Ideas in Control: Architecture and Optimality

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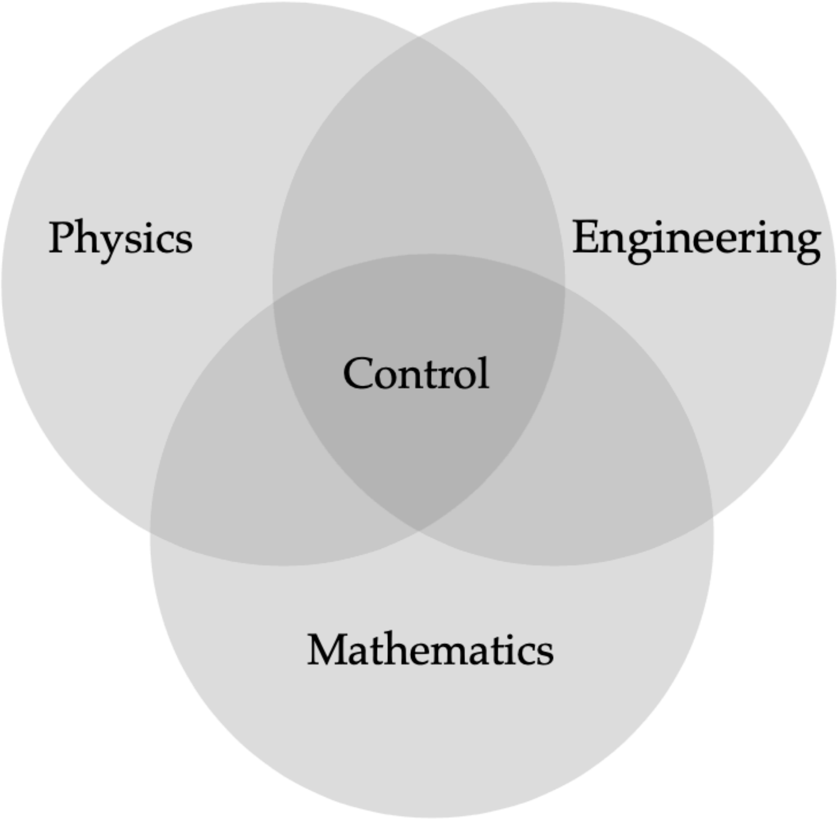
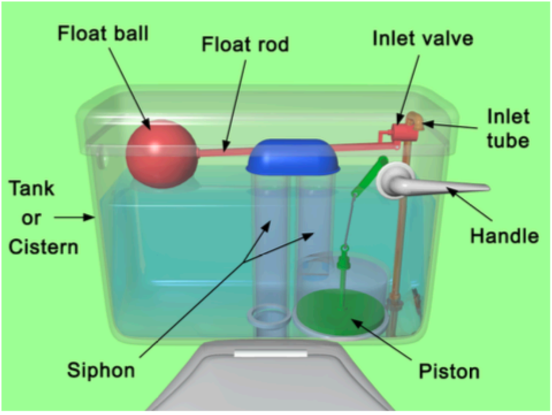
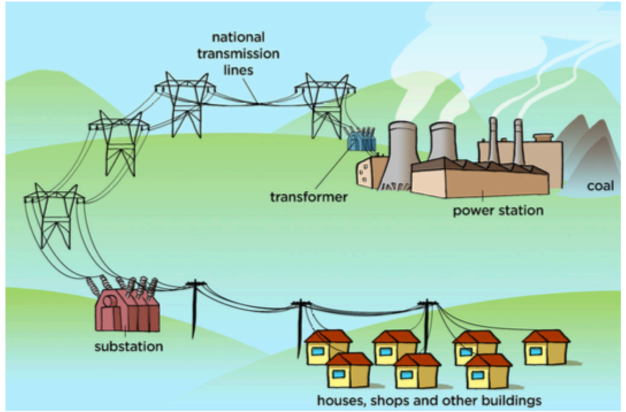
What is Control?

