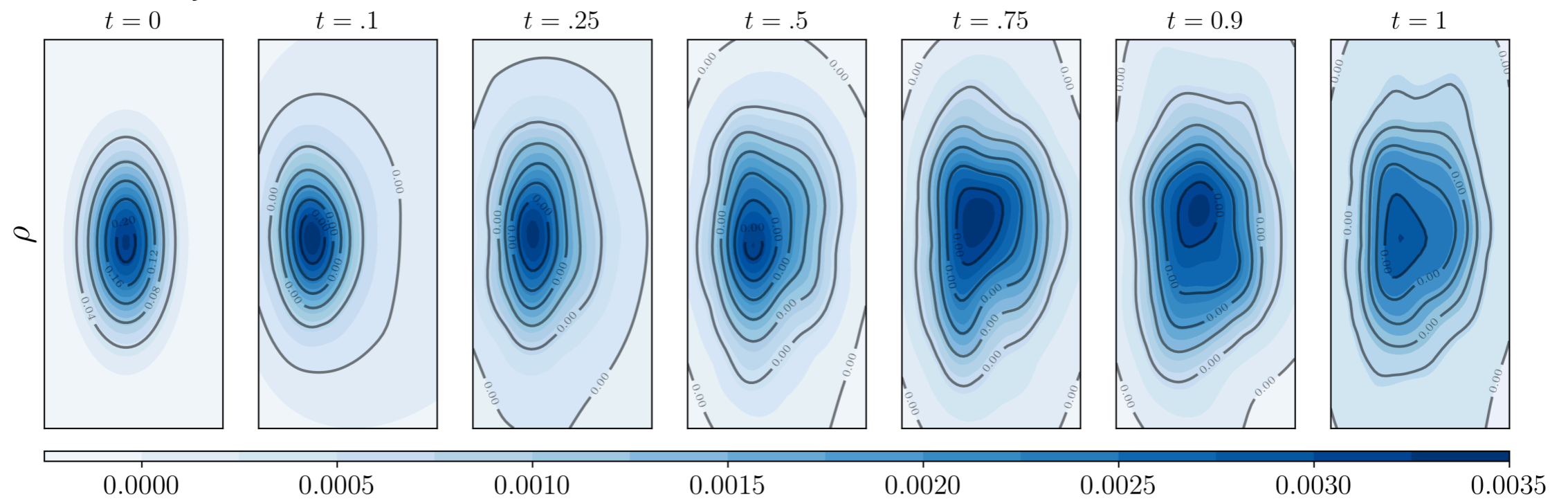
# Feedback Density Control: $f \equiv Ax, B = G, q \equiv 0$



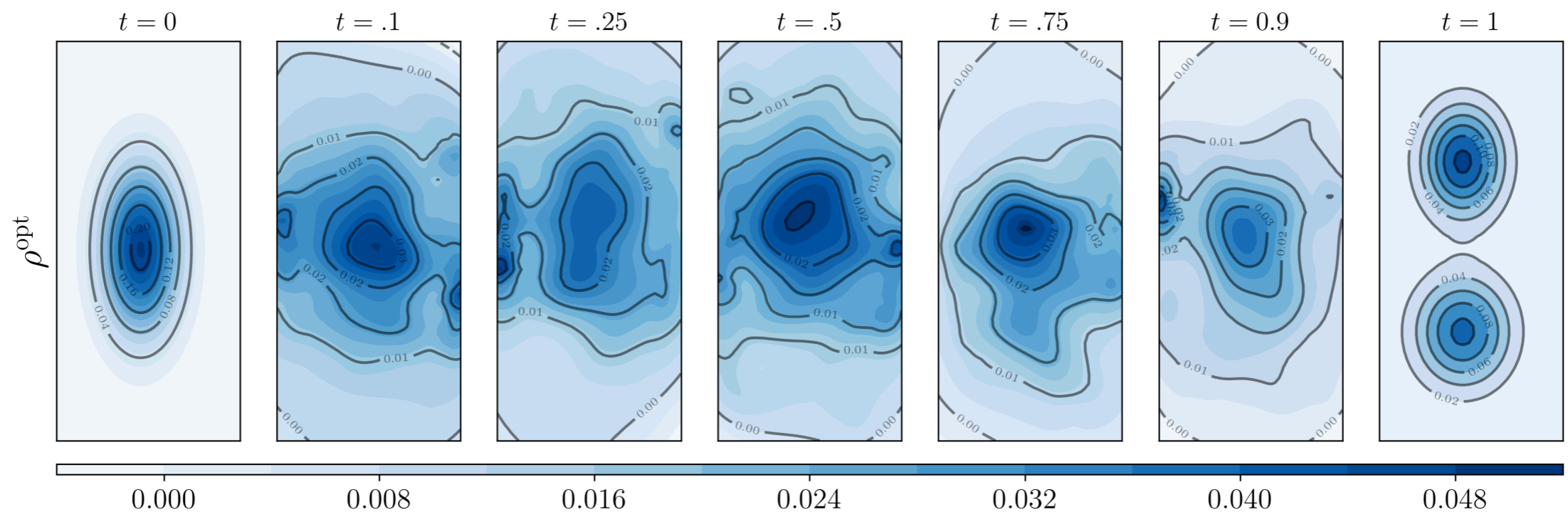
Linear prior dynamics

# Feedback Density Control: Nonlinear Grad. Drift

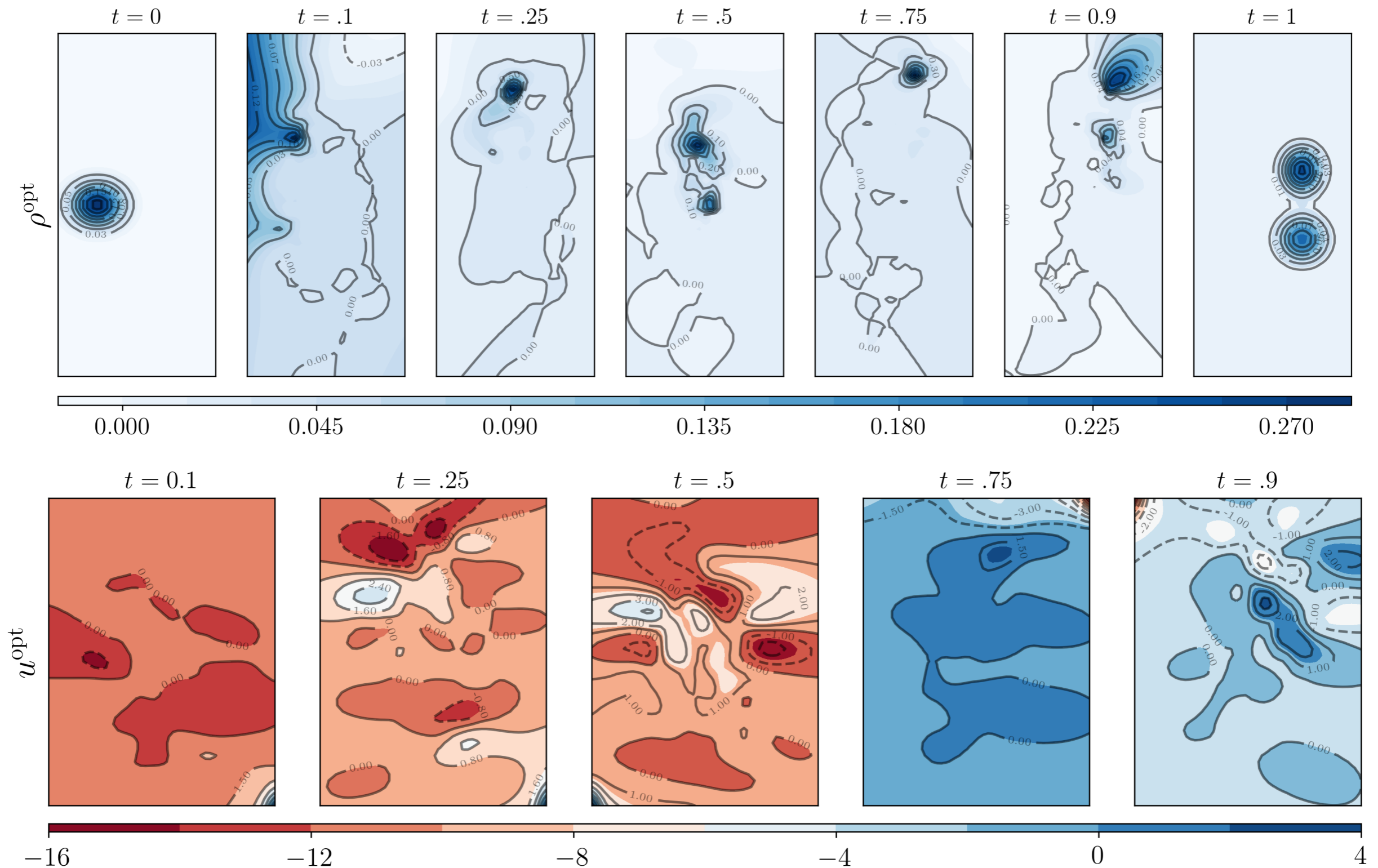
## Uncontrolled joint PDF evolution:



