A Distributed Algorithm for Wasserstein Proximal Operator Splitting

Abhishek Halder

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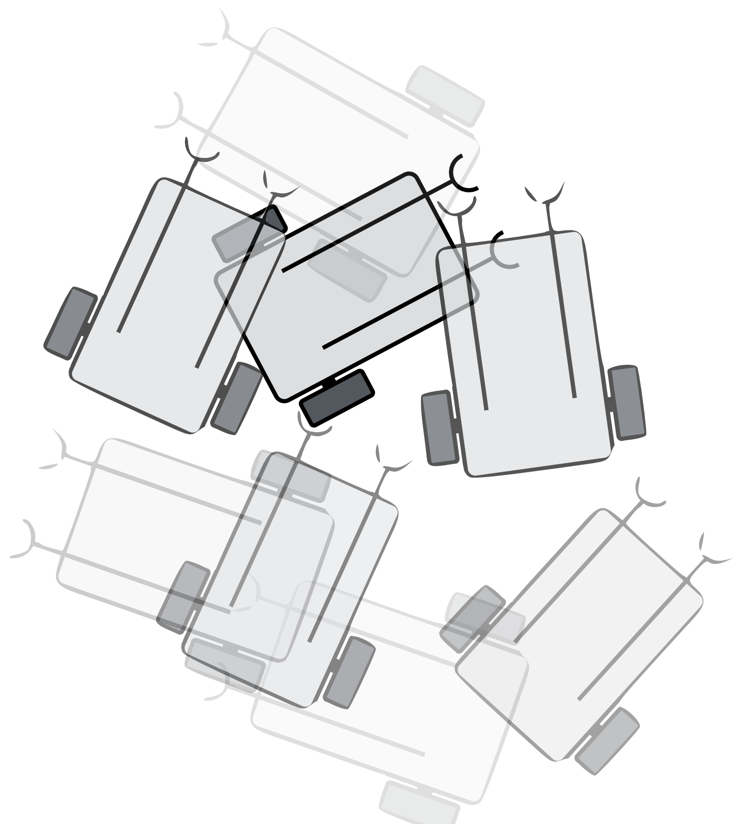
Joint work with I. Nodozi, A.M. Teter (UC Santa Cruz)

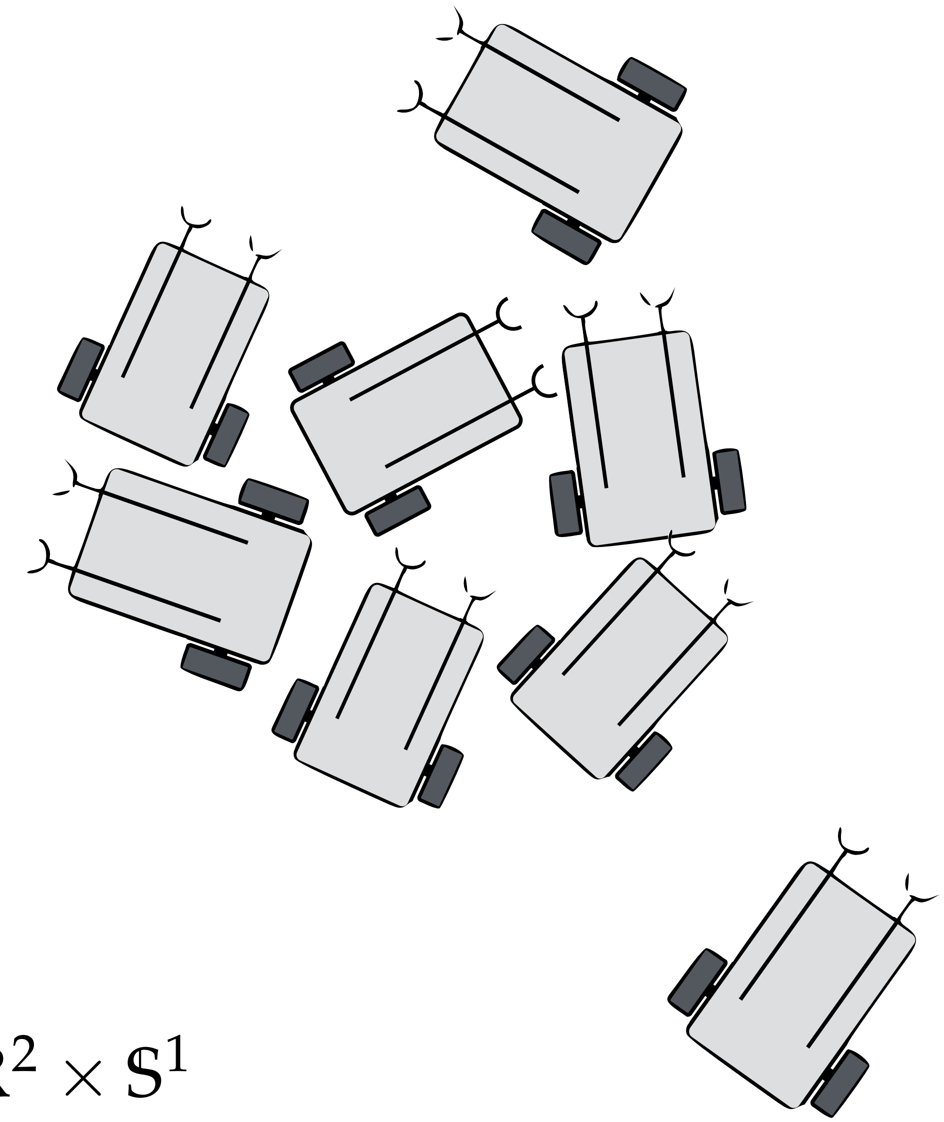
Decision and Control Seminar
Coordinated Science Lab
University of Illinois Urbana-Champaign, November 01, 2023

Topic of this talk

Optimization over the space of measures or distributions

Probability Distribution Population Distribution





$$\boldsymbol{x}(t) = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \in \mathcal{X} \equiv \mathbb{R}^2 \times \mathbb{S}^1$$

$$\rho\left(\boldsymbol{x}, t\right) : \mathcal{X} \times [0, \infty) \mapsto \mathbb{R}_{\geq 0}$$

measure = mass density function

$$\int_{\mathcal{X}} \mathrm{d}\mu = \int_{\mathcal{X}} \rho \, \mathrm{d}\boldsymbol{x} = 1 \quad \text{for all } t \in [0, \infty)$$

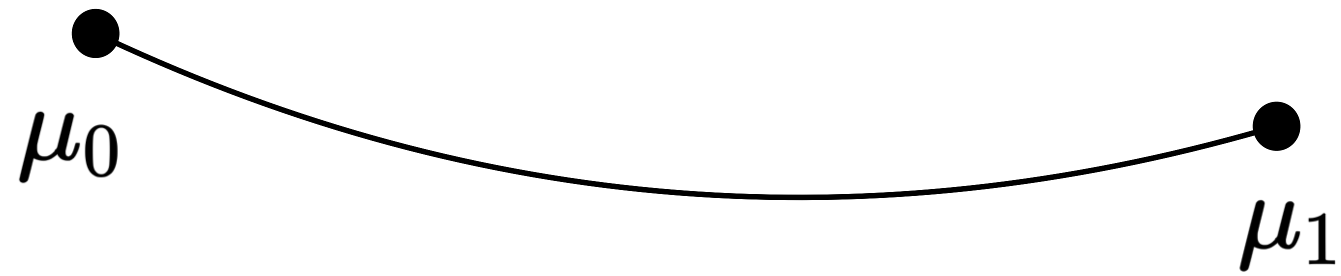
Geometry on the Space of Prob. Measures

Numer. Math. (2000) 84: 375–393
Digital Object Identifier (DOI) 10.1007/s002119900117

Numerische Mathematik

A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem

Jean-David Benamou[1], Yann Brenier[2]