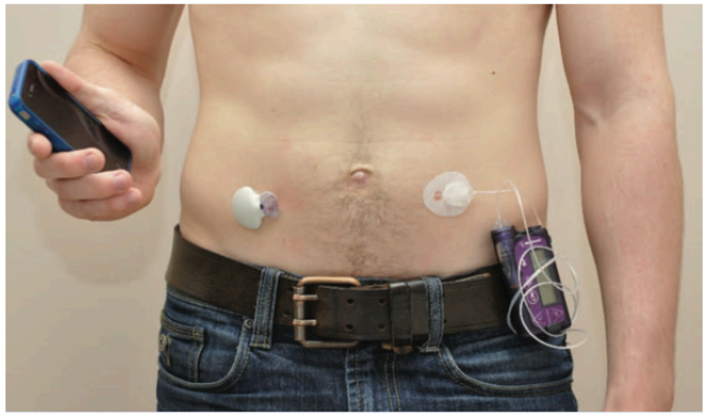
Science

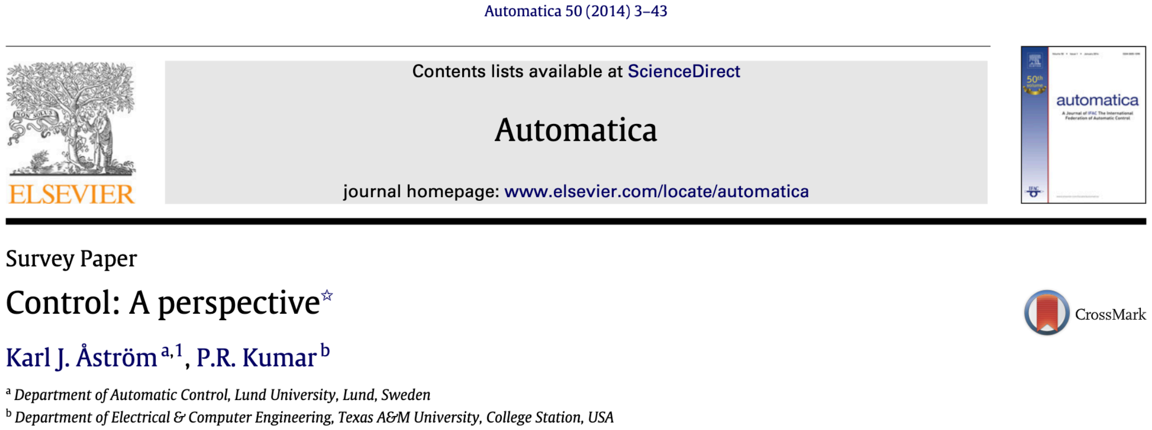
- of decision making and taking actions
- to make systems do what you want them to do
- behind all technologies











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Survey Paper

Control: A perspective☆

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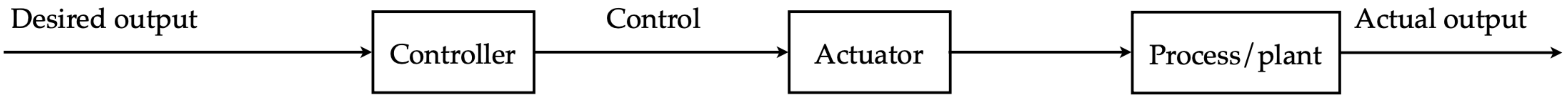
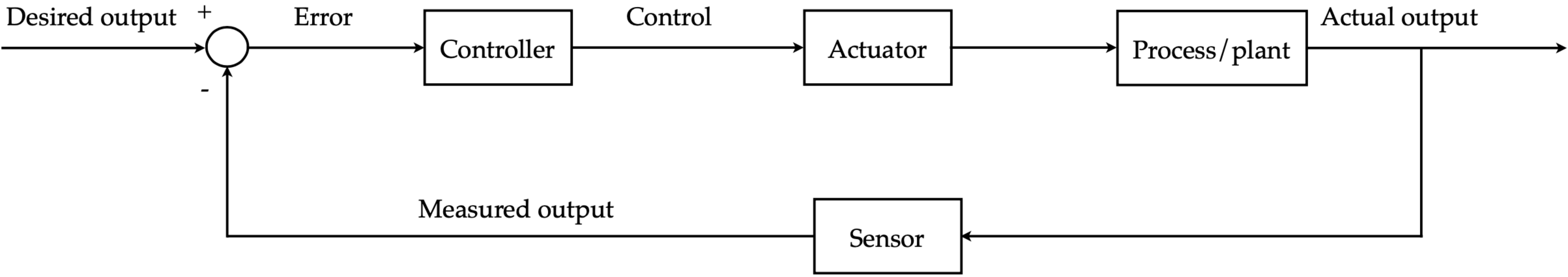
Block Diagrams