## Optimal controlled joint PDF evolution:

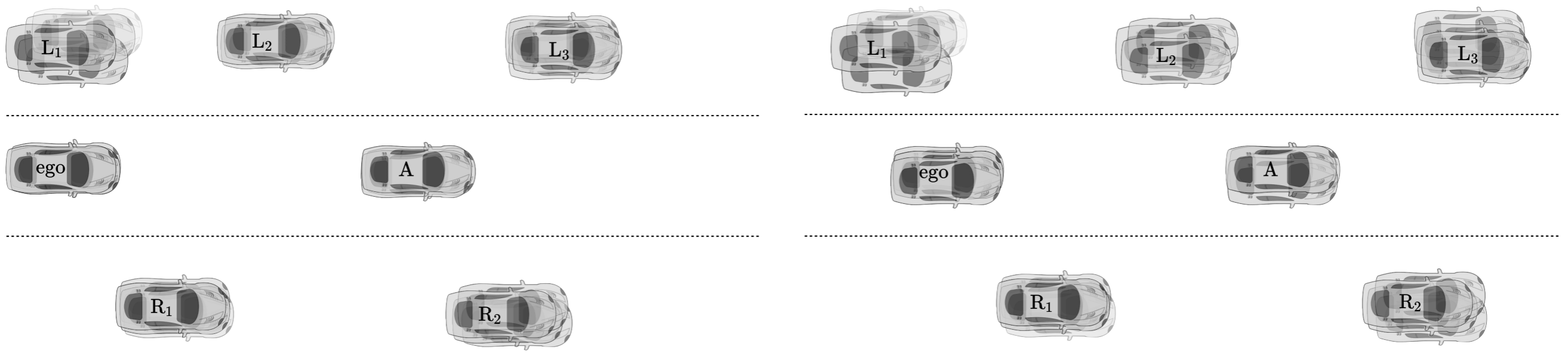


# Feedback Density Control: Mixed Conservative-Dissipative Drift





# Application: Multi-lane Automated Driving

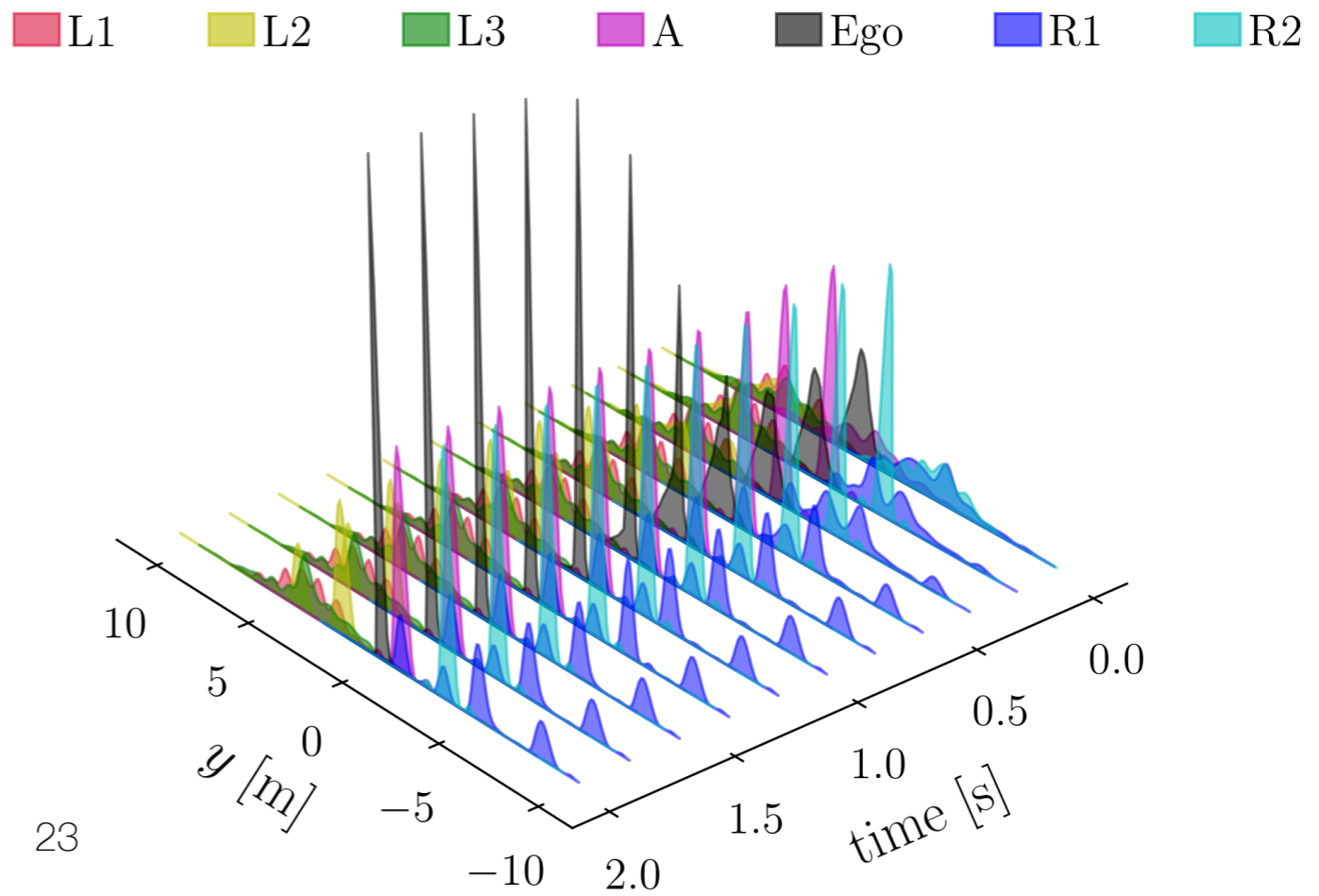
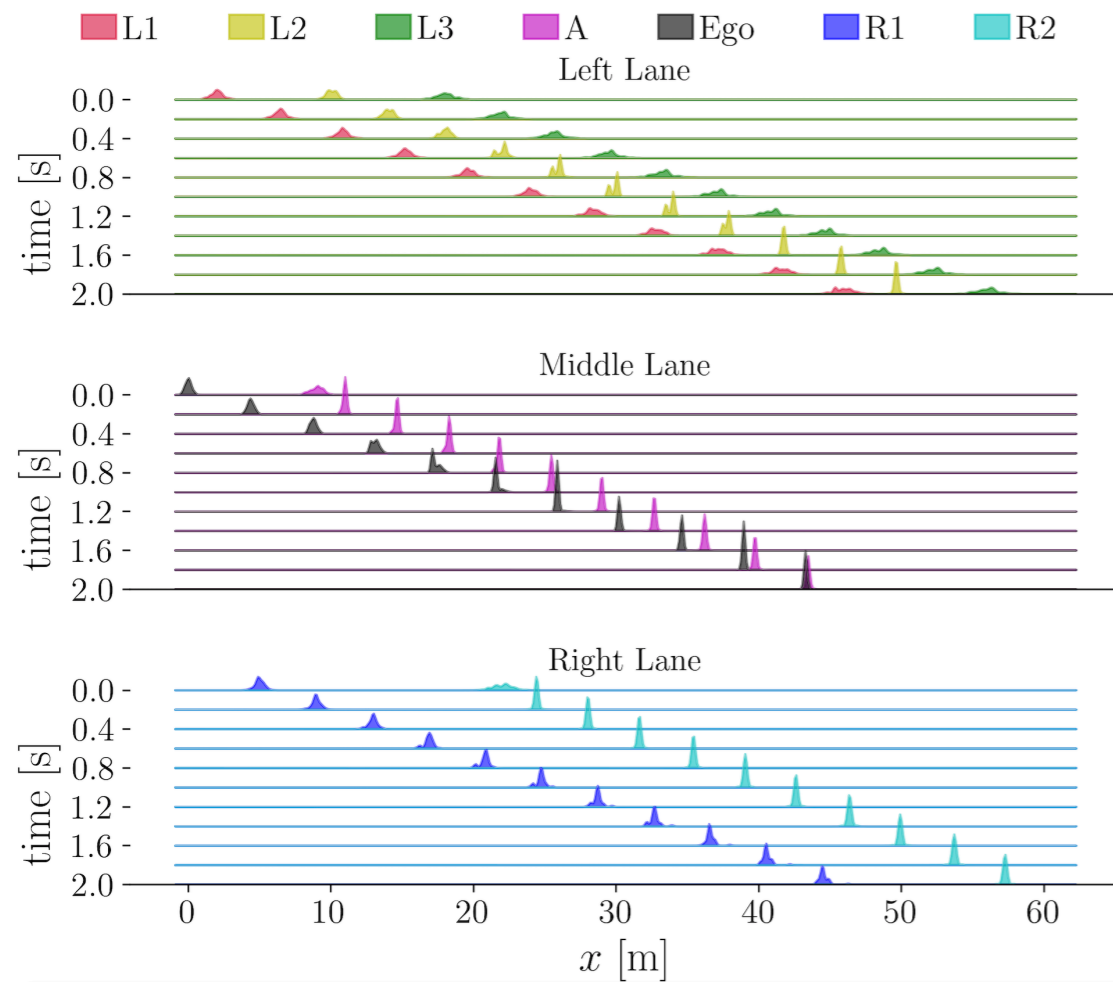


$t_0$

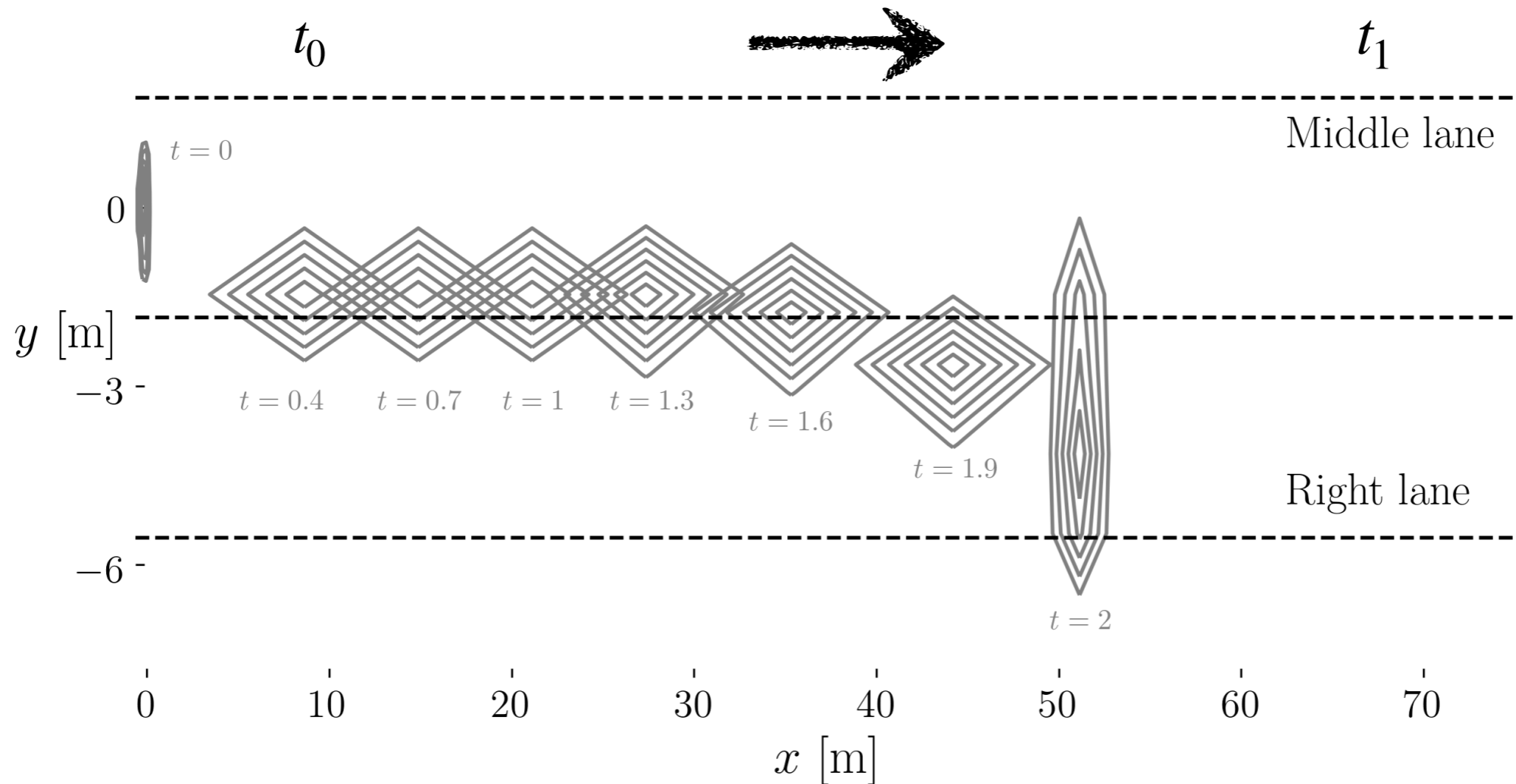
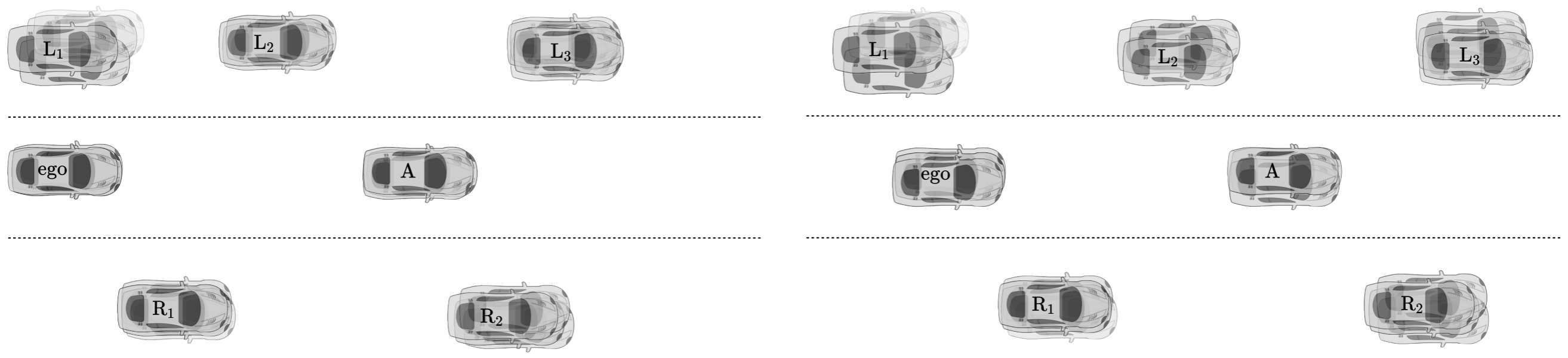
$t_1$