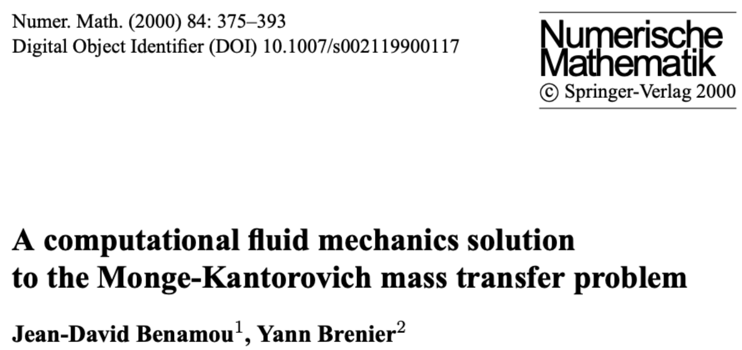


2-Wasserstein distance metric

$$W(\mu_0, \mu_1) := \left(\inf_{\mu, \boldsymbol{v}} \left\{ \frac{1}{2} \int_0^1 \int_{\mathcal{X}} \|\boldsymbol{v}\|^2 \mathrm{d}\mu \, \mathrm{d}t \right\} \right)^{1/2}$$

$$\text{subject to} \quad \frac{\partial \mu}{\partial t} = -\nabla \cdot (\mu \boldsymbol{v}), \;\; \mu(t=0,\cdot) = \mu_0, \;\; \mu(t=1,\cdot) = \mu_1$$

Geometry on the Space of Prob. Measures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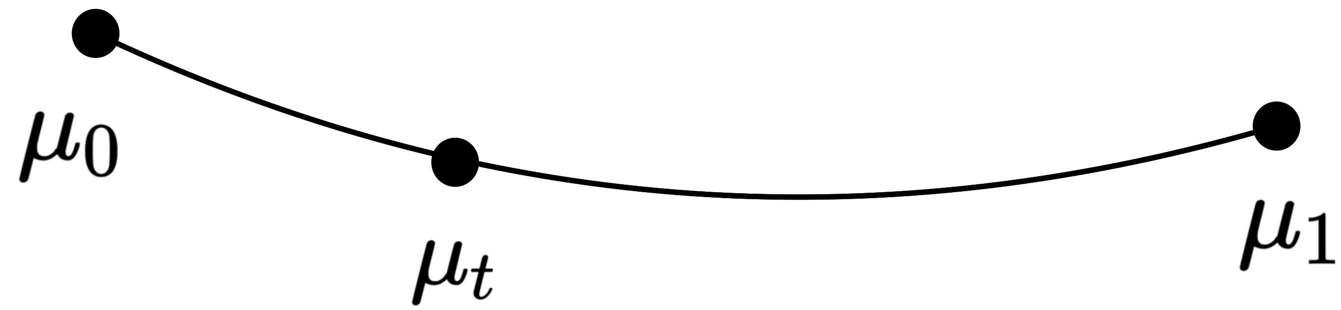

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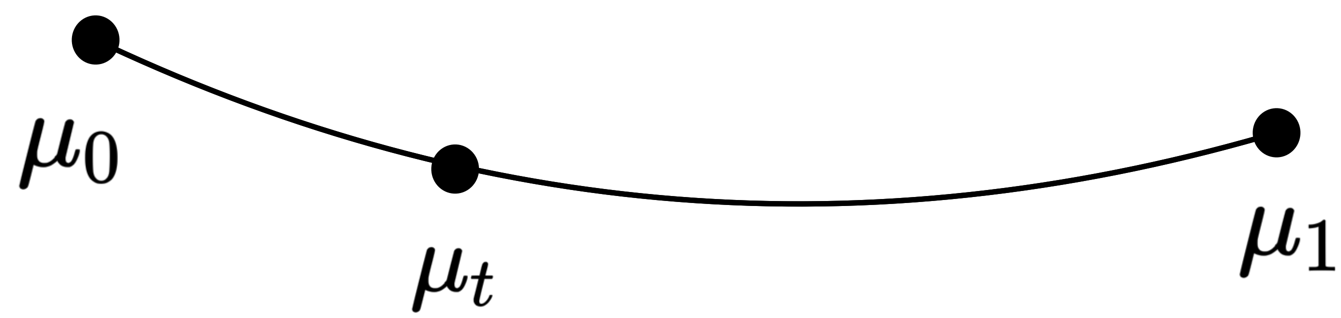
Measure-valued geodesic path for any $t \in [0,1]$

$$\mu_t = \arg\inf_{\nu \in \mathcal{P}_2(\mathcal{X})} \left\{ (1-t) W^2(\mu_0, \nu) + t W^2(\mu_1, \nu) \right\}$$

manifold of probability measures supported on $\mathcal{X}$ with finite second moments

Geometry on the Space of Prob. Measures



2-Wasserstein distance metric

$$W(\mu_0, \mu_1) := \left(\inf_{\mu, \boldsymbol{v}} \left\{ \frac{1}{2} \int_0^1 \int_{\mathcal{X}} \|\boldsymbol{v}\|^2 \mathrm{d}\mu \, \mathrm{d}t \right\} \right)^{1/2}$$

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$(\mu, \boldsymbol{v}) \in \mathrm{AC}((0,1); \mathcal{P}_2(\mathcal{X})) \times L^2(\mu_t, \mathcal{X})$

Measure-valued geodesic path for any $t \in [0,1]$

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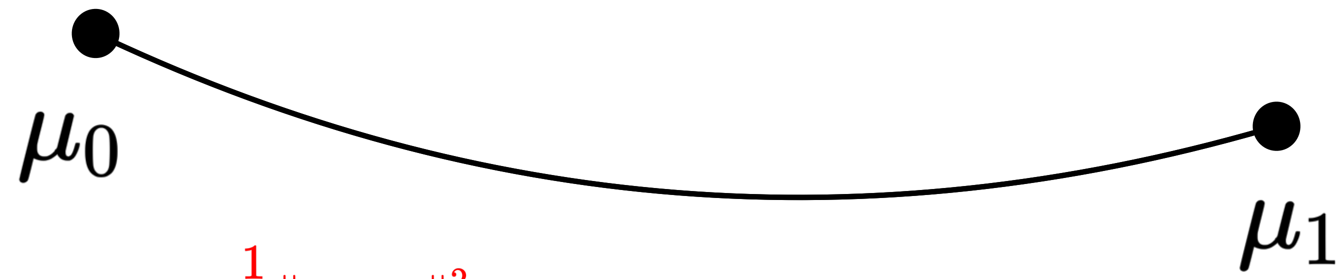
Geometry on the Space of Prob. Measures



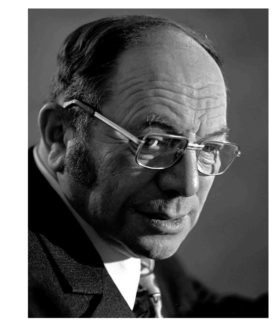
Optimal coupling formulation:

Ground cost, e.g., $\frac{1}{2}\|\boldsymbol{x}-\boldsymbol{y}\|_2^2$

$$W(\mu_0,\mu_1) := \left(\inf_m \int_{\mathcal{X}\times\mathcal{Y}} c(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}m(\boldsymbol{x},\boldsymbol{y})\right)^{1/2}$$

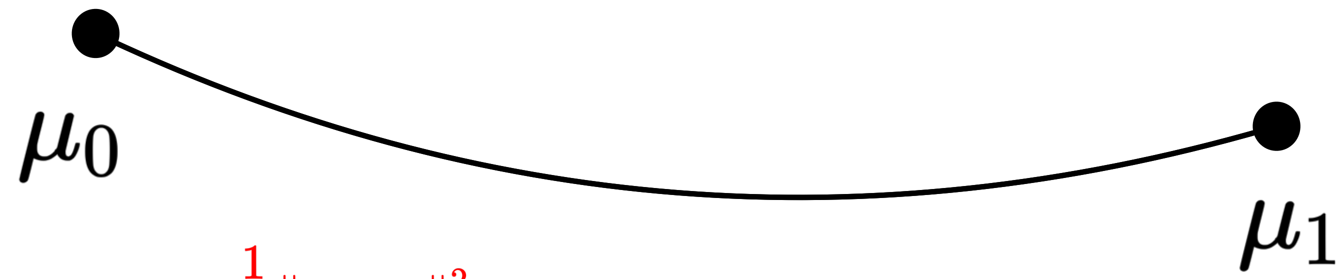
$$\text{subject to} \quad \int_{\mathcal{Y}} \mathrm{d}m = \mu_0(\mathrm{d}\boldsymbol{x}), \quad \int_{\mathcal{X}} \mathrm{d}m = \mu_1(\mathrm{d}\boldsymbol{y})$$