Open loop or feedforward control

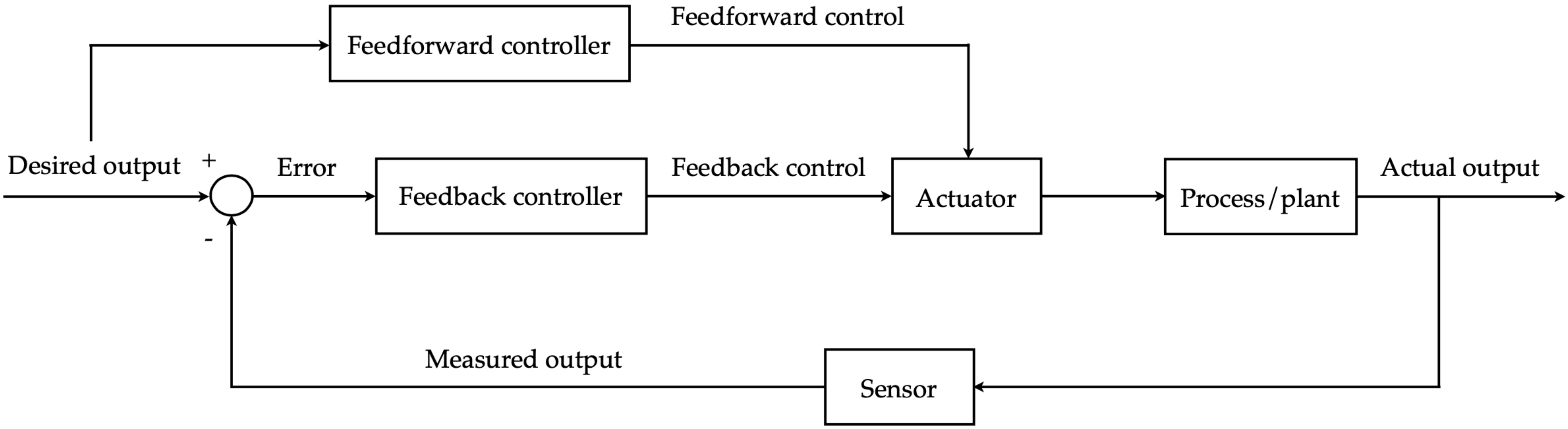


Closed loop or feedback control



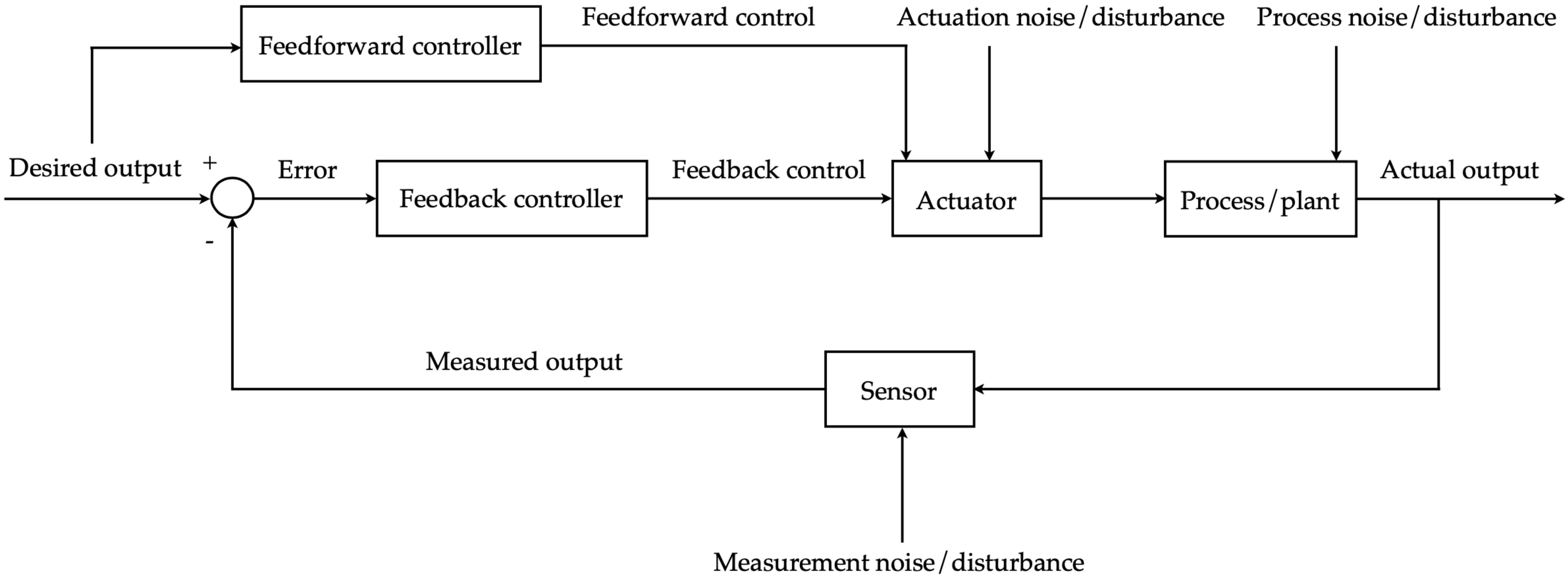
Block Diagrams: Mixed Feedforward-Feedback

Mixed feedforward-feedback control



Block Diagrams: with Noise/Disturbance

Mixed feedforward-feedback control



Open loop/Feedforward vs Closed loop/Feedback

Open loop (feedforward) control

is a “time-table”