$x$  marginals

$y$  marginals



# Application: Multi-lane Automated Driving



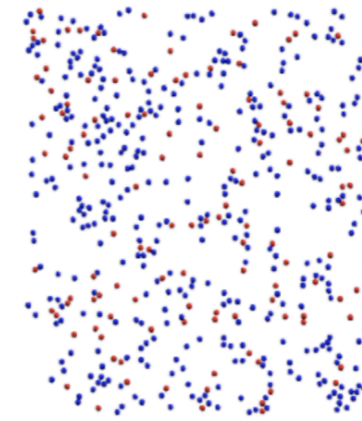
# Control non-affine generalized Schrödinger bridge

No state cost:  $q = 0$

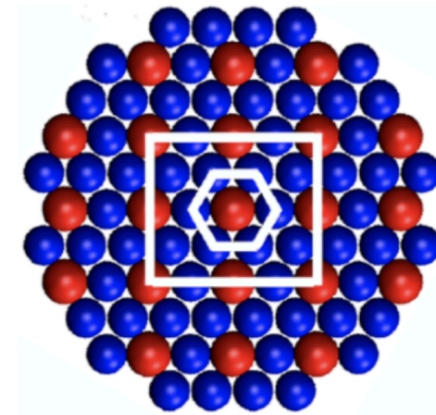
Controlled SDE:

$$d\mathbf{x}_t^u = \mathbf{f}(t, \mathbf{x}_t^u, \mathbf{u})dt + \sqrt{2}\mathbf{g}(t, \mathbf{x}_t^u, \mathbf{u})d\mathbf{w}_t$$

Controlled diffusion tensor:  $\mathbf{G} := \mathbf{g}\mathbf{g}^\top \succeq \mathbf{0}$



Dispersed particles



Ordered structure

Conditions for optimality: system of  $m + 2$  coupled PDEs

$$\frac{\partial \psi}{\partial t} = \frac{1}{2} \|\mathbf{u}_{\text{opt}}\|_2^2 - \langle \nabla_{\mathbf{x}} \psi, \mathbf{f} \rangle - \langle \mathbf{G}, \text{Hess}(\psi) \rangle$$

$$\frac{\partial \rho_{\text{opt}}^u}{\partial t} = -\nabla \cdot (\rho_{\text{opt}}^u \mathbf{f}) + \Delta_{\mathbf{G}} \rho_{\text{opt}}^u$$

$$\mathbf{u}_{\text{opt}} = \nabla_{\mathbf{u}_{\text{opt}}} (\langle \nabla_{\mathbf{x}} \psi, \mathbf{f} \rangle + \langle \mathbf{G}, \text{Hess}(\psi) \rangle)$$

$$\rho_{\text{opt}}^u(0, \mathbf{x}) = \rho_0, \quad \rho_{\text{opt}}^u(T, \mathbf{x}) = \rho_T$$

Known  $\mathbf{f}, \mathbf{g}$

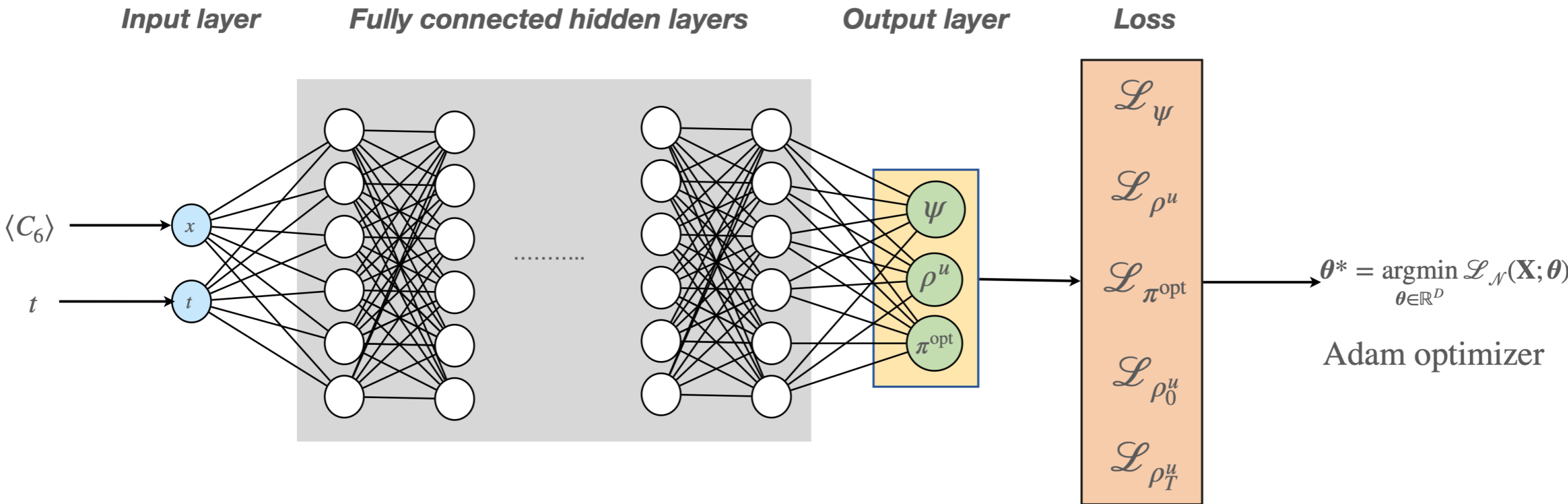
I. Nodozi, J.O'Leary, A. Mesbah,  
and A.H., ACC 2023

🏆 2024 O. Hugo Schuck Best Application Paper Award

Data-driven  $\mathbf{f}, \mathbf{g}$

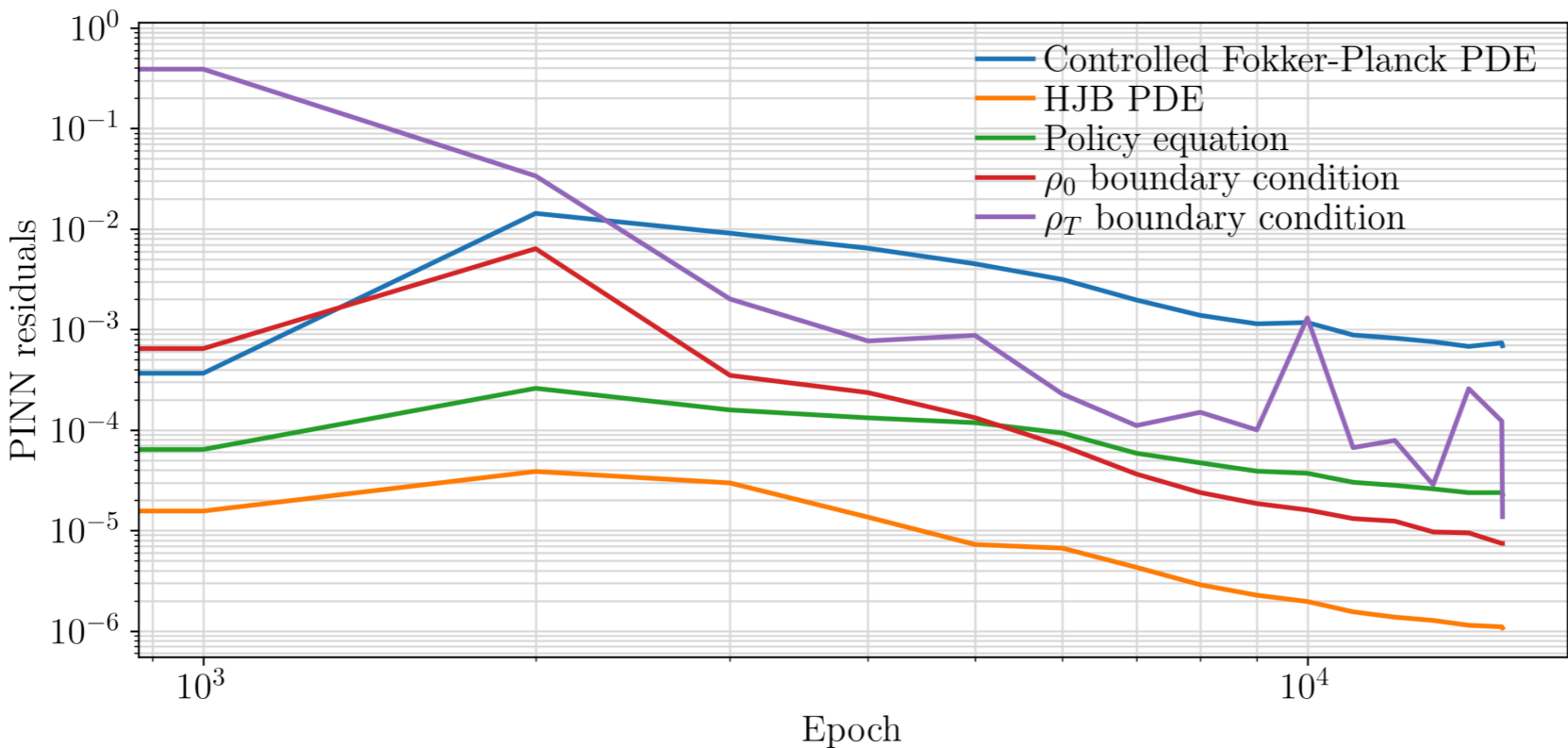
I. Nodozi, C. Yan, M. Khare, A.H.,  
and A. Mesbah, TCST 2024

# Control non-affine generalized Schrödinger bridge



$$\mathcal{L}_\mathcal{N} = \mathcal{L}_\psi + \mathcal{L}_{\rho^u} + \mathcal{L}_{\pi^{\text{opt}}} + \mathcal{L}_{\rho_0^u} + \mathcal{L}_{\rho_T^u}$$

Benchmark controlled self-assembly system: [Y Xue, et al, *IEEE Trans. Control Sys. Technology*, 2014]





# Generalization # 2: hard sample path constraints

**Main idea: path constraints  $\sim$  reflected Itô SDEs**  
modify the controlled sample path dynamics to

$$dx_t^u = \{f(t, x_t^u) + B(t)u(t, x_t^u)\}dt + \sqrt{2\theta}G(t)dw_t + n(x_t^u)d\gamma_t$$