Gaspard Monge



Leonid Kantorovich

Geometry on the Space of Prob. Measures



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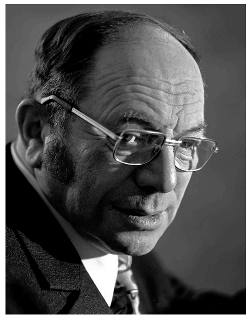

Leonid Kantorovich

Entropic/Sinkhorn regularization:

$$W_\varepsilon(\mu_0,\mu_1) := \left(\inf_m \int_{\mathcal{X}\times\mathcal{Y}} \left\{c(\boldsymbol{x},\boldsymbol{y}) + \varepsilon \log m\right\} \mathrm{d}m(\boldsymbol{x},\boldsymbol{y})\right)^{1/2}, \quad \varepsilon > 0$$

$$\text{subject to} \quad \int_{\mathcal{Y}} \mathrm{d}m = \mu_0(\mathrm{d}\boldsymbol{x}), \quad \int_{\mathcal{X}} \mathrm{d}m = \mu_1(\mathrm{d}\boldsymbol{y})$$

Measure-valued Optimization Problems

$$\underset{\mu \in \mathcal{P}_2(\mathbb{R}^d)}{\arg\inf}\; F(\mu)$$

Space of Borel probability measures on $\mathbb{R}^d$ with finite second moments

2-Wasserstein geodescially convex functional

In many applications, we have additive structure:

$$F(\mu) = F_1(\mu) + F_2(\mu) + \ldots + F_n(\mu)$$

where each $F_i : \mathscr{P}_2\left(\mathbb{R}^d\right) \mapsto (-\infty, +\infty]$ is proper, lsc, and 2-Wasserstein geodescially convex

Connection with Wasserstein Gradient Flows

$$\frac{\partial \mu}{\partial t} = -\underbrace{\nabla^W}_{\text{Wasserstein gradient}} F(\mu) := \nabla \cdot \left(\mu \nabla \frac{\delta F}{\delta \mu} \right) \qquad (\star)$$

Minimizer of $\underset{\mu \in \mathcal{P}_2(\mathbb{R}^d)}{\arg\inf} F(\mu)$ $\leftsquigarrow$ Stationary solution of $(\star)$

Transient solution of $(\star)$ $\rightsquigarrow$ Discrete time-stepping realizing grad. descent of $\underset{\mu \in \mathcal{P}_2(\mathbb{R}^d)}{\arg\inf} F(\mu)$

Connection with Wasserstein Gradient Flows

$$\frac{\partial \mu}{\partial t} = -\nabla^W F(\mu) := \nabla \cdot \left(\mu \nabla \frac{\delta F}{\delta \mu} \right) \qquad (\star)$$

Wasserstein gradient

Minimizer of $\arg\inf_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} F(\mu)$ $\leftsquigarrow$ Stationary solution of $(\star)$

Transient solution of $(\star)$ $\rightsquigarrow$ Discrete time-stepping realizing grad. descent of $\arg\inf_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} F(\mu)$

Wasserstein proximal recursion à la Jordan-Kinderlehrer-Otto (JKO) scheme

Gradient Flows

Gradient Flow in $\mathcal{X}$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -\nabla f(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

Recursion:

$$\begin{aligned}\boldsymbol{x}_k &= \boldsymbol{x}_{k-1} - h\nabla f(\boldsymbol{x}_k)\\ &= \underset{\boldsymbol{x}\in\mathcal{X}}{\arg\min}\left\{\frac{1}{2}\|\boldsymbol{x}-\boldsymbol{x}_{k-1}\|_2^2 + hf(\boldsymbol{x})\right\}\\ &=: \mathrm{prox}_{hf}^{\|\cdot\|_2}(\boldsymbol{x}_{k-1})\end{aligned}$$

Convergence:

$$\boldsymbol{x}_k \to \boldsymbol{x}(t = kh) \quad \text{as} \quad h \downarrow 0$$

f as Lyapunov function:

$$\frac{\mathrm{d}}{\mathrm{d}t}f = -\|\nabla f\|_2^2 \leq 0$$

Gradient Flow in $\mathcal{P}_2(\mathcal{X})$

$$\frac{\partial \mu}{\partial t} = -\nabla^W F(\mu), \quad \mu(\boldsymbol{x}, 0) = \mu_0$$

Recursion:

$$\begin{aligned}\mu_k &= \mu(\cdot, t = kh)\\ &= \underset{\mu\in\mathcal{P}_2(\mathcal{X})}{\arg\min}\left\{\frac{1}{2}W^2(\mu,\mu_{k-1}) + hF(\mu)\right\}\\ &=: \mathrm{prox}_{hF}^{W}(\mu_{k-1})\end{aligned}$$

Convergence:

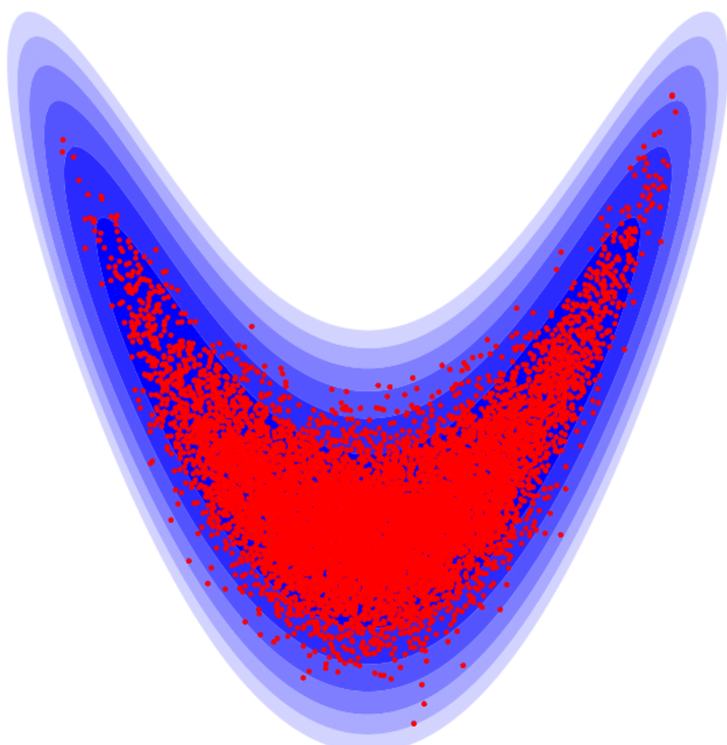
$$\mu_k \to \mu(\cdot, t = kh) \quad \text{as} \quad h \downarrow 0$$

F as Lyapunov functional:

$$\frac{\mathrm{d}}{\mathrm{d}t}F = -\mathbb{E}_\mu\left[\left\|\nabla\frac{\delta F}{\delta\mu}\right\|_2^2\right] \leq 0$$

