Generalized Schrödinger Bridges

Abhishek Halder

Department of Aerospace Engineering, Iowa State University Department of Applied Mathematics, University of California Santa Cruz

Joint work with students and collaborators



Decision Superiority Seminar, Lawrence Livermore National Lab December 05, 2024



What is a bridge

A stochastic process connecting two given states a, b in a given deadline $[t_1, t_2]$



Source: https://medium.com/@christopher.tabori/between-certainty-and-chance-tracing-the-probability-distribution-of-paths-of-brownian-bridges-b1f97eba638d

What is a Schrödinger brid

Prior physics = E

1931 31.3. ÜBER DIE UMKEHRUNG DER NATURGESETZE VON E. SCHRÖDINGER SONDERAUSGABE AUS DEN SITZUNGSBEERCHTEN

DER PREUSSISCHEN AKADEMIE DER WISSENSCHAFTEN PHYS.-MATH. KLASSE. 1981. IX

[1931]

Sur la théorie relativiste de l'électron et l'interprétation de la mécanique quantique

par E. SCHRÖDINGER

I. — Introduction

J'ai l'intention d'exposer dans ces conférences diverses idées concernant la mécanique quantique et l'interprétation qu'on en donne généralement à l'heure actuelle ; je parlerai principalement de la théorie quantique relativiste du mouvement de l'électron. Autant que nous pouvons nous en rendre compte aujourd'hui, il semble à peu près sûr que la mécanique quantique de l'électron, sous sa forme idéale, *que nous ne possédons pas encore*, doit former un jour la base de toute la physique. A cet intérêt tout à fait général, s'ajoute, ici à Paris, un intérêt particulier : vous savez tous que les bases de la théorie moderne de l'électron ont été posées à Paris par votre célèbre compatriote Louis de BROGLIE.

[1932]



I. — Introducti

J'ai l'intention d'exposer dans ces conférences diverses idées concernant la mécanique quantique et l'interprétation qu'on en donne généralement à l'heure actuelle ; je parlerai principalement de la théorie quantique relativiste du mouvement de l'électron. Autant que nous pouvons nous en rendre compte aujourd'hui, il semble à peu près sir que la mécanique quantique de l'électron, sous sa forme idéale, que nous ne possidons pas encore, doit former un jour la base de toute la physique. A cet intérêt tout à fait général, s'ajoute, ici à Paris, un intérêt particulier : vous savez tous que les bases de la théorie moderne de l'électron out été posées à Paris par votre célèbre compatriote Louis de BROGLIE. Les recherches que je vais exposer ne forment nullement une théorie

nette et complètement achevée (¹). Le lien commun, un peu lâche d'ailleurs, qui les rattache les unes aux autres, la source commune dont elles dérivent, est le mécontentement que l'on éprouve quand on considère l'état présent de la théorie et surtout celui de l'*interprétation physique actuelle* de la mécanique quantique. Je voudrais

(1) Les mémoires originaux, qui forment la base de ces conférences, ont été publiés dans les limingribritide de pressistiches Aladomie de Wissenchaften, 1930, p. 418; 1931, pp. 63, 144, 8. Dans les pages qui vont suivre, quelques-uns des asp.ets des problèmes envisagés sont peutre un peu mieux précisés; on y trouvera également des résultats nouveaux (v. Notes I-III),

Annales de l'Institut H. Poincaré.





t = 0

1. E. G. S. S.

ÜBER DIE UMKEHRUNG DER NATURGESETZE vos e. schrödinger

PHYS.-MATH KLASSE, 1981, D

Find the most likely explanation of observation vs prior physics mismatch

3

Sur la théorie relativiste de l'électron l'interprétation de la mécanique quantique

E. SCHRÖDINGER

. . .