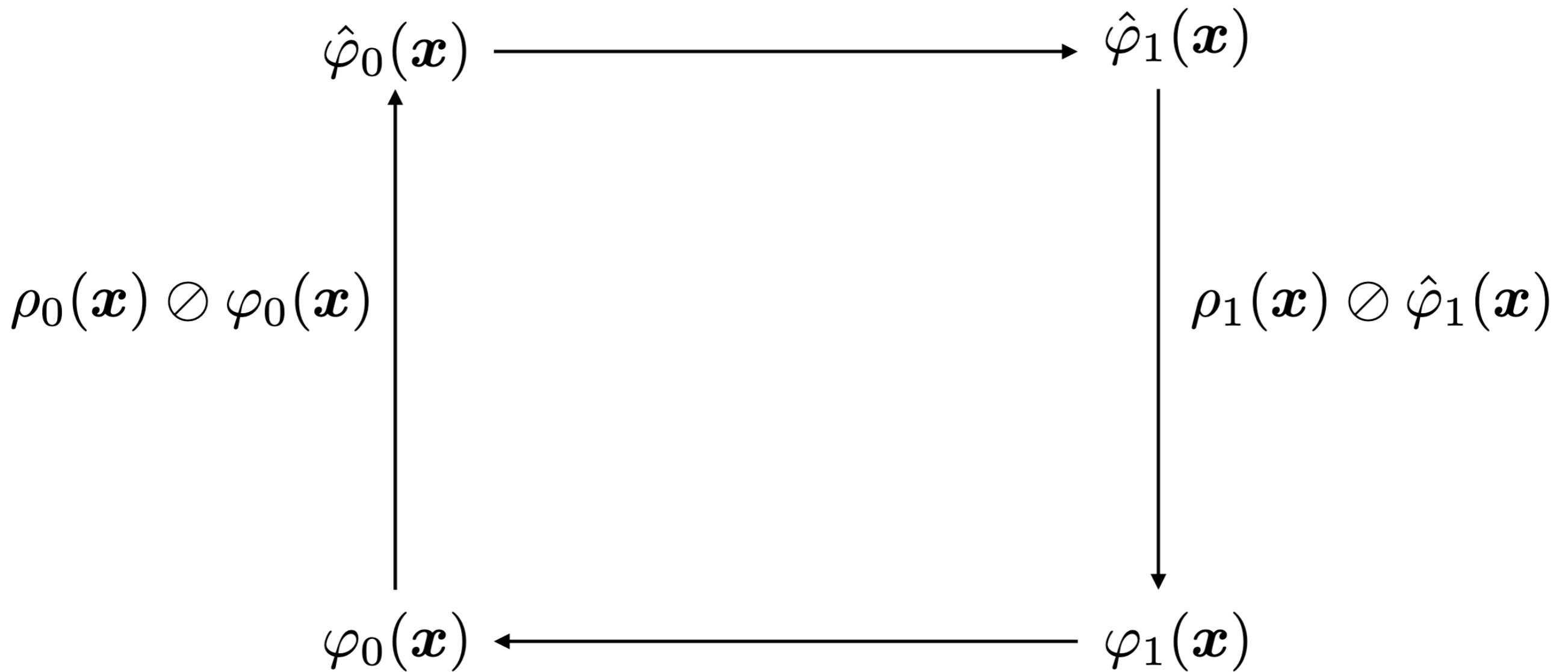
$x_t^u \in \bar{\mathcal{X}} := \mathcal{X} \cup \partial\mathcal{X}$ , closure of connected smooth  $\mathcal{X}$

$n$  is inward unit normal to the boundary  $\partial\mathcal{X}$

$\gamma_t$  is minimal local time stochastic process

# Reflected bridge: Schrödinger factor recursion

$$\int \text{with b.c. } \langle \mathbf{f} \hat{\varphi} - \theta \nabla \hat{\varphi}, \mathbf{n} \rangle |_{\partial \mathcal{X}} = 0$$

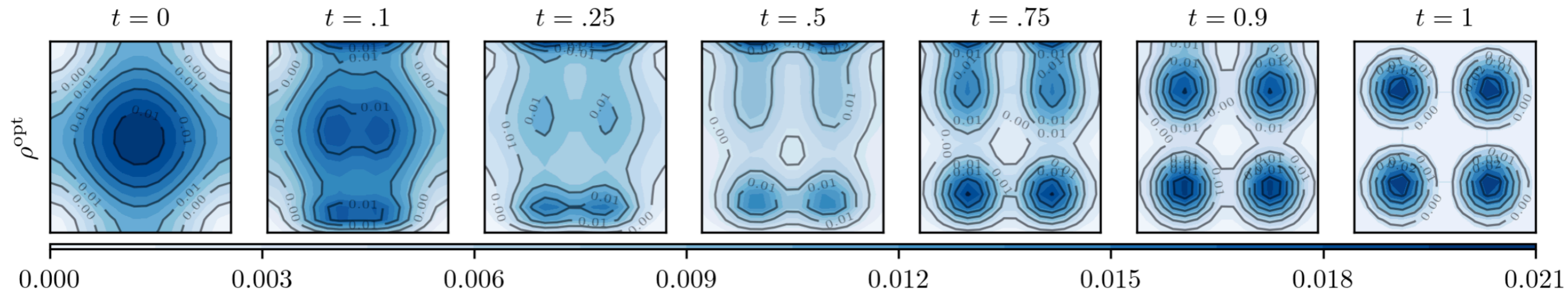


$$\int \text{with b.c. } \langle \nabla \varphi, \mathbf{n} \rangle |_{\partial \mathcal{X}} = 0$$

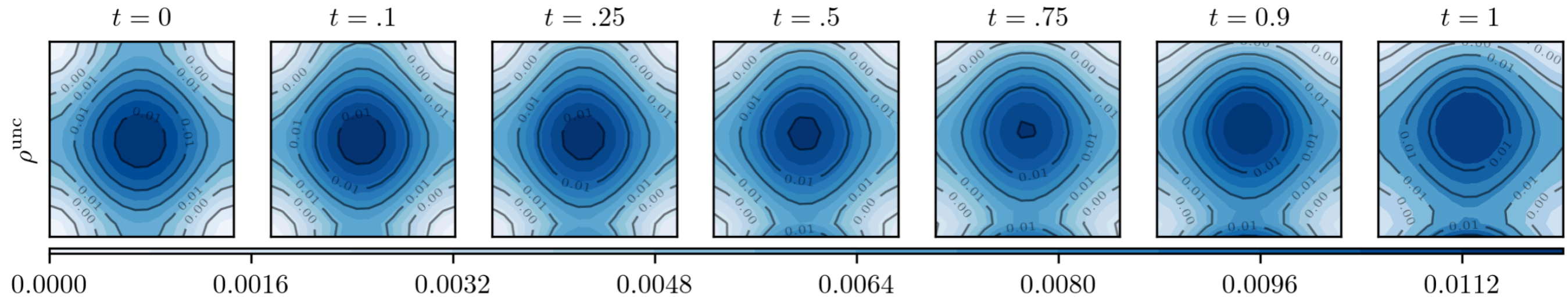
# Reflected bridge: numerics with $\nabla V$ drift

$$V(x_1, x_2) = (x_1^2 + x_2^3)/5, \quad \bar{\mathcal{X}} = [-4, 4]^2$$

Optimal controlled state PDFs:



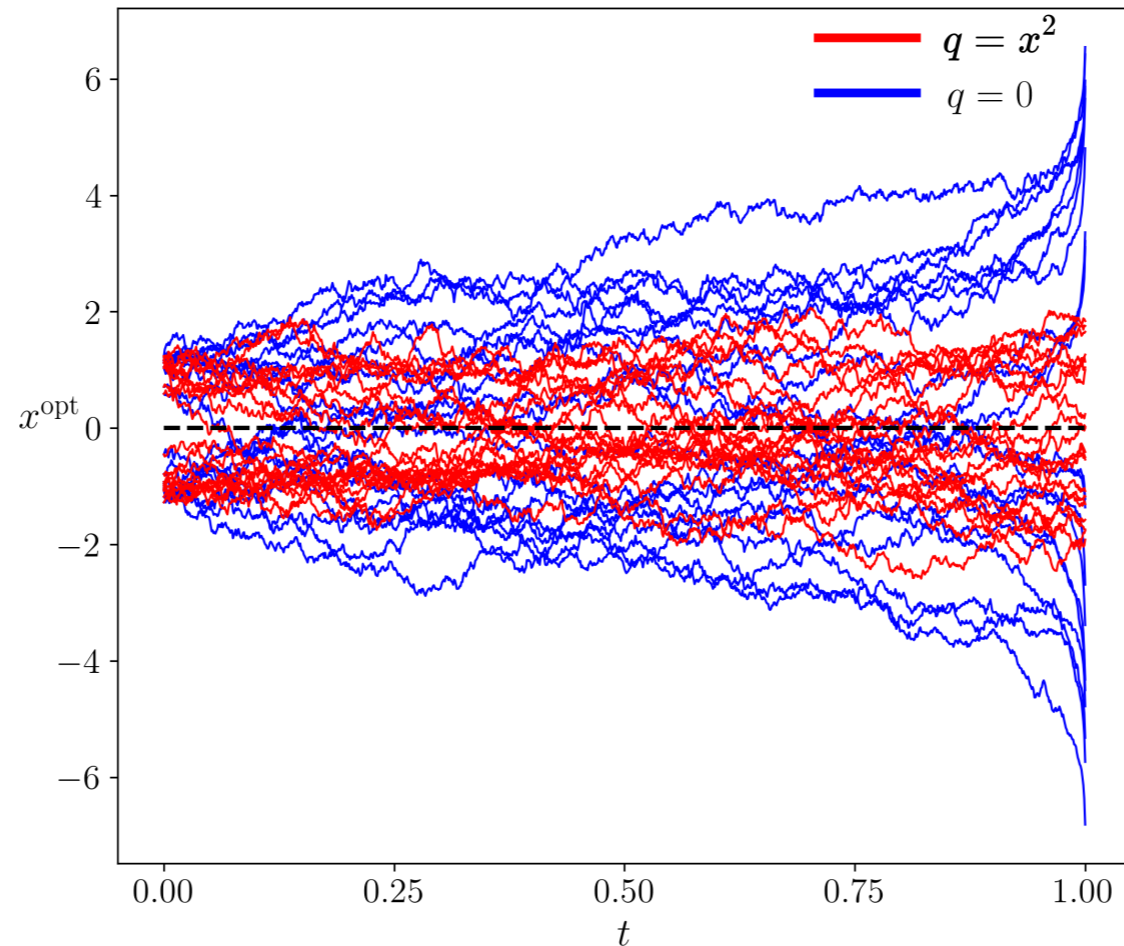
Uncontrolled state PDFs:



# Generalization # 3: additive state cost ( $q \neq 0$ )

**Question.** Where does state cost come from?

**Answer 1.** From extra regularization (e.g., classical LQ optimal control)



**Answer 2.** Problem reformulation (push dynamical nonlinearity to Lagrangian)

**Probabilistic Lambert Problem: Connections with Optimal Mass Transport, Schrödinger Bridge and Reaction-Diffusion PDEs\***

Alexis M.H. Teter<sup>†</sup>, Iman Nodozi<sup>‡</sup>, and Abhishek Halder<sup>§</sup>

A.M. Teter, I. Nodozi, and A.H.,  
*arXiv:2401.07961*



# Schrödinger bridge with quadratic state cost:

$$q(\mathbf{x}) = \mathbf{x}^\top \mathbf{Q} \mathbf{x}, \mathbf{Q} \succeq \mathbf{0}$$

**Solution:**  $\rho^{\text{opt}}(\mathbf{x}, t) = \varphi(\mathbf{x}, t) \hat{\varphi}(\mathbf{x}, t)$

$$\left( \frac{\partial}{\partial t} + \frac{1}{2} \Delta - q \right) \varphi = 0 \quad \text{[Backward reaction-diffusion PDE]}$$

$$\left( \frac{\partial}{\partial t} - \frac{1}{2} \Delta + q \right) \hat{\varphi} = 0 \quad \text{[Forward reaction-diffusion PDE]}$$

# Schrödinger bridge with quadratic state cost:

$$q(\mathbf{x}) = \mathbf{x}^\top \mathbf{Q} \mathbf{x}, \mathbf{Q} \succeq \mathbf{0}$$