Motivating Applications

Langevin sampling from an unnormalized prior



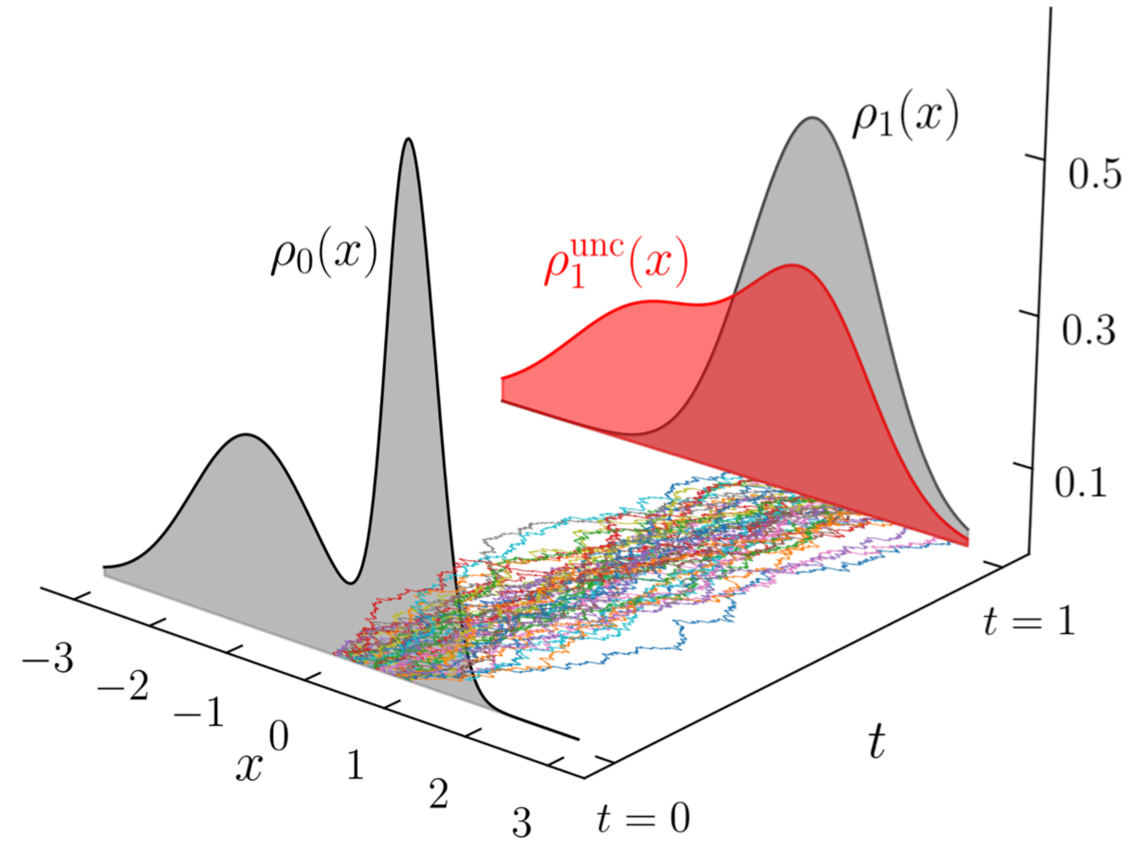
Stramer and Tweedie, *Methodology and Computing in Applied Probability*, 1999

Jarner and Hansen, *Stochastic Processes and their Applications*, 2000

Roberts and Stramer, *Methodology and Computing in Applied Probability*, 2002

Vempala and Wibisino, *NeurIPS*, 2019

Optimal control of distributions a.k.a. Schrödinger bridge problems



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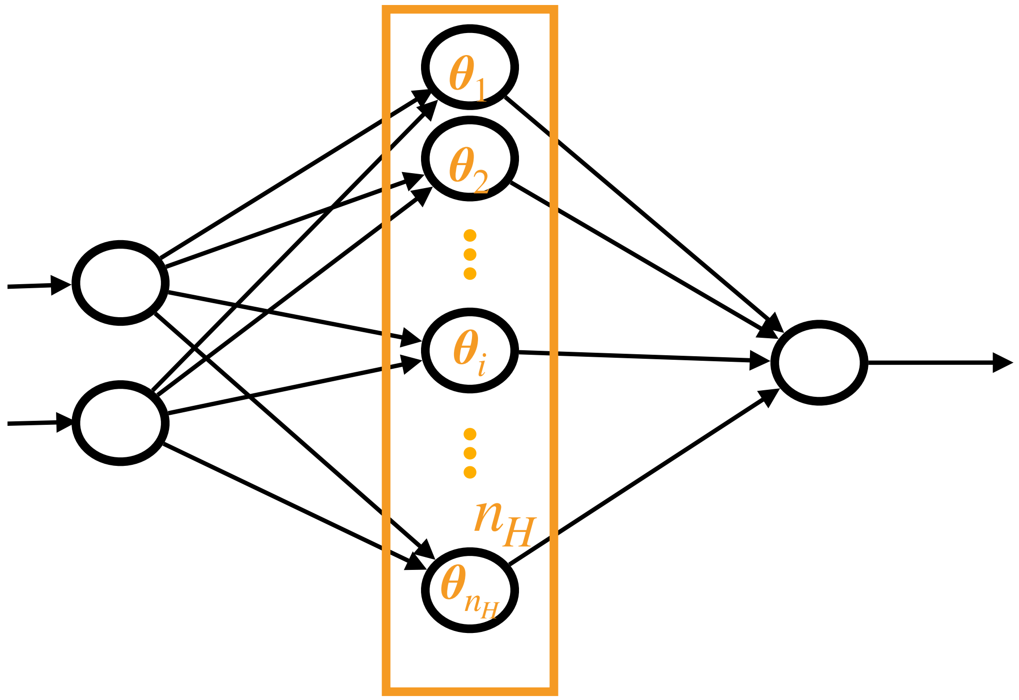
Chen, Georgiou and Pavon, *Journal on Optimization Theory and Applications*, 2016

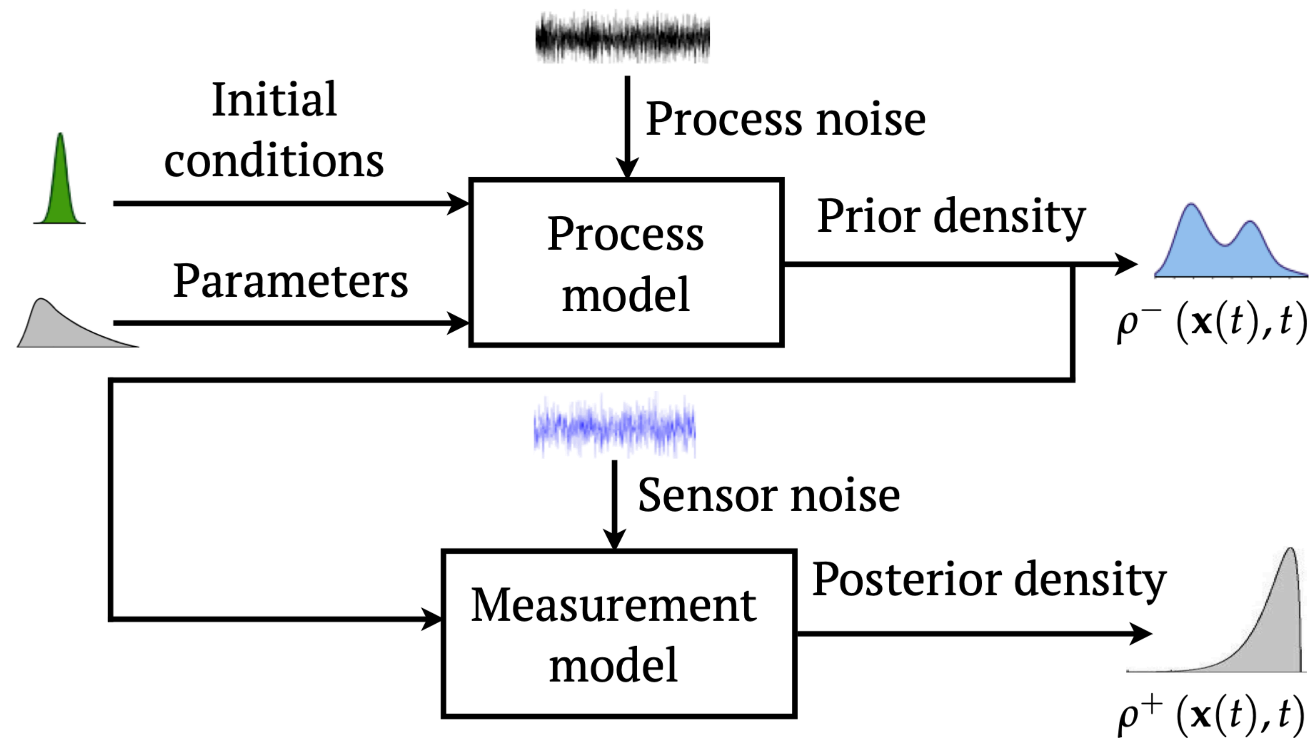
Caluya and Halder, *IEEE Transactions on Automatic Control*, 2021

Motivating Applications (contd.)