] si l'intention d'exposer dans ces contérences diverses idées concernant la micanique quantique el l'interprétation qu'on en donne génralement la l'heure actuelle ; je parleral principalement de la théorie quantique relativisté du nouvement de l'électron. Attant que nous pouvons mous en reedre compte aujourd'hui, il semble à peu près de neux de pouvoien par source, d'autor de la terre de la présique. A cet intérêt tout à fait général, s'ajoute, ici à Paris, un intérêt particulier : vous saver tous que le base de la théorie moderne de l'électron out dé posées à Paris par votre célèbre compatriote. Louis de l'électron out de posées à Paris par votre célèbre compatriote. Louis de l'électron out de posées à Paris par votre célèbre compatriote. Louis de l'électron out de posées à Paris par votre célèbre compatriote. Louis de l'électron out de posées à Paris par votre célèbre compatriote. Louis de l'électron out de posées à Paris par votre célèbre compatriote. Louis de l'électron out de posées à Paris par votre célèbre compatriote. Louis de l'électron out de posées à Paris par votre célèbre compatriote. Louis de l'électron out de posées à Paris par votre célèbre compa-

Les recherches què je vais exposer ne forment nuflement une bénéri nette et complétenent achevie (P). Le lies commun. en peut liéé d'ailleurs, qui les rattache les unes aux autres, la source commun dont elles détivent, est le mécontentement que les roin éprove quan on considére l'état présent de la théorie et surtout celui de l'ér terpérabilité physique adaulté de la méconique quantique, le vondir (e) Les ménéres orgèneux, qui fernes la las consideres, est és padré dun l Sampérabel de possibilité d'articue de l'érreste de la ferreste de la définité de la méconique de la méconique quantique, le vondir (e) Les ménéres orgèneux qui fernes la las consideres, est és padré dun l Sampérabel de possibilité d'articue de l'érreste de la source de l'érreste de la source de l'érreste de la de l'érreste

What is a Schrödinger bridge



Most parsimonious correction of prior physics

Constrained maximum likelihood problem on measure-valued paths

What is a Schrödinger bridge

Schrödinger bridge as large deviation principle: Sanov's theorem [1957]

 $\lim_{N\uparrow\infty}\log(ext{empiricial prob}_N ext{ under }\mathbb{W}\in\Pi_{01}) \asymp - \inf_{\mathbb{P}\in\Pi_{01}} D_{ ext{KL}}(\mathbb{P}\parallel\mathbb{W})$ KL div as rate function

Schrödinger bridge as stochastic optimal control: [1990s]

$$egin{aligned} & ext{minimize} \, \mathbb{E}\left[\int_{t_0}^{t_1} rac{1}{2} \|oldsymbol{u}(t,oldsymbol{x}_t^u)\|_2^2 \mathrm{d}t
ight] \ & ext{subject to} \ & ext{d}oldsymbol{x}_t^u = oldsymbol{u}(t,oldsymbol{x}_t^u) \mathrm{d}t + \mathrm{d}oldsymbol{w}_t \ & oldsymbol{x}_t^u(t=t_0) \sim
ho_0, \quad oldsymbol{x}_t^u(t=t_1) \sim
ho_1 \end{aligned}$$

What is a Schrödinger bridge

Schrödinger bridge as large deviation principle: Sanov's theorem [1957]

 $\lim_{N\uparrow\infty}\log(ext{empiricial prob}_N ext{ under }\mathbb{W}\in\Pi_{01}) \asymp - \inf_{\mathbb{P}\in\Pi_{01}} D_{ ext{KL}}(\mathbb{P}\parallel\mathbb{W})$ KL div as rate function

Schrödinger bridge as stochastic optimal control: [1990s]

$$\begin{split} & \underset{\boldsymbol{u} \in \mathcal{U}}{\text{minimize}} \mathbb{E} \left[\int_{t_0}^{t_1} \frac{1}{2} \| \boldsymbol{u}(t, \boldsymbol{x}_t^u) \|_2^2 \mathrm{d}t \right] \\ & \text{subject to} \\ & \mathbf{d} \boldsymbol{x}_t^u = \boldsymbol{u}(t, \boldsymbol{x}_t^u) \mathrm{d}t + \mathrm{d} \boldsymbol{w}_t \\ & \boldsymbol{x}_t^u(t = t_0) \sim \rho_0, \quad \boldsymbol{x}_t^u(t = t_1) \sim \rho_1 \end{split}$$

Resurgence of Schrödinger bridge in ML/AI

Learn most likely progression of medical condition



Learn joint stochastic time-varying hardware resource availability



G.A. Bondar, R. Gifford, L.T.X. Phan, and A.H., *ACC* 2024, *arXiv*:2310.00604 *arXiv*:2405.12463

7

Connections with graphical models

Learn most likely progression of medical condition



Learn joint stochastic time-varying hardware resource availability



G.A. Bondar, R. Gifford, L.T.X. Phan, and A.H., *ACC* 2024, *arXiv*:2310.00604 *arXiv*:2405.12463