We know:  $\rho^{\text{opt}}(\mathbf{x}, t) = \varphi(\mathbf{x}, t) \hat{\varphi}(\mathbf{x}, t)$

$$\left( \frac{\partial}{\partial t} + \cancel{\frac{1}{2}} \Delta - q \right) \varphi = 0 \quad \text{[Backward reaction-diffusion PDE]}$$

$$\left( \frac{\partial}{\partial t} - \cancel{\frac{1}{2}} \Delta + q \right) \hat{\varphi} = 0 \quad \text{[Forward reaction-diffusion PDE]}$$

Need kernel / Green's function  $\kappa(0, \mathbf{x}; t, \mathbf{y})$

for IVP solutions to use in Schrödinger factor recursion:

$$\frac{\partial \hat{\varphi}}{\partial t} = \underbrace{\mathcal{L}_{\text{forward}}}_{(\Delta - \mathbf{x}^\top \mathbf{Q} \mathbf{x})} \hat{\varphi}, \quad \hat{\varphi}(t=0, \mathbf{x}) = \hat{\varphi}_0 \quad \Leftrightarrow \quad \hat{\varphi}(\mathbf{x}, t) = \int_{\mathbb{R}^n} \kappa(0, \mathbf{x}; t, \mathbf{z}) \hat{\varphi}_0(\mathbf{z}) d\mathbf{z}$$

# Schrödinger bridge with quadratic state cost:

$$q(\mathbf{x}) = \mathbf{x}^\top \mathbf{Q} \mathbf{x}, \mathbf{Q} \succ \mathbf{0}$$

**Thm.** Eig. decomposition:  $\mathbf{Q} = \mathbf{V} \mathbf{D} \mathbf{V}^\top$

Then,  $\hat{\varphi}(\mathbf{x}, t) = \eta(\mathbf{y} = \mathbf{V} \mathbf{x}, t)$  where  $\eta(\mathbf{y}, t) = \int_{\mathbb{R}^n} \kappa(0, \mathbf{y}; t, \mathbf{z}) \eta_0(\mathbf{z}) d\mathbf{z}$

and

$$\kappa(0, \mathbf{y}; t, \mathbf{z}) = \frac{(\det(\mathbf{D}))^{1/4}}{\sqrt{(2\pi)^n \det(\sinh(2t\sqrt{\mathbf{D}}))}} \exp\left(-\frac{1}{2} (\mathbf{y} \quad \mathbf{z}) \mathbf{M} \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix}\right)$$

$$\mathbf{M} := \begin{bmatrix} \mathbf{D}^{1/4} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{D}^{1/4} \end{bmatrix} \mathbf{M}_1 \mathbf{M}_2 \begin{bmatrix} \mathbf{D}^{1/4} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{D}^{1/4} \end{bmatrix}, \mathbf{M}_1 := \begin{bmatrix} \cosh(2t\sqrt{\mathbf{D}}) & -\mathbf{I}_n \\ -\mathbf{I}_n & \cosh(2t\sqrt{\mathbf{D}}) \end{bmatrix}, \mathbf{M}_2 := \begin{bmatrix} \operatorname{csch}(2t\sqrt{\mathbf{D}}) & \mathbf{0} \\ \mathbf{0} & \operatorname{csch}(2t\sqrt{\mathbf{D}}) \end{bmatrix}$$

$$\eta_0(\mathbf{y}) = \hat{\varphi}_0(\mathbf{V}^\top \mathbf{x})$$

$\mathbf{Q} = \mathbf{I}$  recovers the multivariate Mehler kernel in quantum harmonic oscillator

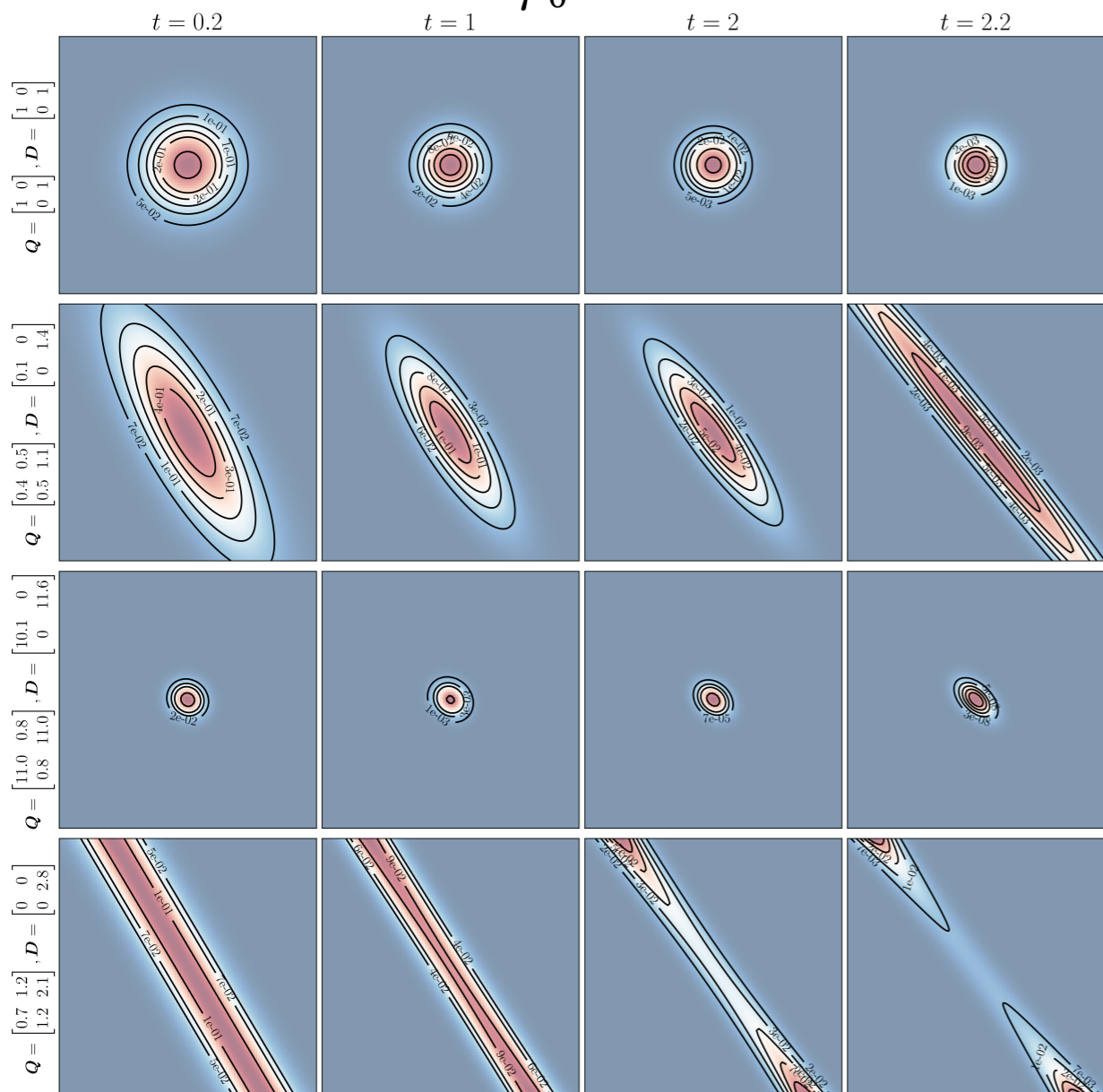
# Schrödinger bridge with quadratic state cost:

$$q(x) = x^\top Qx, Q \succeq 0$$

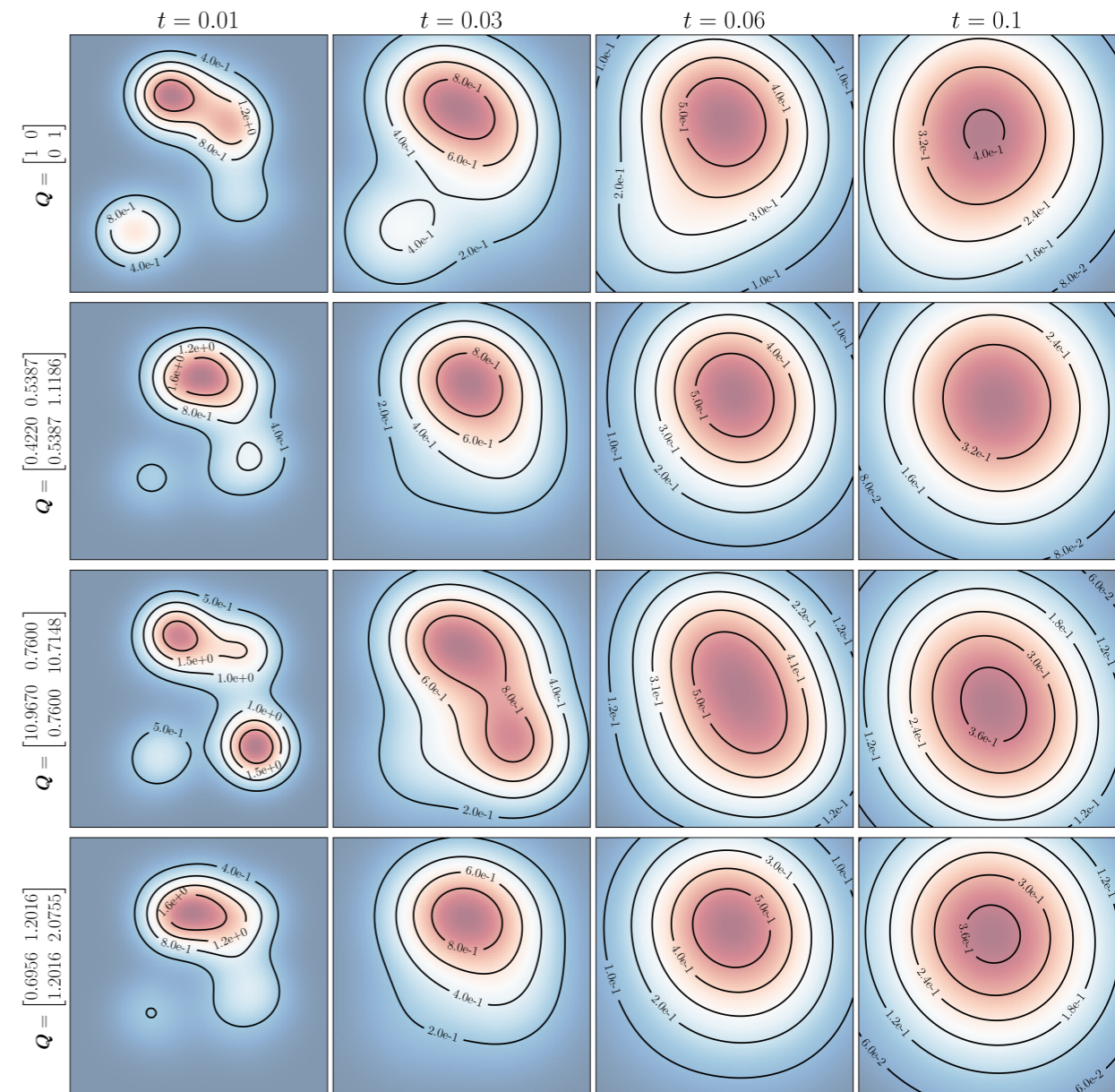
Thm.  $\kappa(0, \mathbf{y}; t, \mathbf{z}) = \underbrace{\kappa_+(0, \mathbf{y}_{[i_1:i_{n-p}]}; t, \mathbf{z}_{[i_1:i_{n-p}]})}_{\text{derived pos def kernel in } n-p \text{ variables}} \underbrace{\kappa_0(0, \mathbf{y}_{[i_{n-p+1}:i_n]}; t, \mathbf{z}_{[i_{n-p+1}:i_n]})}_{\text{heat kernel in } p \text{ variables}}$