Mean field learning dynamics in neural networks



Prediction and estimation of time-varying joint state probability densities



Mei, Montanari and Nguyen, *Proceedings of the National Academy of Sciences*, 2018

Chizat and Bach, *NeurIPS*, 2018

Rotskoff and Vanden-Eijnden, *NeurIPS*, 2018

Sirignano and Spiliopoulos, *Stochastic Processes and their Applications*, 2020

Caluya and Halder, *IEEE Transactions on Automatic Control*, 2019

Halder and Georgiou, *CDC*, 2019

Halder and Georgiou, ACC, 2018

Halder and Georgiou, *CDC*, 2017

Many Recently Proposed Algorithms to Solve Measure-valued Optimization Problems

Peyré, *SIAM Journal on Imaging Sciences*, 2015

Benamou, Carlier and Laborde, *ESAIM: Proceedings and Surveys*, 2016

Carlier, Duval, Peyré and Schimtzer, *SIAM Journal on Mathematical Analysis*, 2017

Karlsson and Ringh, *SIAM Journal on Imaging Sciences*, 2017

Caluya and Halder, *IEEE Transactions on Automatic Control*, 2019

Carrillo, Craig, Wang and Wei, *Foundations of Computational Mathematics*, 2021

Mokrov, Korotin, Li, Gnevay, Solomon, and Burnaev, *NeurIPS*, 2021

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Wang, and Li, *Journal of Scientific Computing*, 2022

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Wang, and Li, *Journal of Scientific Computing*, 2022

But all require centralized computing

Centralized Computing Can Become Intensive: Mean Field SGD Dynamics in NN Classification

Free energy functional: $F(\mu) = R\left(\hat{f}\left(\boldsymbol{x}, \mu\right)\right)$

For quadratic loss:

$$F(\mu) = F_0 + \int_{\mathbb{R}^p} V(\boldsymbol{\theta})\mathrm{d}\mu(\boldsymbol{\theta}) + \int_{\mathbb{R}^{2p}} U(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}})\mathrm{d}\mu(\boldsymbol{\theta})\mathrm{d}\mu(\tilde{\boldsymbol{\theta}})$$

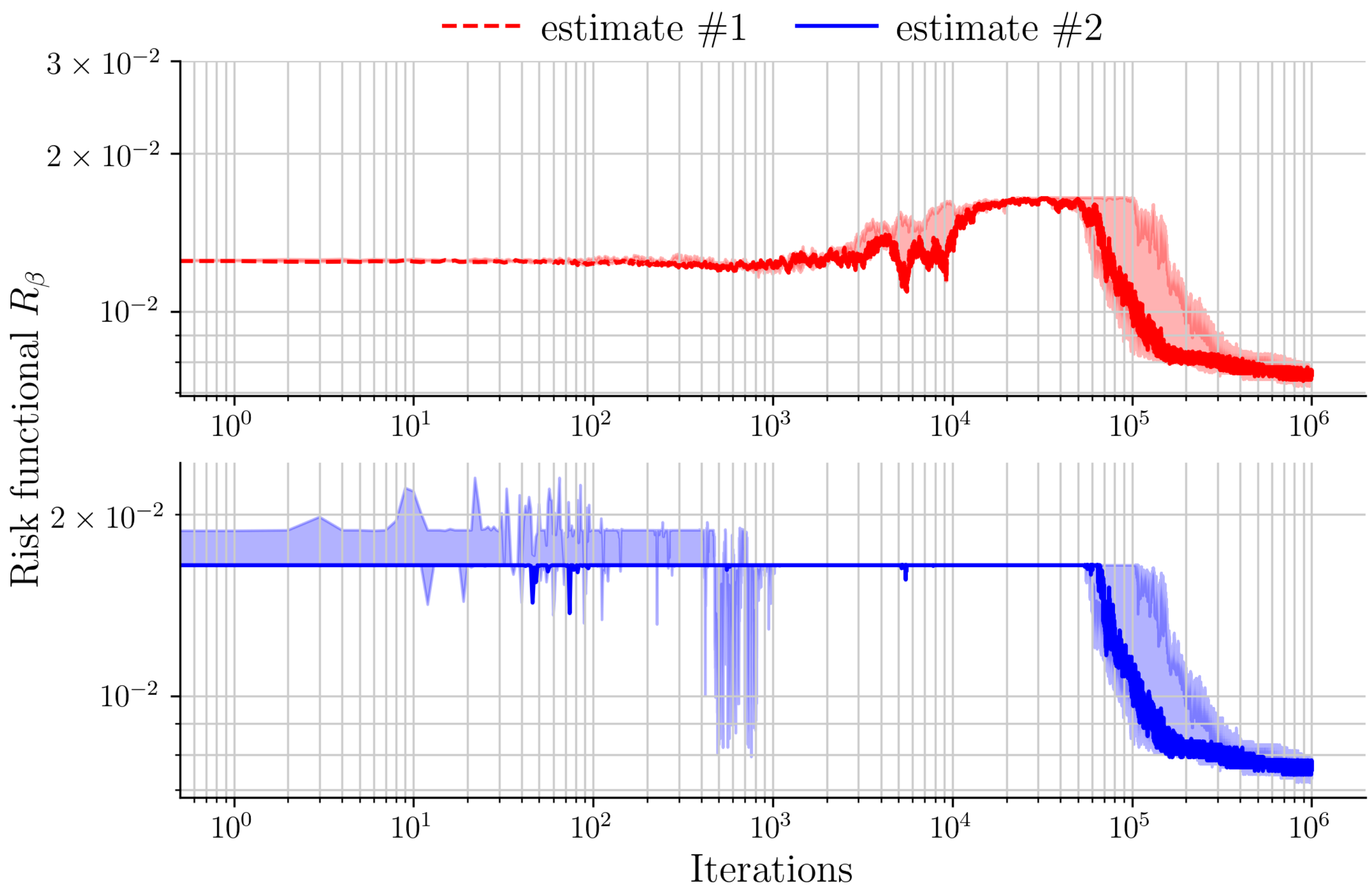
depend on activation functions of the NN

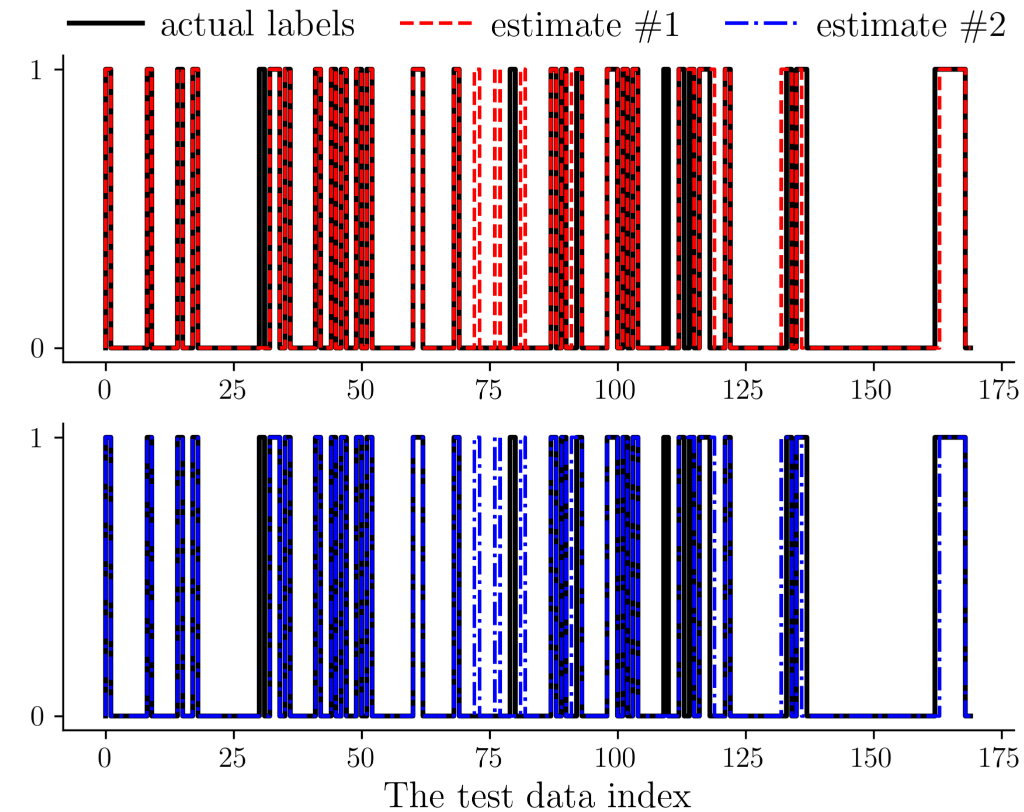
Neuronal population measure dynamics: $\dfrac{\partial \mu}{\partial t} = \nabla \cdot \left(\mu \nabla \dfrac{\delta F}{\delta \mu}\right) =: -\nabla^{W_2} F(\mu)$