Resurgence of Schrödinger bridge in ML/AI

Diffusion models for generative AI

Source: https://yang-song.net/blog/2021/score/



UAI 2023

Aligned Diffusion Schrödinger Bridges

Vignesh Ram Somnath^{*1,2} Matteo Pariset*1,3 Ya-Ping Hsieh¹ Maria Rodriguez Martinez² Andreas Krause¹ Charlotte Bunne¹ ¹Department of Computer Science, ETH Zürich ²IBM Research Zürich ³Department of Computer Science, EPFL alignment ligand π^{\star} protein Brownian bridge $\mathrm{d}X_t = g_t^2 \frac{\mathbf{x}_1 - X_t}{\varphi} \,\mathrm{d}t + g_t \,\mathrm{d}\mathbb{W}_t$ ligand receptor unbound bound SBALIGN receptor $\mathrm{d}X_{t} = g_{t}^{2} \left[b_{t}^{\theta} \left(X_{t} \right) + \nabla \log h_{t}^{\theta} \left(X_{t} \right) \right] \mathrm{d}t + g_{t} \, \mathrm{d}\mathbb{W}_{t}$

NeurIPS 2021

Diffusion Schrödinger Bridge with Applications to Score-Based Generative Modeling

Valentin De Bortoli Department of Statistics, University of Oxford, UK

Jeremy Heng ESSEC Business School, Singapore James Thornton Department of Statistics, University of Oxford, UK

Arnaud Doucet Department of Statistics, University of Oxford, UK

NeurIPS 2024

Diffusion Schrödinger Bridge Matching



9

This talk: generalized Schrödinger bridges

#1. general controlled dynamics

#2. extra sample path constraints

#3. additive state cost

Generalization #1: more general controlled dyn.



Motivating applications

Distribution ~ **Probability**

bering based Machensal bering parachutes is "Historical in the second s concern with high Mach number deployments, utecdeployments in cregions where the heating is onhinofeinerteald sEstern a Arexampler Gillishis libe Le Somtewhere between two 37d 2hFee.

brid welge and many the set was developed that are small differed. Mary Phoenix Europe, on the oner Hand, used emulting of these process of the set of the set of the set of the naviewer these process of the set of the set of the set of the set of the naviewer the set of the a Clarin Good and Canada eaw hyd shup any proposed network and a contract in the soliton in thorizontal used a

wes-a pure range trigger. When originally proposed, MSL (known then as Mars Smalt) above the *MOLA* reference areoid. It was argued no parachute inflation put the parachute at a hi aute during paragring innances paragring paragring at a tighter isk of Fail used due of the factors: (1) the paragring preserves for each effective paragring of the star log the star log the star of the tight of the star log the star of the star

Distribution ~ **Population**

Dynamic shaping of swarms

 $\leq \mu$

Feedback sync. and desync. of neuronal population

Control of ensemble

Generalized Schrödinger bridge

Diffusion tensor: $D := GG^{\top}$

Hessian operator w.r.t. state: Hess

$$egin{aligned} &\inf_{(
ho,oldsymbol{u})\in\mathcal{P}_{01} imes\mathcal{U}} \int_{\mathbb{R}^n} \int_0^1 igg(rac{1}{2}\|oldsymbol{u}(t,oldsymbol{x}_t^u)\|_2^2 + q(t,oldsymbol{x}_u^u)igg)
ho(t,oldsymbol{x}_t^u) \mathrm{d}t\,\mathrm{d}oldsymbol{x}_t^u \ & ext{subject to} \ &rac{\partial
ho}{\partial t} +
abla \cdot ((oldsymbol{f}+oldsymbol{B}oldsymbol{u})
ho) = \Delta_{oldsymbol{D}}
ho \ &
ho(t=0,oldsymbol{x}_0^u) =
ho_0, \quad
ho(t=1,oldsymbol{x}_1^u) =
ho_1 \end{aligned}$$

Controlled Fokker-Planck or Kolmogorov's forward PDE

Zero process noise --> Generalized OMT

Diffusion tensor: $D := GG^{\top}$

$$P_{o1} := \left\{ \begin{array}{c} & & \\ P_{o} & & \\ P_{o} & & \\ \end{array} \right\}$$