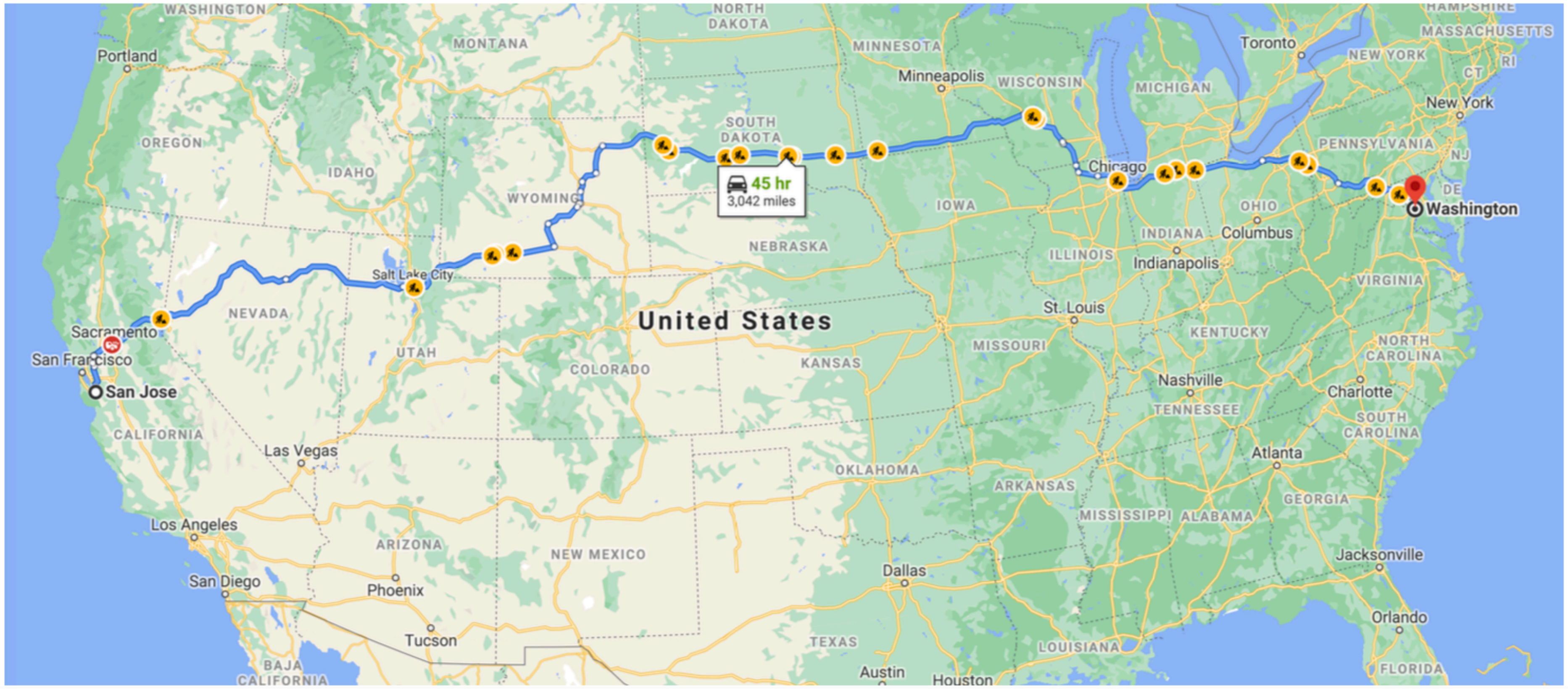
no real-time sensing

Closed loop (feedback) control

is an “output-table”

requires real-time sensing

Feedback is necessary to handle uncertainties: **a motivating example**



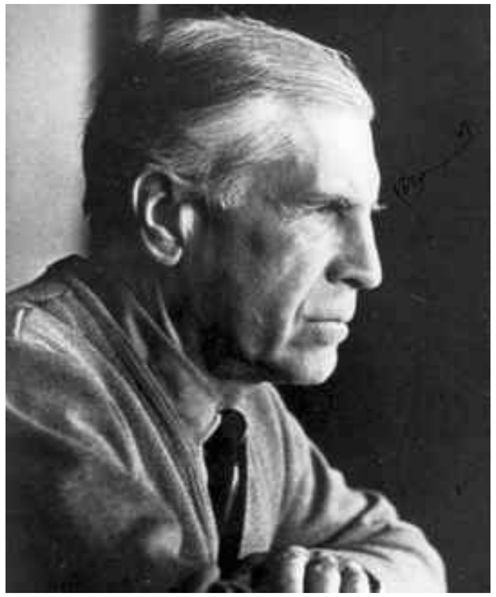
Control $\neq$ Controller

Control	**Controller**
is a signal	is an algorithm
is along an arrow (in the block diagram)	is a box (in the block diagram)
is also called “input” / “action”	is also called “policy” / “rule”

Control (signal) is the output of the controller block

Optimal Control: Deterministic

2 parallel strands of development during cold war



Pontryagin’s Maximum Principle (PMP)

- in Soviet Union (late 1950s)
- by **Lev Pontryagin** and his students
- presented in ICM 1958 at Edinburgh

Dynamic Programming (DP)

- in United States (late 1950s)
- by **Richard Bellman** at RAND corporation
- 1979 IEEE Medal of Honor for this work

Deterministic Optimal Control Problem (OCP)

Basic template:

$$\underset{\boldsymbol{u}(\cdot)\in\mathcal{U}([0,T])}{\text{minimize}} \quad \phi\left(T, \boldsymbol{x}(T)\right) + \int_0^T L(t, \boldsymbol{x}, \boldsymbol{u})\,\mathrm{d}t$$

$$\text{subject to} \quad \dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}),$$

$$\boldsymbol{\psi}\left(T, \boldsymbol{x}(T)\right) = \boldsymbol{0}$$

State $\boldsymbol{x} : [0, T] \mapsto \mathbb{R}^n$

Initial state $\boldsymbol{x}_0 := \boldsymbol{x}(0)$ given

Control $\boldsymbol{u} : [0, T] \mapsto \mathbb{R}^m$

Deterministic Optimal Control Problem (OCP)