Wasserstein proximal recursion: $\mu_{k+1} = \mathrm{prox}_{hF}^{W}(\mu_k)$

Centralized Computing can become intensive: Mean Field SGD Dynamics in NN Classification

Case study: Wisconsin Breast Cancer (Diagnostic) Data Set





Classification accuracy for the WBDC dataset		
β	Estimate #1	Estimate #2
0.03	91.17%	92.35%
0.05	92.94%	92.94%
0.07	78.23%	92.94%

CPU: 3.4 GHz 6 core intel i5 8GB RAM (≈ 33 hrs runtime)

GPU: Jetson TX2 NVIDIA Pascal GPU 256 CUDA cores, 64 bit NVIDIA Denver + ARM Cortex A57 CPUs (≈ 2 hrs runtime)

Teter, Nodozi and Halder, *arXiv: 2210.13879*, 2022

Specific Instances of Additive Objective

$$\operatorname*{arg\,inf}_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} \; F_1(\mu) + F_2(\mu) + \ldots + F_n(\mu)$$

Maximum likelihood deconvolution

$$Y_i = X_i + Z_i, \qquad X \sim \mu \text{ (unknown)}, \;\text{PDF of } Z \text{ is } \rho_Z \text{ (known)}$$

$$F_i(\mu) = -\log\left(\int \rho_Z(Y_i - x)\mathrm{d}\mu(x)\right)$$

If $\rho_Z = \mathcal{N}(0, \varepsilon^2)$

then the optimizer is the projection:

$$\operatorname*{arg\,inf}_{\mu \in \mathcal{P}_2} \; W_\varepsilon^2\left(\mu, \frac{1}{n}\sum_{i=1}^n \delta_{Y_i}\right)$$

C. R. Acad. Sci. Paris, Ser. I 356 (2018) 1228–1235

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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ELSEVIER

Statistics

Entropic optimal transport is maximum-likelihood deconvolution

Le transport optimal entropique correspond à l'estimateur du maximum de vraisemblance en déconvolution

Philippe Rigollet, Jonathan Weed

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA

Specific Instances of Additive Objective

$$\arg\inf_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} F_1(\mu) + F_2(\mu) + \ldots + F_n(\mu)$$

Wasserstein Barycenter of measures

Unregularized: $F_i(\mu) = w_i W^2(\mu, \mu_i), \quad w_i \geq 0$

Sinkhorn-regularized: $F_i(\mu) = w_i W_\varepsilon^2(\mu, \mu_i), \quad w_i \geq 0$