Hessian operator w.r.t. state: Hess

$$egin{aligned} &\inf_{(
ho,oldsymbol{u})\in\mathcal{P}_{01} imes\mathcal{U}} \int_{\mathbb{R}^n} \int_0^1 igg(rac{1}{2}\|oldsymbol{u}(t,oldsymbol{x}_t^u)\|_2^2 + q(t,oldsymbol{x}_u^u)igg)
ho(t,oldsymbol{x}_t^u) \mathrm{d}t\,\mathrm{d}oldsymbol{x}_t^u \ & ext{subject to} \ &rac{\partial
ho}{\partial t} +
abla \cdot ((oldsymbol{f}+oldsymbol{B}oldsymbol{u})
ho) = \Delta_{oldsymbol{D}}
ho \ &
ho(t=0,oldsymbol{x}_0^u) =
ho_0, \quad
ho(t=1,oldsymbol{x}_1^u) =
ho_1 \end{aligned}$$

Controlled Liouville PDE

Necessary Conditions of Optimality (Assuming $G \equiv B$)

Coupled nonlinear PDEs + linear boundary conditions

Controlled Fokker-Planck or Kolmogorov's forward PDE

$$\frac{\partial \rho^{\text{opt}}}{\partial t} + \nabla \cdot \left(\left(f + D \nabla \psi \right) \rho^{\text{opt}} \right) = \Delta_D \rho^{\text{opt}}$$

Hamilton-Jacobi-Bellman-like PDE

$$\frac{\partial \psi}{\partial t} + \langle \nabla \psi, f \rangle + \langle D, \text{Hess}(\psi) \rangle + \frac{1}{2} \langle \nabla \psi, D \nabla \psi \rangle = q$$

Boundary conditions:

$$\rho^{\text{opt}}(\cdot, t = 0) = \rho_0, \quad \rho^{\text{opt}}(\cdot, t = 1) = \rho_1$$

Optimal control: $u^{\text{opt}} = B^{\top} \nabla \psi$

Feedback synthesis via the Schrödinger factors

Hopf-Cole a.k.a. Fleming's logarithmic transform:

 $(\rho^{\text{opt}}, \psi) \mapsto (\widehat{\varphi}, \varphi) - \text{Schrödinger factors} \quad \widehat{\varphi}(x, t) = \rho^{\text{opt}}(x, t) \exp(-\psi(x, t))$

$$\varphi(\boldsymbol{x},t) = \exp\left(\psi\left(\boldsymbol{x},t\right)\right)$$

2 coupled nonlinear PDEs \rightarrow boundary-coupled linear PDEs!!

Uncontrolled forward-backward advection-reaction-diffusion PDEs:

$$egin{aligned} & \mathcal{L}_{ ext{forward}}\widehat{arphi} \ & \widehat{arphi} = -
abla \cdot (\widehat{arphi}oldsymbol{f}) + \Delta_{oldsymbol{D}}\widehat{arphi} - q\widehat{arphi}, & \widehat{arphi}_0 arphi_0 =
ho_0 \ & \mathcal{L}_{ ext{backward}}arphi \ & rac{\partial arphi}{\partial t} = -\langle
abla arphi, oldsymbol{f} - \Delta_{oldsymbol{D}}\widehat{arphi} + q\widehat{arphi}, & \widehat{arphi}_1 =
ho_1 \end{aligned}$$

Optimal controlled joint state PDF:

Optimal control:

$$\rho^{\text{opt}}(\boldsymbol{x},t) = \widehat{\varphi}(\boldsymbol{x},t)\varphi(\boldsymbol{x},t)$$
$$\boldsymbol{u}^{\text{opt}}(\boldsymbol{x},t) = 2\boldsymbol{B}^{\top}\nabla_{\boldsymbol{x}}\log\varphi(\boldsymbol{x},t)$$

What exactly are Schrödinger factors?