## Action of kernel in $x$ coordinates

$$\varphi_0 = 1$$

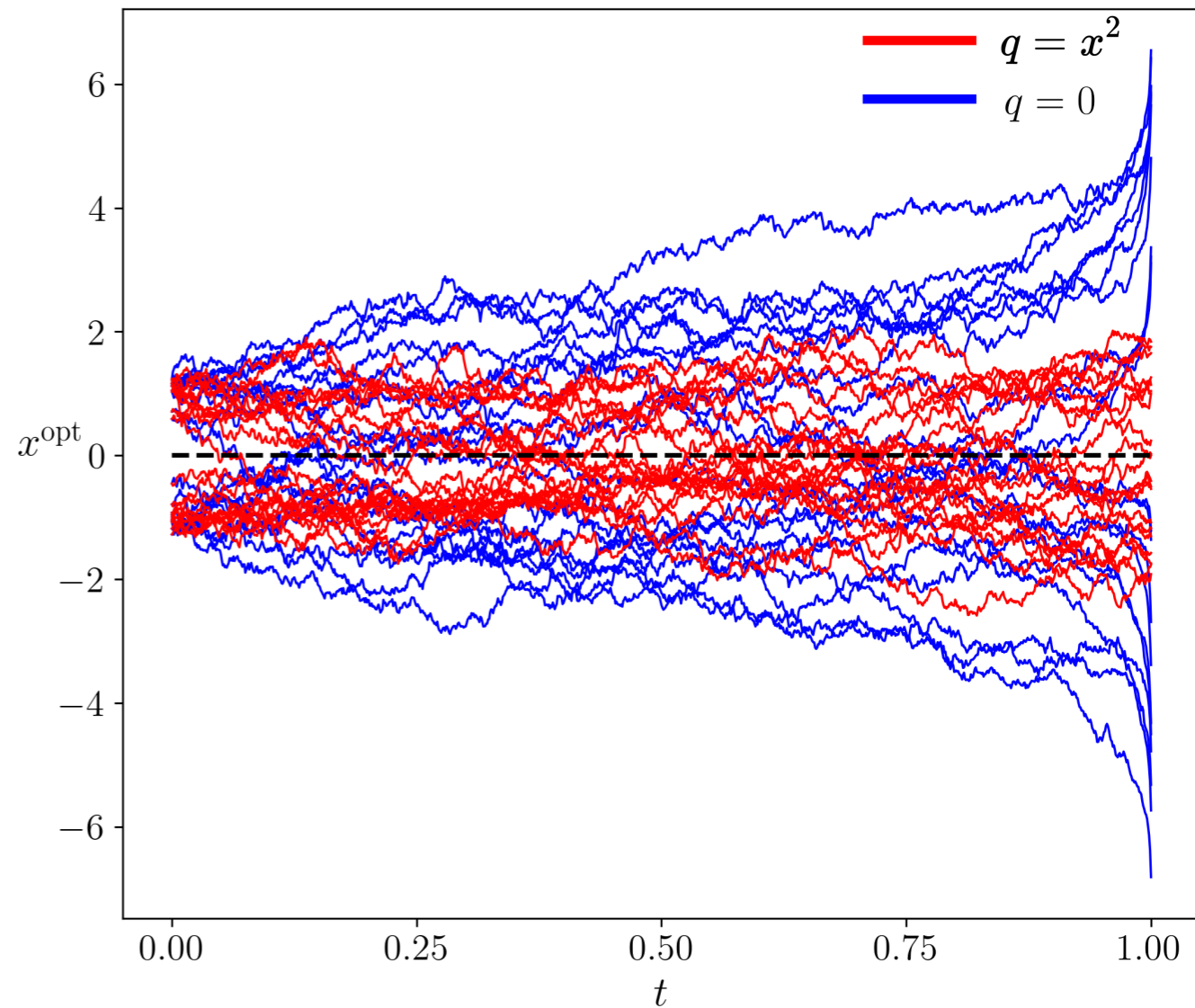
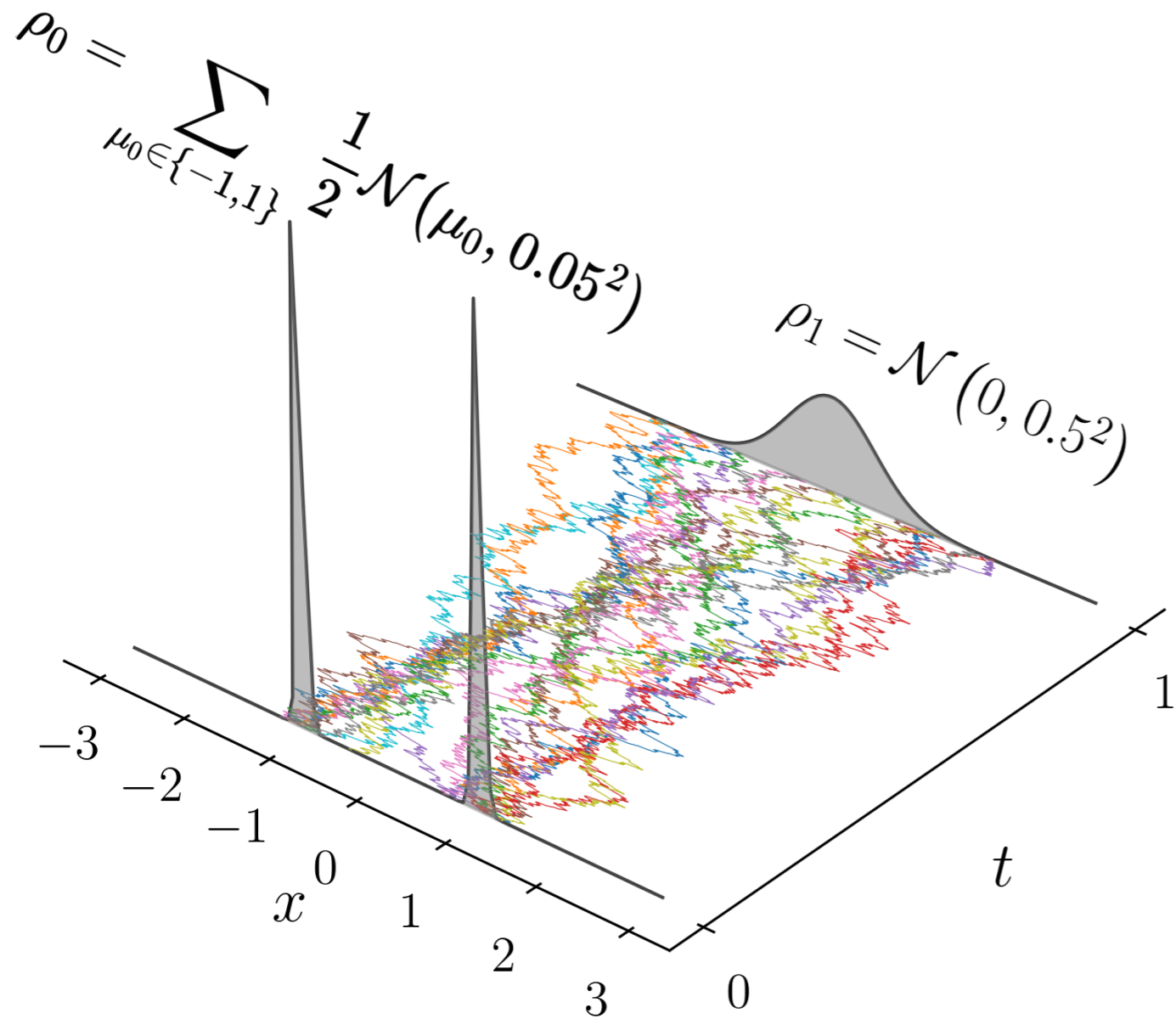


$$\varphi_0 \propto \exp \left\{ -\frac{1}{25} \left( \left( (10x_1 - 5)^2 + 10x_2 - 16 \right)^2 + \left( 10x_1 - 12 + (10x_2 - 5)^2 \right)^2 \right) \right\}$$