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terminal cost

Lagrangian

$$\text{subject to} \quad \dot{\boldsymbol{x}} = \boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{u}),$$

controlled vector field

$$\boldsymbol{\psi}(T,\boldsymbol{x}(T)) = \boldsymbol{0}$$

terminal constraint

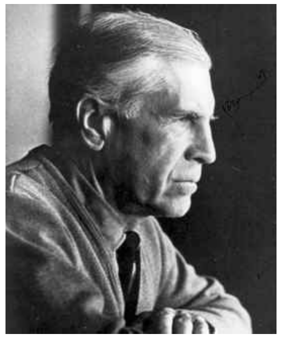
final time T may be fixed OR free

State $\boldsymbol{x} : [0,T] \mapsto \mathbb{R}^n$

Initial state $\boldsymbol{x}_0 := \boldsymbol{x}(0)$ given

Control $\boldsymbol{u} : [0,T] \mapsto \mathbb{R}^m$

Solution of Deterministic OCP by PMP



Necessary conditions for optimality

costate

Hamiltonian $$H(t, \boldsymbol{x}(t), \boldsymbol{\lambda}(t), \boldsymbol{u}(t)) := L + \langle \boldsymbol{\lambda}, \boldsymbol{f} \rangle$$

State equation $$\dot{\boldsymbol{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}}, \quad \boldsymbol{x}_0 \text{ known}$$

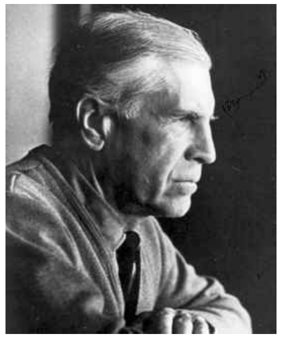
Costate equation $$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \boldsymbol{x}},$$

PMP $$\mathbf{0} = \frac{\partial H}{\partial \boldsymbol{u}},$$

constant Langrange multipliers

Transversality condition $$0 = \left\langle \left(\nabla_{\boldsymbol{x}} \phi + (\nabla_{\boldsymbol{x}} \boldsymbol{\psi})^\top \boldsymbol{\nu} - \boldsymbol{\lambda} \right) \Big|_{t=T}, \mathrm{d}\boldsymbol{x}(T) \right\rangle + \left(\frac{\partial \phi}{\partial t} + \left(\frac{\partial \boldsymbol{\psi}}{\partial t} \right)^\top \boldsymbol{\nu} + H \right) \Big|_{t=T} \mathrm{d}T$$

Geometric Interpretation of PMP



Optimal trajectory in cotangent bundle evolves as per a Hamiltonian vector field

$$\begin{pmatrix} \dot{\boldsymbol{x}}^{\text{opt}} \\ \dot{\boldsymbol{\lambda}}^{\text{opt}} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{I}_n \\ -\boldsymbol{I}_n & \mathbf{0} \end{bmatrix} \nabla_{\begin{pmatrix} \boldsymbol{x}^{\text{opt}} \\ \boldsymbol{\lambda}^{\text{opt}} \end{pmatrix}} H$$

Solution of Deterministic OCP by DP

Value function $V\left(t,\boldsymbol{x}\right) \in \mathcal{C}^{1,1}\left([0,T];\mathbb{R}^{n}\right)$

= Optimal cost-to-go from a generic $\left(t,\boldsymbol{x}\right)$

Hamilton-Jacobi-Bellman (HJB) PDE initial value problem

$$\frac{\partial V}{\partial t} + \underbrace{\inf_{\boldsymbol{u}\in\mathcal{U}}\left\{L(t,\boldsymbol{x},\boldsymbol{u}) + \langle\nabla_{\boldsymbol{x}}V,\boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{u})\rangle\right\}}_{\text{optimized Hamiltonian } H^{\text{opt}}(t,\boldsymbol{x},\nabla_{\boldsymbol{x}}V)} = 0$$

$$V\left(T,\boldsymbol{x}\right) = \phi\left(T,\boldsymbol{x}\right)$$

Solution of Deterministic OCP by DP

Value function $V\left(t,\boldsymbol{x}\right)\in\mathcal{C}^{1,1}\left([0,T];\mathbb{R}^{n}\right)$