Consider Schrödinger's original case: f = 0, B = D = I

Classical:

$$ho^{ ext{opt}}(oldsymbol{x},t) = arphi(oldsymbol{x},t) \widehat{arphi}(oldsymbol{x},t)$$

$$egin{aligned} &\left(rac{\partial}{\partial t}+rac{1}{2}\Delta-q
ight)arphi=0\ &\left(rac{\partial}{\partial t}-rac{1}{2}\Delta+q
ight)\widehat{arphi}=0 \end{aligned}$$

 \mathbf{i}

[Backward reaction-diffusion PDE]

[Forward reaction-diffusion PDE]

Qu

antum:
$$ho^{\text{opt}}(\boldsymbol{x},t) = \Psi(\boldsymbol{x},t)\widehat{\Psi}(\boldsymbol{x},t)$$
 [Born's relation]
wave function
 $\sqrt{-1}\frac{\partial}{\partial t} + \frac{1}{2}\Delta - q \Psi = 0$ [Schrödinger PDE]
 $\sqrt{-1}\frac{\partial}{\partial t} - \frac{1}{2}\Delta + q \widehat{\Psi} = 0$ [Adjoint Schrödinger PDE]

This recursion is contractive in the Hilbert's projective metric!!

$$\mathbf{n}_{\mathbf{r}}(\mathbf{r}) \oslash (\mathbf{n}_{\mathbf{r}})^{18} \qquad \mathbf{n}_{\mathbf{r}}(\mathbf{r}) \oslash (\mathbf{r})^{18}$$

Feedback Density Control: $f \equiv 0, B = G \equiv I, q \equiv 0$

Zero prior dynamics

Feedback Density Control: $f \equiv Ax, B = G, q \equiv 0$

Feedback Density Control: Nonlinear Grad. Drift

Uncontrolled joint PDF evolution:

Optimal controlled joint PDF evolution:

K.F. Caluya, and A.H., *TAC* 2022 K.F. Caluya, and A.H., *TAC* 2019

Feedback Density Control: Mixed Conservative-Dissipative Drift

K.F. Caluya, and A.H., TAC 2019

Application: Multi-lane Automated Driving

Application: Multi-lane Automated Driving

Control non-affine generalized Schrödinger bridge

No state cost: q = 0

Controlled SDE:

$$d\boldsymbol{x}_t^{\boldsymbol{u}} = \boldsymbol{f}(t, \boldsymbol{x}_t^{\boldsymbol{u}}, \boldsymbol{u}) dt + \sqrt{2} \boldsymbol{g}(t, \boldsymbol{x}_t^{\boldsymbol{u}}, \boldsymbol{u}) d\boldsymbol{w}_t$$

Controlled diffusion tensor: $\boldsymbol{G} := \boldsymbol{g} \boldsymbol{g}^{\top} \succeq \boldsymbol{0}$

Dispersed particles

Ordered structure

Conditions for optimality: system of m + 2 coupled PDEs

$$\left(\begin{array}{c} \displaystyle \frac{\partial \psi}{\partial t} = \displaystyle \frac{1}{2} \| \boldsymbol{u}_{\mathrm{opt}} \|_{2}^{2} - \langle \nabla_{\boldsymbol{x}} \psi, \boldsymbol{f} \rangle - \langle \boldsymbol{G}, \mathrm{Hess}(\psi) \rangle \\ \displaystyle \frac{\partial \rho_{\mathrm{opt}}^{\boldsymbol{u}}}{\partial t} = \displaystyle -\nabla \cdot \left(\rho_{\mathrm{opt}}^{\boldsymbol{u}} \boldsymbol{f} \right) + \Delta_{\boldsymbol{G}} \rho_{\mathrm{opt}}^{\boldsymbol{u}} \\ \boldsymbol{u}_{\mathrm{opt}} = \nabla_{\boldsymbol{u}_{\mathrm{opt}}} \left(\langle \nabla_{\boldsymbol{x}} \psi, \boldsymbol{f} \rangle + \langle \boldsymbol{G}, \mathrm{Hess}(\psi) \rangle \right) \\ \rho_{\mathrm{opt}}^{\boldsymbol{u}}(0, \boldsymbol{x}) = \rho_{0}, \quad \rho_{\mathrm{opt}}^{\boldsymbol{u}}(T, \boldsymbol{x}) = \rho_{T} \end{array} \right) \\ \begin{array}{c} \mathsf{Known} \boldsymbol{f}, \boldsymbol{g} \\ \mathsf{I. Nodozi, J.O'Leary, A. Mesbah, \\ \mathsf{and A.H., ACC 2023} \\ \mathfrak{V} = 2024 \ 0. \ \mathsf{Hugo Schuck Best Application Paper Award} \\ \mathbf{Data-driven} \boldsymbol{f}, \boldsymbol{g} \\ \mathsf{I. Nodozi, C. Yan, M. Khare, A.H., \\ \mathsf{and A. Mesbah, TCST 2024} \end{array} \right)$$