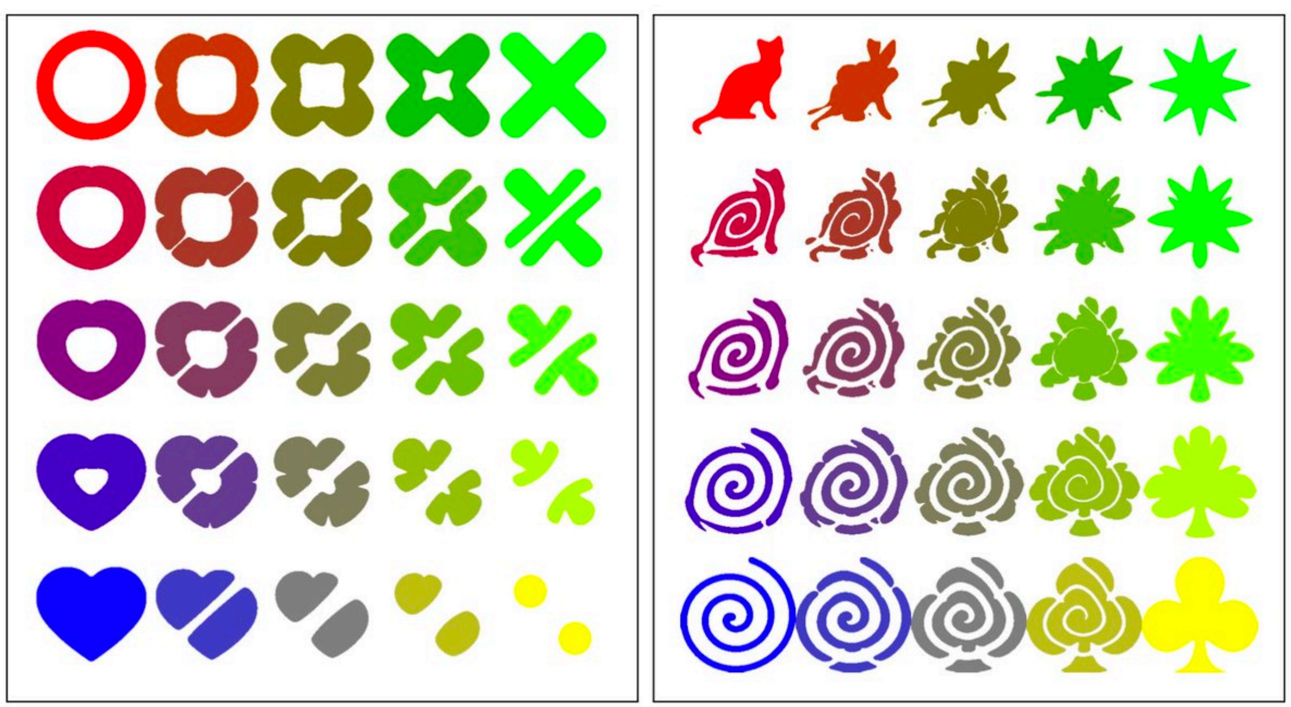

Image credit: G. Pyre

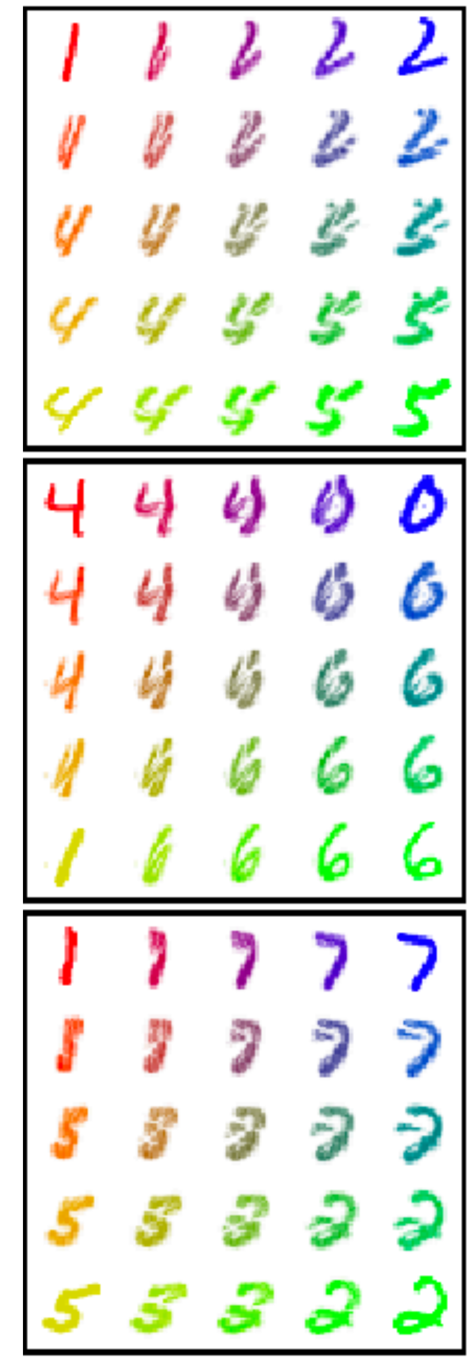

Image credit: Simon and Aberdam, *CVPR 2019*

Our Present Work: Distributed Algorithm

$$\operatorname*{arg\,inf}_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} \; F_1(\mu) + F_2(\mu) + \ldots + F_n(\mu)$$

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$$\operatorname*{arg\,inf}_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} \; F_1(\mu) + F_2(\mu) + \ldots + F_n(\mu)$$

Main idea:

$$\rightsquigarrow \text{ re-write}$$

$$\begin{array}{ll} \operatorname*{arg\,inf}_{(\mu_1,\ldots,\mu_n,\zeta) \in \mathcal{P}_2^{n+1}(\mathbb{R}^d)} & F_1(\mu_1) + F_2(\mu_2) + \ldots + F_n(\mu_n) \\ \text{subject to} & \mu_i = \zeta \quad \text{for all } i \in [n] \end{array}$$

Our Present Work: Distributed Algorithm

$$\underset{\mu \in \mathcal{P}_2(\mathbb{R}^d)}{\arg\inf}\; F_1(\mu) + F_2(\mu) + \ldots + F_n(\mu)$$

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$$\begin{aligned} &\underset{(\mu_1,\ldots,\mu_n,\zeta) \in \mathcal{P}_2^{n+1}(\mathbb{R}^d)}{\arg\inf} && F_1(\mu_1) + F_2(\mu_2) + \ldots + F_n(\mu_n) \\ &\text{subject to} && \mu_i = \zeta \quad \text{for all } i \in [n] \end{aligned}$$

Define Wasserstein augmented Lagrangian:

$$L_\alpha(\mu_1,\ldots,\mu_n,\zeta,\nu_1,\ldots,\nu_n) := \sum_{i=1}^{n} \left\{ F_i(\mu_i) + \frac{\alpha}{2} W^2(\mu_i,\zeta) + \int_{\mathbb{R}^d} \nu_i(\boldsymbol{\theta})(\mathrm{d}\mu_i - \mathrm{d}\zeta) \right\}$$

regularization > 0 Lagrange multipliers

Proposed Consensus ADMM

$$
\begin{aligned}
\mu_i^{k+1} &= \operatorname*{arg\,inf}_{\mu_i \in \mathcal{P}_2(\mathbb{R}^d)} L_\alpha\big(\mu_1, \ldots, \mu_n, \zeta^k, \nu_1^k, \ldots, \nu_n^k\big) \\
\zeta^{k+1} &= \operatorname*{arg\,inf}_{\zeta \in \mathcal{P}_2(\mathbb{R}^d)} L_\alpha\big(\mu_1^{k+1}, \ldots, \mu_n^{k+1}, \zeta, \nu_1^k, \ldots, \nu_n^k\big) \\
\nu_i^{k+1} &= \nu_i^k + \alpha\big(\mu_i^{k+1} - \zeta^{k+1}\big)
\end{aligned}
$$

where $i \in [n]$, $k \in \mathbb{N}_0$

Proposed Consensus ADMM

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\zeta^{k+1} &= \operatorname*{arg\,inf}_{\zeta \in \mathcal{P}_2(\mathbb{R}^d)} L_\alpha\left(\mu_1^{k+1}, \ldots, \mu_n^{k+1}, \zeta, \nu_1^k, \ldots, \nu_n^k\right) \\
\nu_i^{k+1} &= \nu_i^k + \alpha\left(\mu_i^{k+1} - \zeta^{k+1}\right) \qquad \text{where } i \in [n],\ k \in \mathbb{N}_0
\end{aligned}
$$

Define

$$
\nu_{\text{sum}}^k(\boldsymbol{\theta}) := \sum_{i=1}^n \nu_i^k(\boldsymbol{\theta}), \quad k \in \mathbb{N}_0
$$

and simplify the recursions to

$$
\begin{aligned}
\mu_i^{k+1} &= \operatorname{prox}^W_{\frac{1}{\alpha}\left(F_i(\cdot) + \int \nu_i^k \,\mathrm{d}(\cdot)\right)}\left(\zeta^k\right) \\
\zeta^{k+1} &= \operatorname*{arg\,inf}_{\zeta \in \mathcal{P}_2(\mathbb{R}^d)} \left\{ \left(\sum_{i=1}^n W^2\left(\mu_i^{k+1}, \zeta\right) \right) - \frac{2}{\alpha} \int_{\mathbb{R}^d} \nu_{\text{sum}}^k(\boldsymbol{\theta}) \mathrm{d}\zeta \right\} \\
\nu_i^{k+1} &= \nu_i^k + \alpha\left(\mu_i^{k+1} - \zeta^{k+1}\right)
\end{aligned}
$$

Proposed Consensus ADMM (contd.)

$$\boxed{\mu_i^{k+1} = \mathrm{prox}^{W}_{\frac{1}{\alpha}\left(F_i(\cdot)+\int \nu_i^k \,\mathrm{d}(\cdot)\right)}\left(\zeta^k\right)}$$

$$\zeta^{k+1} = \underset{\zeta\in\mathcal{P}_2\left(\mathbb{R}^d\right)}{\arg\inf}\left\{\left(\sum_{i=1}^{n} W^2\left(\mu_i^{k+1},\zeta\right)\right) - \frac{2}{\alpha}\int_{\mathbb{R}^d} \nu_{\mathrm{sum}}^k(\boldsymbol{\theta})\mathrm{d}\zeta\right\}$$

$$\nu_i^{k+1} = \nu_i^k + \alpha\left(\mu_i^{k+1} - \zeta^{k+1}\right)$$

Split free energy functionals: $\Phi_i(\mu_i) := F_i(\mu_i) + \int_{\mathbb{R}^d} \nu_i^k \,\mathrm{d}\mu_i$

$\therefore$ Distributed Wasserstein prox $\approx$ time updates of $\dfrac{\partial\tilde{\mu}_i}{\partial t} = -\nabla^W \Phi_i(\tilde{\mu}_i)$

Proposed Consensus ADMM (contd.)

$$\boxed{\mu_i^{k+1} = \operatorname{prox}^{W}_{\frac{1}{\alpha}\left(F_i(\cdot)+\int \nu_i^k \,\mathrm{d}(\cdot)\right)}\left(\zeta^k\right)}$$

$$\zeta^{k+1} = \underset{\zeta\in\mathcal{P}_2\left(\mathbb{R}^d\right)}{\arg\inf}\left\{\left(\sum_{i=1}^{n} W^2\left(\mu_i^{k+1},\zeta\right)\right) - \frac{2}{\alpha}\int_{\mathbb{R}^d}\nu^k_{\text{sum}}(\boldsymbol{\theta})\mathrm{d}\zeta\right\}$$

$$\nu_i^{k+1} = \nu_i^k + \alpha\left(\mu_i^{k+1} - \zeta^{k+1}\right)$$

Split free energy functionals: $\Phi_i(\mu_i) := F_i(\mu_i) + \int_{\mathbb{R}^d} \nu_i^k \,\mathrm{d}\mu_i$