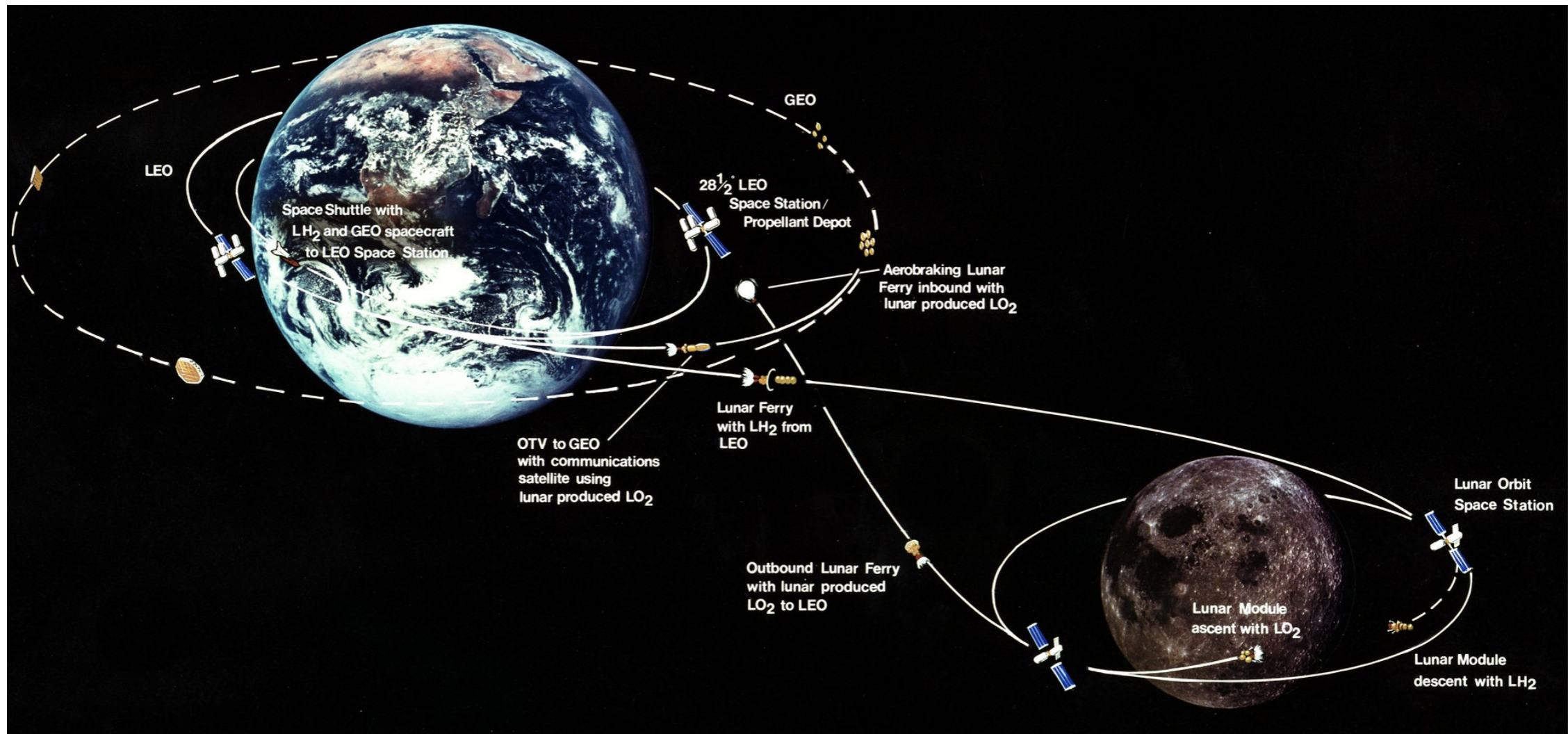


# Schrödinger bridge in 1D: with vs without quadratic state cost



A.M. Teter, W. Wang, and A.H.,  
*arXiv:2406.00503*  
*arXiv:2407.15245*

# Lambert's Problem



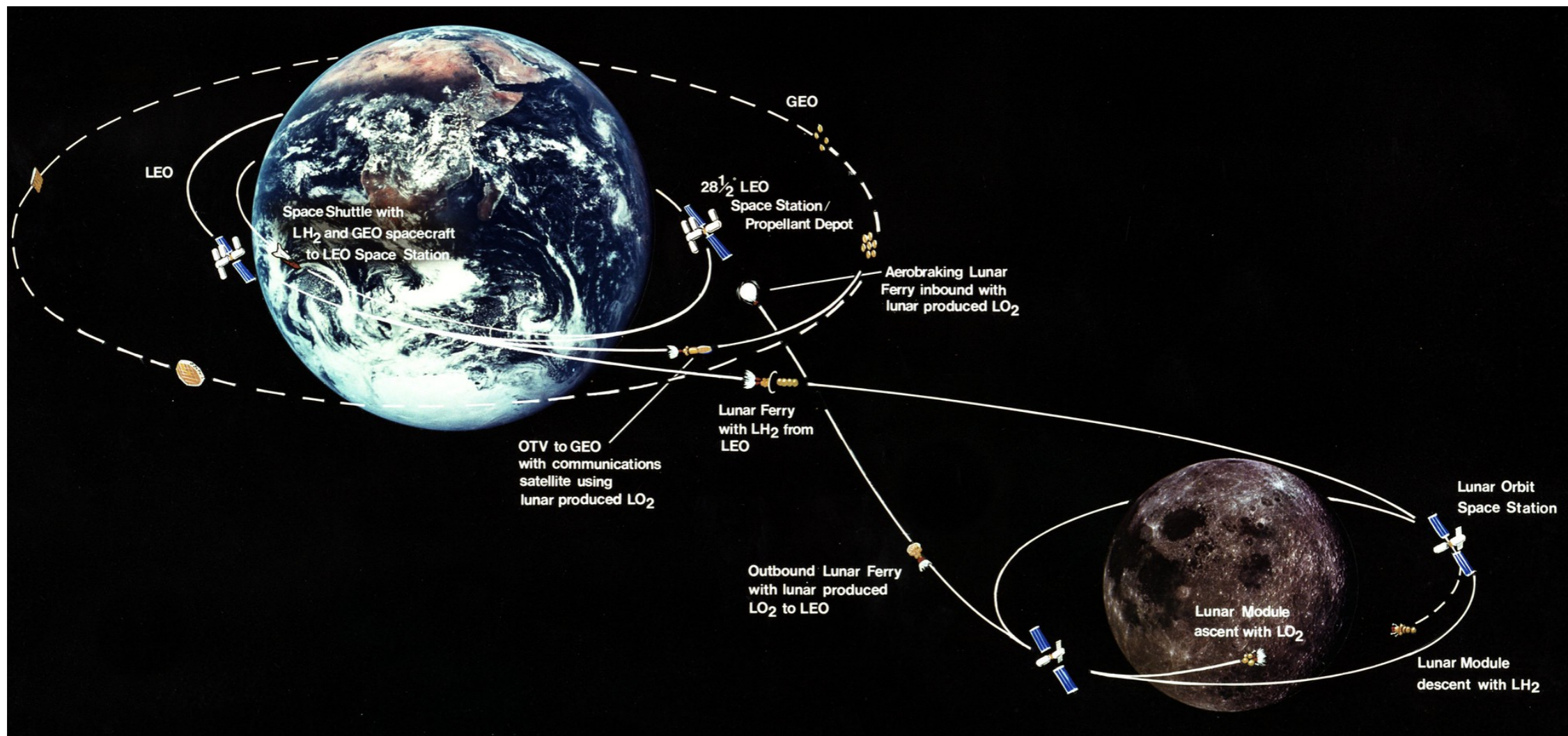
3D position coordinate  $\mathbf{r} := \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

Find velocity control policy  $\dot{\mathbf{r}} := \mathbf{v}(t, \mathbf{r})$  such that

$$\ddot{\mathbf{r}} = -\nabla_{\mathbf{r}} V(\mathbf{r}), \quad \mathbf{r}(t = t_0) = \mathbf{r}_0(\text{ given } ), \quad \mathbf{r}(t = t_1) = \mathbf{r}_1(\text{ given } )$$



# Lambert's Problem



3D position coordinate  $\mathbf{r} := \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

**ODE is 2nd order but endpoint boundary conditions are first order**

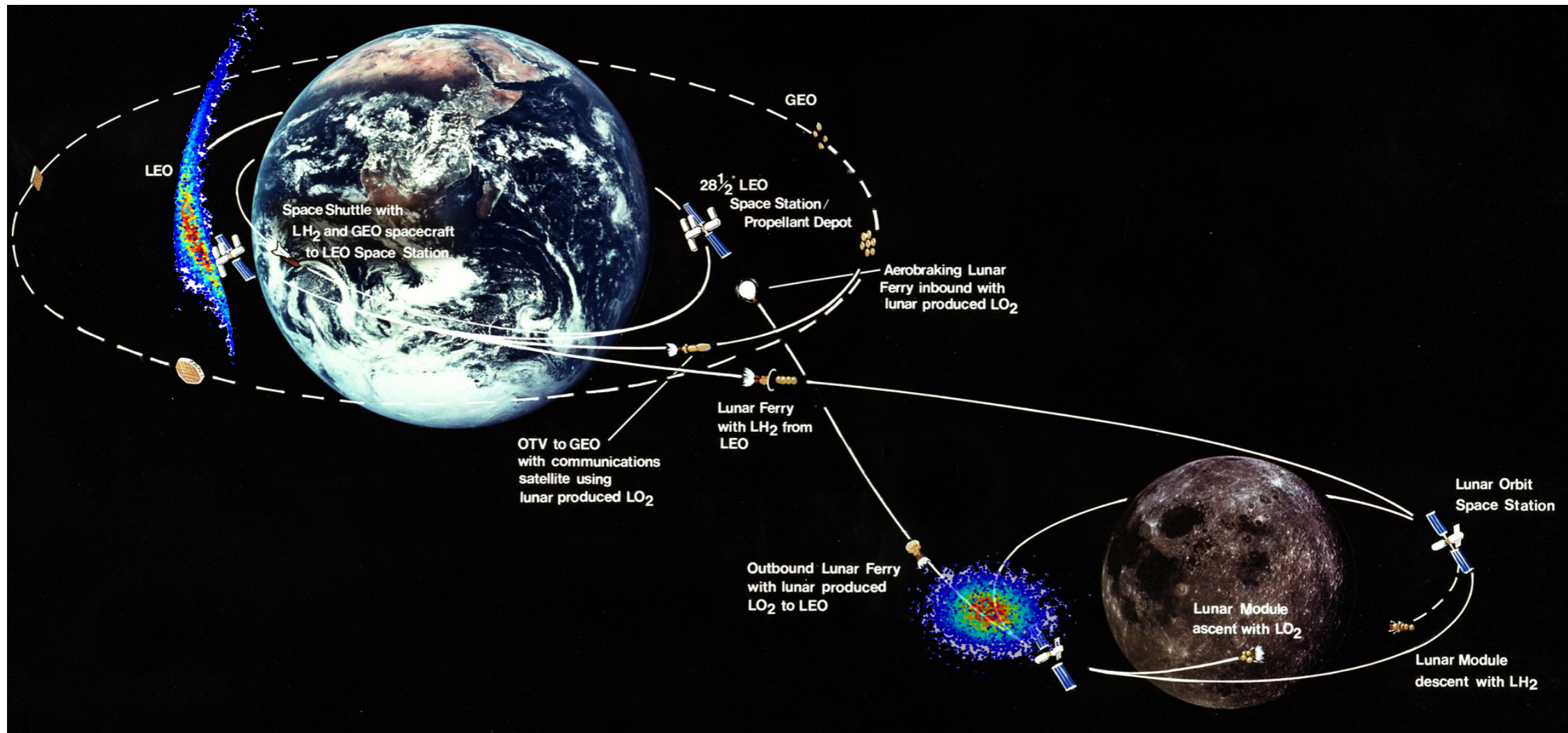
**↪ partially specified TPBVP**

Find velocity control policy  $\dot{\mathbf{r}} := \mathbf{v}(t, \mathbf{r})$  such that

$$\ddot{\mathbf{r}} = -\nabla_{\mathbf{r}} V(\mathbf{r}), \quad \mathbf{r}(t = t_0) = \mathbf{r}_0(\text{ given } ), \quad \mathbf{r}(t = t_1) = \mathbf{r}_1(\text{ given } )$$



# Probabilistic Lambert's Problem



3D position coordinate  $\mathbf{r} := \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

Find velocity control policy  $\dot{\mathbf{r}} := \mathbf{v}(t, \mathbf{r})$  such that

$$\dot{\mathbf{r}} = -\nabla_{\mathbf{r}} V(\mathbf{r}), \quad \mathbf{r}(t = t_0) \sim \rho_0 \text{ (given)}, \quad \mathbf{r}(t = t_1) \sim \rho_1 \text{ (given)}$$



# Probabilistic Lambert problem is OMT

$$\arg \inf_{(\rho, \mathbf{v})} \int_{t_0}^{t_1} \mathbb{E}_{\rho} \left[ \frac{1}{2} \|\mathbf{v}\|_2^2 - V(\mathbf{r}) \right] dt$$

$$\dot{\mathbf{r}} = \mathbf{v},$$

$$\mathbf{r}(t = t_0) \sim \rho_0 \text{ (given)}, \quad \mathbf{r}(t = t_1) \sim \rho_1 \text{ (given)}$$



$$\arg \inf_{(\rho, \mathbf{v})} \int_{t_0}^{t_1} \mathbb{E}_{\rho} \left[ \frac{1}{2} \|\mathbf{v}\|_2^2 - V(\mathbf{r}) \right] dt$$

$$\frac{\partial \rho}{\partial t} + \nabla_{\mathbf{r}} \cdot (\rho \mathbf{v}) = 0, \quad \text{— Liouville PDE}$$

$$\rho(t = t_0, \cdot) = \rho_0, \quad \rho(t = t_1, \cdot) = \rho_1$$

# Connection to SBP with state cost

$$\arg \inf_{(\rho, \mathbf{v}) \in \mathcal{P}_{01} \times \mathcal{V}} \int_{t_0}^{t_1} \int_{\mathbb{R}^n} \left( \frac{1}{2} |\mathbf{v}|^2 - V(\mathbf{x}) \right) \rho(\mathbf{x}, t) d\mathbf{x} dt$$

$$\frac{\partial \rho}{\partial t} + \nabla_r \cdot (\rho \mathbf{v}) = 0, \quad \text{— Liouville PDE}$$

$$\rho(t = t_0, \cdot) = \rho_0, \quad \rho(t = t_1, \cdot) = \rho_1$$

⇓ **Lambertian SBP (L-SBP)**

$$\arg \inf_{(\rho, \mathbf{v}) \in \mathcal{P}_{01} \times \mathcal{V}} \int_{t_0}^{t_1} \int_{\mathbb{R}^n} \left( \frac{1}{2} |\mathbf{v}|^2 - V(\mathbf{x}) \right) \rho(\mathbf{x}, t) d\mathbf{x} dt$$