Control non-affine generalized Schrödinger bridge

$$\mathscr{L}_{\mathscr{N}} = \mathscr{L}_{\psi} + \mathscr{L}_{\rho^{u}} + \mathscr{L}_{\pi^{\mathrm{opt}}} + \mathscr{L}_{\rho^{u}_{0}} + \mathscr{L}_{\rho^{\mu}_{T}}$$

Benchmark controlled self-assembly system: [Y Xue, et al, IEEE Trans. Control Sys. Technology, 2014]

Generalization #2: hard sample path constraints

Main idea: path constraints \sim reflected Itô SDEs modify the controlled sample path dynamics to

$$\mathrm{d} \mathbf{x}_t^{\boldsymbol{u}} = \{ f(t, \mathbf{x}_t^{\boldsymbol{u}}) + \boldsymbol{B}(t) \boldsymbol{u}(t, \mathbf{x}_t^{\boldsymbol{u}}) \} \mathrm{d} t + \sqrt{2\theta} \boldsymbol{G}(t) \mathrm{d} \boldsymbol{w}_t + \boldsymbol{n}(\mathbf{x}_t^{\boldsymbol{u}}) \mathrm{d} \boldsymbol{\gamma}_t$$

 $x_t^u \in \overline{\mathcal{X}} := \mathcal{X} \cup \partial \mathcal{X}$, closure of connected smooth \mathcal{X} *n* is inward unit normal to the boundary $\partial \mathcal{X}$ γ_t is minimal local time stochastic process

Reflected bridge: Schrödinger factor recursion

K.F. Caluya, and A.H., ACC 2021

Reflected bridge: numerics with ∇V **drift**

$$V(x_1, x_2) = (x_1^2 + x_2^3)/5, \quad \overline{\mathcal{X}} = [-4, 4]^2$$

Optimal controlled state PDFs:

Uncontrolled state PDFs:

Generalization #3: additive state cost ($q \neq 0$)

Question. Where does state cost come from?

Answer 1. From extra regularization (e.g., classical LQ optimal control)

Answer 2. Problem reformulation (push dynamical nonlinearity to Lagrangian)

Probabilistic Lambert Problem: Connections with Optimal Mass Transport, Schrödinger Bridge and Reaction-Diffusion PDEs*

Alexis M.H. Teter † , Iman Nodozi ‡ , and Abhishek Halder §

A.M. Teter, I. Nodozi, and A.H., *arXiv*:2401.07961

Schrödinger bridge with quadratic state cost: $q(x) = x^{\top}Qx, Q \ge 0$

Solution:

$$ho^{ ext{opt}}(oldsymbol{x},t) = arphi(oldsymbol{x},t) \widehat{arphi}(oldsymbol{x},t)$$

$$egin{aligned} &\left(rac{\partial}{\partial t}+rac{1}{2}\Delta-q
ight)arphi=0\ &\left(rac{\partial}{\partial t}-rac{1}{2}\Delta+q
ight)\widehat{arphi}=0 \end{aligned}$$

[Backward reaction-diffusion PDE]

[Forward reaction-diffusion PDE]

Schrödinger bridge with quadratic state cost: $q(x) = x^{\top}Qx, Q \ge 0$

We know:

$$ho^{ ext{opt}}(oldsymbol{x},t) = arphi(oldsymbol{x},t) \widehat{arphi}(oldsymbol{x},t)$$

[Backward reaction-diffusion PDE]

[Forward reaction-diffusion PDE]

Need kernel/Green's function $\kappa(0, x; t, y)$

for IVP solutions to use in Schrödinger factor recursion:

$$rac{\partial \widehat{arphi}}{\partial t} = \underbrace{\mathcal{L}_{ ext{forward}}}_{(\Delta - oldsymbol{x}^ op oldsymbol{Q} x)} \widehat{arphi}, \quad \widehat{arphi}(t=0,oldsymbol{x}) = \widehat{arphi}_0 \quad \Leftrightarrow \quad \widehat{arphi}(oldsymbol{x},t) = \int_{\mathbb{R}^n} oldsymbol{\kappa}(0,oldsymbol{x};t,oldsymbol{z}) \widehat{arphi}_0(oldsymbol{z}) \mathrm{d}oldsymbol{z}$$