= Optimal cost-to-go from a generic $(t,\boldsymbol{x})$

Hamilton-Jacobi-Bellman (HJB) PDE initial value problem

pointwise minimization over control, NOT over controllers

$$\frac{\partial V}{\partial t}+\underbrace{\inf_{\boldsymbol{u}\in\mathcal{U}}\left\{L(t,\boldsymbol{x},\boldsymbol{u})+\langle\nabla_{\boldsymbol{x}}V,\boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{u})\rangle\right\}}_{\text{optimized Hamiltonian } H^{\text{opt}}(t,\boldsymbol{x},\nabla_{\boldsymbol{x}}V)}=0$$

$$V\left(T,\boldsymbol{x}\right)=\phi\left(T,\boldsymbol{x}\right)$$

1st order nonlinear PDE IVP

Stochastic OCP

Basic template:

$$\underset{\boldsymbol{u}(\cdot)\in\mathcal{U}(0,T])}{\text{minimize}} \quad \mathbb{E}\left[\phi(T,\boldsymbol{x}(T)) + \int_0^T L(t,\boldsymbol{x},\boldsymbol{u})\mathrm{d}t\right]$$

$$\text{subject to} \quad \mathrm{d}\boldsymbol{x} = \underbrace{\boldsymbol{f}}_{\text{drift}}(t,\boldsymbol{x},\boldsymbol{u})\mathrm{d}t + \underbrace{\boldsymbol{G}}_{\text{diffusion}}(t,\boldsymbol{x},\boldsymbol{u})\mathrm{d}\boldsymbol{w}$$

State $\boldsymbol{x} : [0,T] \mapsto \mathbb{R}^n$

Initial state $\boldsymbol{x}_0 := \boldsymbol{x}(0)$ given

Control $\boldsymbol{u} : [0,T] \mapsto \mathbb{R}^m$

Solution of Stochastic OCP by DP

Value function $V(t,\boldsymbol{x}) \in \mathcal{C}^{1,2}\left([0,T];\mathbb{R}^{n}\right)$

= Optimal cost-to-go from a generic $(t,\boldsymbol{x})$

Hamilton-Jacobi-Bellman (HJB) PDE initial value problem

pointwise minimization over control, NOT over controllers

$$\frac{\partial V}{\partial t} + \underbrace{\inf_{\boldsymbol{u}\in\mathcal{U}}\left\{L(t,\boldsymbol{x},\boldsymbol{u}) + \langle\nabla_{\boldsymbol{x}}V, \boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{u})\rangle + \frac{1}{2}\langle\boldsymbol{G}\boldsymbol{G}^{\top}, \mathrm{Hess}_{\boldsymbol{x}}\,V\rangle\right\}}_{\text{optimized Hamiltonian } H^{\text{opt}}\,(t,\boldsymbol{x},\nabla_{\boldsymbol{x}}V,\mathrm{Hess}_{\boldsymbol{x}}\,V)} = 0$$

$$V\left(T,\boldsymbol{x}\right) = \phi\left(T,\boldsymbol{x}\right)$$

2nd order nonlinear PDE IVP

Numerical Computation: Deterministic OCP

Using PMP

Direct method: discretize the problem + call NLP solver

Indirect method: multiple shooting, not used in practice

Using DP

Method of characteristics, Hopf-Lax formula for structured problems

In general, curse-of-dimensionality

Numerical Computation: Stochastic OCP

Use logarithmic transform to convert the 2nd order nonlinear HJB PDE to 2nd order linear backward Kolmogorov PDE

Apply Feynman-Kac path integral computation for the transformed linear PDE IVP

Approximate conditional expectation of a nonlinear function

Good news: partially parallelizable

Not-so-good news: randomized function approximation of a deterministic function

Thank You

Next time: Optimal Transport and Schrödinger Bridge