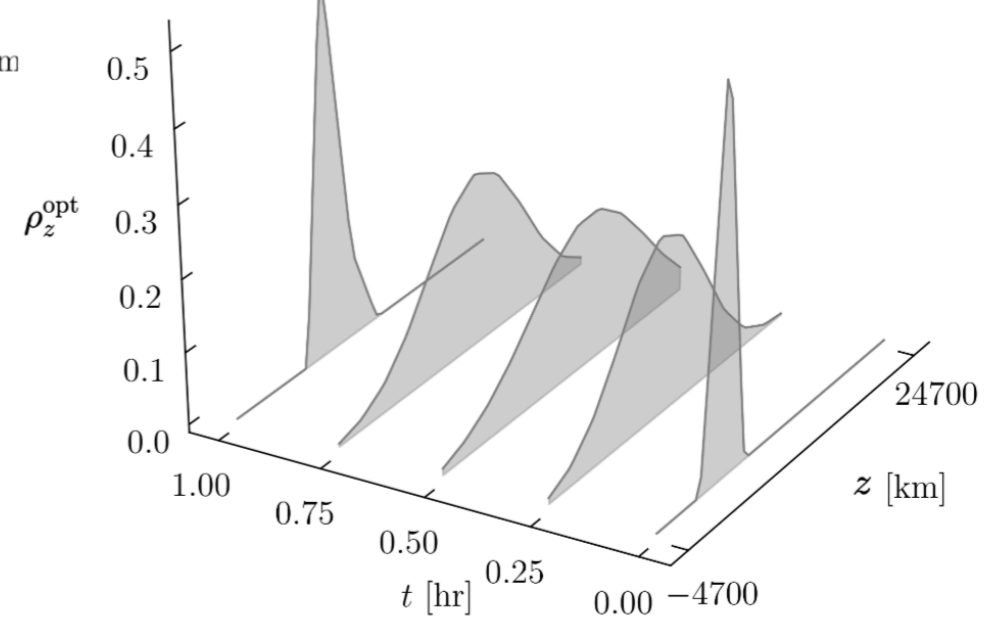
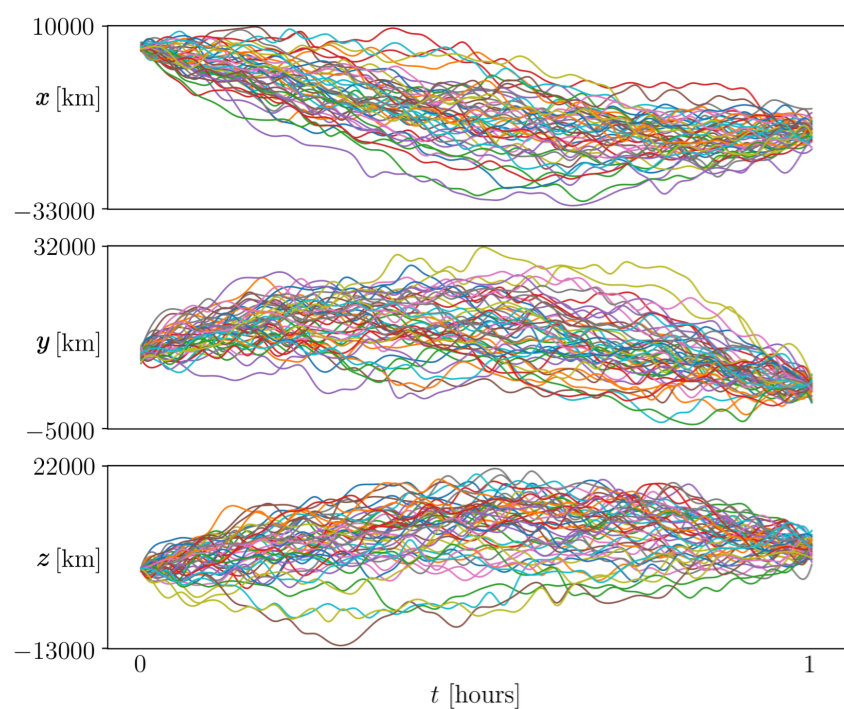
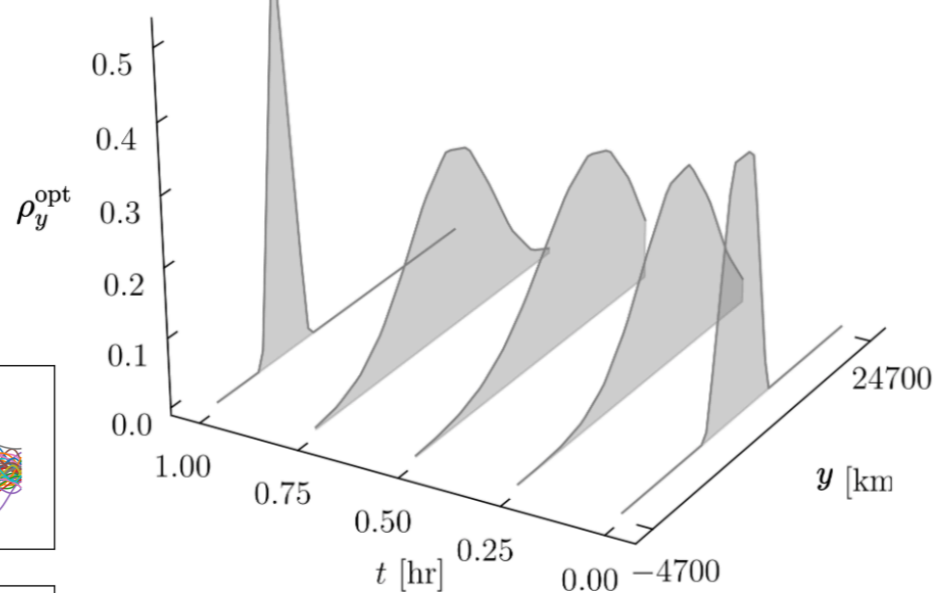
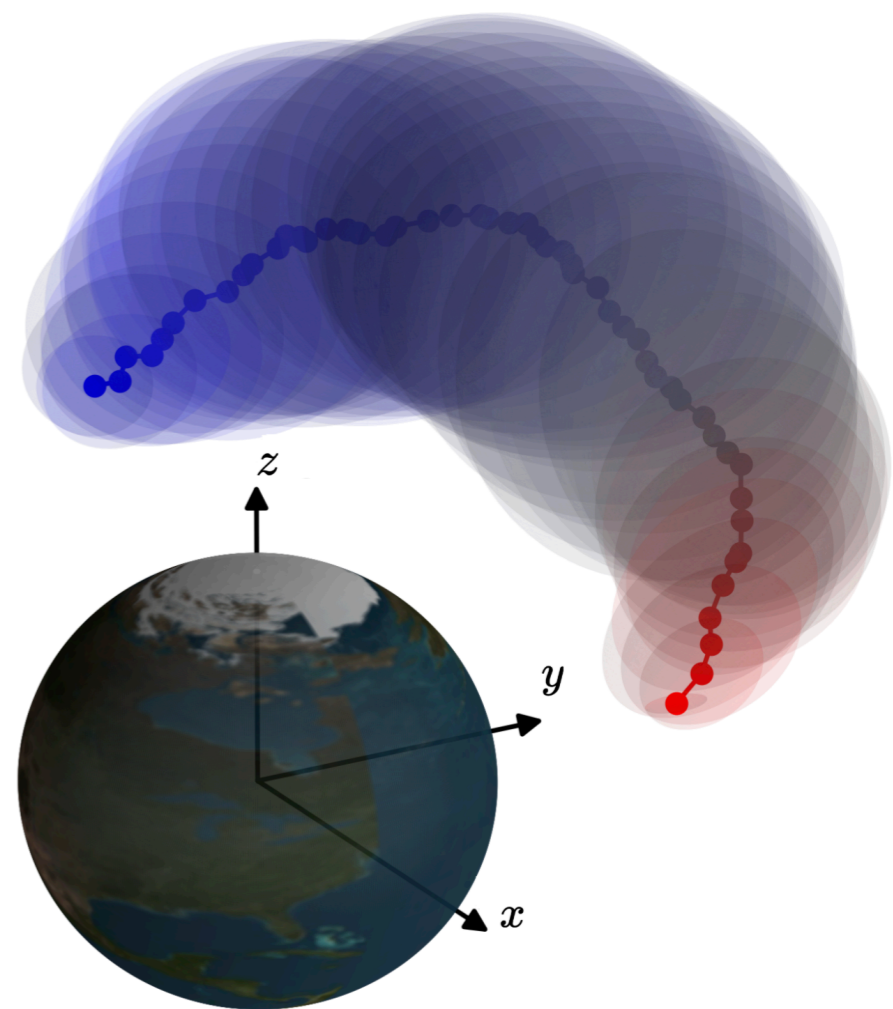
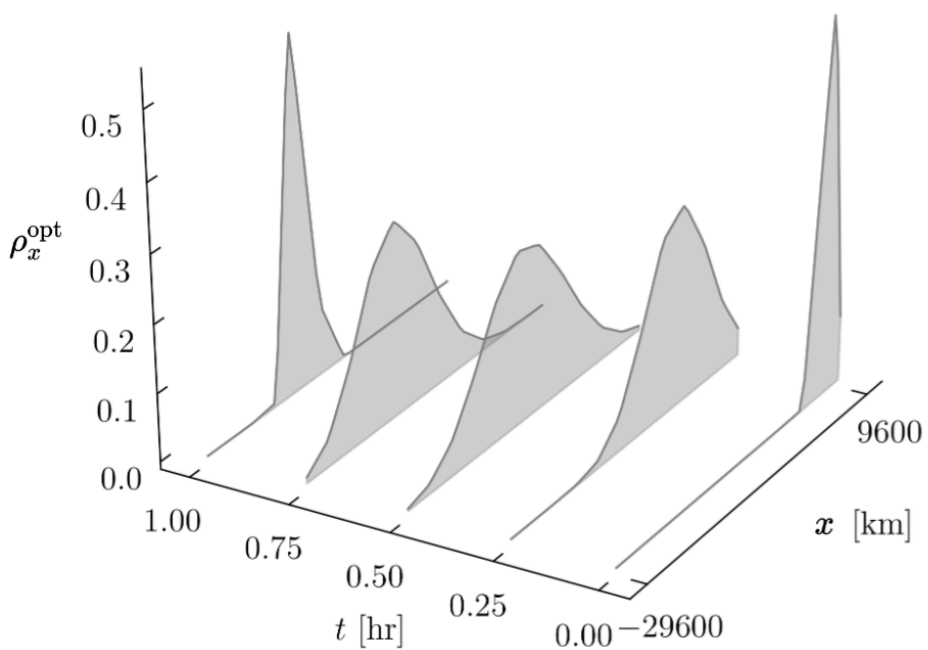
**Regularization > 0**

$$\frac{\partial \rho}{\partial t} + \nabla_r \cdot (\rho \mathbf{v}) = \varepsilon \Delta_r \rho, \quad \text{— Fokker-Planck-Kolmogorov PDE}$$

$$\rho(t = t_0, \cdot) = \rho_0, \quad \rho(t = t_1, \cdot) = \rho_1$$

# Numerical Case Study

Univariate marginals for optimally controlled joint PDFs



# Outlook

- Theory and applications of Schrödinger bridge are undergoing rapid developments
- Lots of mathematics, algorithms, and applications to be done
- Growing interdisciplinary community
- Strong intersections with: physics, control, probability / statistics, PDE, AI/ML, information theory, signal processing, robotics, biology



# Thank You

Support:



CITRIS  
PEOPLE AND  
ROBOTS