Schrödinger bridge with quadratic state cost: $q(x) = x^{\top}Qx, Q > 0$

Thm. Eig. decomposition: $\boldsymbol{Q} = \boldsymbol{V} \boldsymbol{D} \boldsymbol{V}^{\top}$

Then, $\widehat{\varphi}(\boldsymbol{x},t) = \eta(\boldsymbol{y} = \boldsymbol{V}\boldsymbol{x},t)$ where $\eta(\boldsymbol{y},t) = \int_{\mathbb{R}^n} \kappa(0,\boldsymbol{y};t,\boldsymbol{z})\eta_0(\boldsymbol{z})\mathrm{d}\boldsymbol{z}$

and
$$\kappa(0, \boldsymbol{y}; t, \boldsymbol{z}) = rac{(\det(\boldsymbol{D}))^{1/4}}{\sqrt{(2\pi)^n \det\left(\sinh(2t\sqrt{\boldsymbol{D}})\right)}} \exp\left(-rac{1}{2}(\boldsymbol{y} \quad \boldsymbol{z})\boldsymbol{M}\begin{pmatrix}\boldsymbol{y}\\\boldsymbol{z}\end{pmatrix}\right)$$

$$oldsymbol{M} := egin{bmatrix} oldsymbol{D}^{1/4} & oldsymbol{0}_{n imes n} \ oldsymbol{0}_{n imes n} & oldsymbol{D}^{1/4} \end{bmatrix} oldsymbol{M}_1 M_2 egin{bmatrix} oldsymbol{D}^{1/4} & oldsymbol{0}_{n imes n} \ oldsymbol{D}^{1/4} \end{bmatrix}, \ oldsymbol{M}_1 := egin{bmatrix} \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & -oldsymbol{I}_n \ -oldsymbol{I}_n & \cosh\left(2t\sqrt{oldsymbol{D}}
ight) \end{bmatrix}, \ oldsymbol{M}_2 := egin{bmatrix} \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & oldsymbol{0} \ oldsymbol{0} & \cosh\left(2t\sqrt{oldsymbol{D}}
ight) \end{bmatrix}, \ oldsymbol{M}_2 := egin{bmatrix} \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & oldsymbol{0} \ oldsymbol{0} & \cosh\left(2t\sqrt{oldsymbol{D}}
ight) \end{bmatrix}, \ oldsymbol{M}_2 := egin{bmatrix} \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & oldsymbol{0} \ oldsymbol{0} & \cosh\left(2t\sqrt{oldsymbol{D}}
ight) \end{bmatrix}, \ oldsymbol{M}_2 := egin{bmatrix} \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & oldsymbol{0} \ oldsymbol{0} & \cosh\left(2t\sqrt{oldsymbol{D}}
ight) \end{bmatrix}, \ oldsymbol{M}_2 := egin{bmatrix} \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & oldsymbol{0} \ oldsymbol{0} & \cosh\left(2t\sqrt{oldsymbol{D}}
ight) \end{bmatrix}, \ oldsymbol{M}_2 := egin{bmatrix} \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & oldsymbol{0} \ oldsymbol{0} & \cosh\left(2t\sqrt{oldsymbol{D}}
ight) \end{bmatrix}, \ oldsymbol{M}_2 := egin{bmatrix} \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & oldsymbol{0} \ oldsymbol{0} & \cosh\left(2t\sqrt{oldsymbol{D}}
ight) \end{bmatrix}, \ oldsymbol{M}_2 := egin{bmatrix} \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & \cosh\left(2t\sqrt{oldsymbol{D}}
ight) \end{bmatrix}, \ oldsymbol{M}_2 := egin{bmatrix} \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & \cosh\left(2t\sqrt{oldsymbol{D}}
ight) \end{bmatrix}, \ oldsymbol{M}_2 := egin{bmatrix} \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & \cosh\left(2t\sqrt{oldsymbol{D}}
ight) \end{bmatrix}, \ oldsymbol{M}_2 := egin{bmatrix} \cosh\left(2t\sqrt{oldsymbol{D}}
ight) & \cosh\left(2t\sqrt{oldsymbol{D}}$$

Q = I recovers the multivariate Mehler kernel in quantum harmonic oscillator

Schrödinger bridge with quadratic state cost: $q(x) = x^{\top}Qx, Q \ge 0$

Schrödinger bridge in 1D: with vs without quadratic state cost

A.M. Teter, W. Wang, and A.H., arXiv:2406.00503 arXiv:2407.15245

Lambert's Problem

3D position coordinate $\boldsymbol{r} := \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

Find velocity control policy $\dot{\boldsymbol{r}} := \boldsymbol{v}(t, \boldsymbol{r})$ such that

$$m{\ddot{r}} = -
abla_{m{r}} V(m{r}), \hspace{0.2cm} \left(m{r}(t=t_0)=m{r}_0(ext{ given }), \hspace{0.2cm} m{r}(t=t_1)=m{r}_1(ext{ given })
ight)$$

Lambert's Problem

3D position coordinate $\boldsymbol{r} := \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^3$ ODE is 2nd order but endpoint boundary conditions are first order **~>** partially specified TPBVP

Find velocity control policy $\dot{\boldsymbol{r}} := \boldsymbol{v}(t, \boldsymbol{r})$ such that

 $m{\ddot{r}} = abla_{m{r}}V(m{r}), \hspace{0.2cm} m{r}(t=t_0) = m{r}_0(ext{ given }), \hspace{0.2cm} m{r}(t=t_1) = m{r}_1(ext{ given }) m{r}_1(ext{ given })$

Probabilistic Lambert's Problem

3D position coordinate $\boldsymbol{r} := \left(\begin{array}{c} y \end{array} \right)$

$$\in egin{pmatrix} x \ y \ z \end{pmatrix} \in \mathbb{R}^3$$

1 ~ \

Find velocity control policy $\dot{\boldsymbol{r}} := \boldsymbol{v}(t, \boldsymbol{r})$ such that

 $\ddot{m{r}} = abla_{m{r}} V(m{r}), \ \left(m{r}(t=t_0) \sim
ho_0 \ (ext{given}), \ \ m{r}(t=t_1) \sim
ho_1 \ (ext{given})
ight)$

Probabilistic Lambert problem is OMT

$$egin{argsinf} & \displaystyle \int_{t_0}^{t_1} \mathbb{E}_{
ho} igg[rac{1}{2} \|oldsymbol{v}\|_2^2 - V(oldsymbol{r}) igg] \,\mathrm{d}t \ oldsymbol{\dot{r}} = oldsymbol{v}, & oldsymbol{r}(t = t_0) \sim
ho_0 \ (ext{given}), & oldsymbol{r}(t = t_1) \sim
ho_1 \ (ext{given}) \ & igg] \ & \displaystyle \inf_{(
ho,oldsymbol{v})} & \displaystyle \int_{t_0}^{t_1} \mathbb{E}_{
ho} igg[rac{1}{2} \|oldsymbol{v}\|_2^2 - V(oldsymbol{r}) igg] \,\mathrm{d}t \ & \displaystyle rac{\partial
ho}{\partial t} +
abla_{oldsymbol{r}} \cdot (
hooldsymbol{v}) = oldsymbol{0}, & oldsymbol{r}(t = t_1, \cdot) =
ho_1 \ \end{array}$$

Connection to SBP with state cost

$$rginf_{(
ho,oldsymbol{v})\in\mathcal{P}_{01} imes\mathcal{V}} \int_{t_0}^{t_1} \int_{\mathbb{R}^n} \left(rac{1}{2}|oldsymbol{v}|^2 - V(oldsymbol{x})
ight)
ho(oldsymbol{x},t) \, doldsymbol{x} dt$$

$$egin{aligned} &rac{\partial
ho}{\partial t} +
abla_{m{r}} \cdot (
ho m{v}) = 0, \ - extsf{Liouville PDE} \ &
ho(t = t_0, \cdot) =
ho_0, \quad
ho(t = t_1, \cdot) =
ho_t \end{aligned}$$

Lambertian SBP (L-SBP)

Numerical Case Study

Univariate marginals for optimally controlled joint PDFs

Outlook

- Theory and applications of Schrödinger bridge are undergoing rapid developments

- Lots of mathematics, algorithms, and applications to be done

- Growing interdisciplinary community

- Strong intersections with: physics, control, probability/statistics, PDE, AI/ML, information theory, signal processing, robotics, biology

Thank You