$\therefore$ Distributed Wasserstein prox $\approx$ time updates of $\dfrac{\partial\tilde{\mu}_i}{\partial t} = -\nabla^W \Phi_i(\tilde{\mu}_i)$

Examples:

$\Phi_i(\cdot) = F_i(\cdot) + \int \nu_i^k \mathrm{d}(\cdot)$	PDE	Name
$\int_{\mathbb{R}^d}\left(V(\boldsymbol{\theta}) + \nu_i^k(\boldsymbol{\theta})\right)\mathrm{d}\mu_i(\boldsymbol{\theta})$	$\dfrac{\partial\widetilde{\mu}_i}{\partial t} = \nabla\cdot\left(\widetilde{\mu}_i\left(\nabla V + \nabla\nu_i^k\right)\right)$	Liouville equation
$\int_{\mathbb{R}^d}\left(\nu_i^k(\boldsymbol{\theta}) + \beta^{-1}\log\mu_i(\boldsymbol{\theta})\right)\mathrm{d}\mu_i(\boldsymbol{\theta})$	$\dfrac{\partial\widetilde{\mu}_i}{\partial t} = \nabla\cdot\left(\widetilde{\mu}_i\nabla\nu_i^k\right) + \beta^{-1}\Delta\widetilde{\mu}_i$	Fokker-Planck equation
$\int_{\mathbb{R}^d}\nu_i^k(\boldsymbol{\theta})\mathrm{d}\mu_i(\boldsymbol{\theta}) + \int_{\mathbb{R}^{2d}} U(\boldsymbol{\theta},\boldsymbol{\sigma})\mathrm{d}\mu_i(\boldsymbol{\theta})\mathrm{d}\mu_i(\boldsymbol{\sigma})$	$\dfrac{\partial\widetilde{\mu}_i}{\partial t} = \nabla\cdot\left(\widetilde{\mu}_i\left(\nabla\nu_i^k + \nabla\left(U\circledast\widetilde{\mu}_i\right)\right)\right)$	Propagation of chaos equation
$\int_{\mathbb{R}^d}\left(\nu_i^k(\boldsymbol{\theta}) + \frac{\beta^{-1}}{m-1}\mathbf{1}^\top\mu_i^m\right)\mathrm{d}\mu_i(\boldsymbol{\theta}), m>1$	$\dfrac{\partial\widetilde{\mu}_i}{\partial t} = \nabla\cdot\left(\widetilde{\mu}_i\nabla\nu_i^k\right) + \beta^{-1}\Delta\widetilde{\mu}_i^m$	Porous medium equation

Discrete Version of the Proposed ADMM

Euclidean distance matrix

$$\begin{aligned}
\boldsymbol{\mu}_i^{k+1} &= \operatorname{prox}^{W}_{\frac{1}{\alpha}\left(F_i(\boldsymbol{\mu}_i)+\langle \boldsymbol{\nu}_i^k,\boldsymbol{\mu}_i\rangle\right)}\left(\boldsymbol{\zeta}^k\right) \\
&= \operatorname*{arg\,inf}_{\boldsymbol{\mu}_i\in\Delta^{N-1}}\left\{\min_{\boldsymbol{M}\in\Pi_N\left(\boldsymbol{\mu}_i,\zeta^k\right)} \frac{1}{2}\langle \boldsymbol{C}, \boldsymbol{M}\rangle + \frac{1}{\alpha}\left(F_i(\boldsymbol{\mu}_i) + \langle \boldsymbol{\nu}_i^k, \boldsymbol{\mu}_i\rangle\right)\right\} \\
\boldsymbol{\zeta}^{k+1} &= \operatorname*{arg\,inf}_{\zeta\in\Delta^{N-1}}\left\{\left(\sum_{i=1}^{n}\min_{\boldsymbol{M}_i\in\Pi_N\left(\boldsymbol{\mu}_i^{k+1},\zeta\right)} \frac{1}{2}\langle \boldsymbol{C}, \boldsymbol{M}_i\rangle\right) - \frac{2}{\alpha}\langle \boldsymbol{\nu}_{\text{sum}}^k, \boldsymbol{\zeta}\rangle\right\} \\
\boldsymbol{\nu}_i^{k+1} &= \boldsymbol{\nu}_i^k + \alpha\left(\boldsymbol{\mu}_i^{k+1} - \boldsymbol{\zeta}^{k+1}\right)
\end{aligned}$$

where N is the number of samples

Discrete Version of the Proposed ADMM

$$
\begin{aligned}
\boldsymbol{\mu}_i^{k+1} &= \mathrm{prox}^{W}_{\frac{1}{\alpha}(F_i(\boldsymbol{\mu}_i)+\langle \boldsymbol{\nu}_i^k,\boldsymbol{\mu}_i\rangle)}(\boldsymbol{\zeta}^k) \\
&= \operatorname*{arg\,inf}_{\boldsymbol{\mu}_i\in\Delta^{N-1}} \left\{ \min_{\boldsymbol{M}\in\Pi_N(\boldsymbol{\mu}_i,\zeta^k)} \frac{1}{2}\langle \boldsymbol{C},\boldsymbol{M}\rangle + \frac{1}{\alpha}\big(F_i(\boldsymbol{\mu}_i) + \langle \boldsymbol{\nu}_i^k,\boldsymbol{\mu}_i\rangle\big) \right\} \\
\boldsymbol{\zeta}^{k+1} &= \operatorname*{arg\,inf}_{\boldsymbol{\zeta}\in\Delta^{N-1}} \left\{ \left(\sum_{i=1}^{n} \min_{\boldsymbol{M}_i\in\Pi_N(\boldsymbol{\mu}_i^{k+1},\boldsymbol{\zeta})} \frac{1}{2}\langle \boldsymbol{C},\boldsymbol{M}_i\rangle \right) - \frac{2}{\alpha}\langle \boldsymbol{\nu}_{\mathrm{sum}}^k,\boldsymbol{\zeta}\rangle \right\} \\
\boldsymbol{\nu}_i^{k+1} &= \boldsymbol{\nu}_i^k + \alpha\big(\boldsymbol{\mu}_i^{k+1} - \boldsymbol{\zeta}^{k+1}\big)
\end{aligned}
$$

With Sinkhorn regularization:

Discrete Sinkhorn divergence

$$
\begin{aligned}
\boldsymbol{\mu}_i^{k+1} &= \mathrm{prox}^{W_\varepsilon}_{\frac{1}{\alpha}(F_i(\boldsymbol{\mu}_i)+\langle \boldsymbol{\nu}_i^k,\boldsymbol{\mu}_i\rangle)}(\boldsymbol{\zeta}^k) \\
&= \operatorname*{arg\,inf}_{\boldsymbol{\mu}_i\in\Delta^{N-1}} \left\{ \min_{\boldsymbol{M}\in\Pi_N(\boldsymbol{\mu}_i,\zeta^k)} \left\langle \frac{1}{2}\boldsymbol{C} + \varepsilon\log\boldsymbol{M},\boldsymbol{M}\right\rangle + \frac{1}{\alpha}\big(F_i(\boldsymbol{\mu}_i) + \langle \boldsymbol{\nu}_i^k,\boldsymbol{\mu}_i\rangle\big) \right\} \\
\boldsymbol{\zeta}^{k+1} &= \operatorname*{arg\,inf}_{\boldsymbol{\zeta}\in\Delta^{N-1}} \left\{ \left(\sum_{i=1}^{n} \min_{\boldsymbol{M}_i\in\Pi_N(\boldsymbol{\mu}_i^{k+1},\boldsymbol{\zeta})} \left\langle \frac{1}{2}\boldsymbol{C} + \varepsilon\log\boldsymbol{M}_i,\boldsymbol{M}_i\right\rangle \right) - \frac{2}{\alpha}\langle \boldsymbol{\nu}_{\mathrm{sum}}^k,\boldsymbol{\zeta}\rangle \right\} \\
\boldsymbol{\nu}_i^{k+1} &= \boldsymbol{\nu}_i^k + \alpha\big(\boldsymbol{\mu}_i^{k+1} - \boldsymbol{\zeta}^{k+1}\big)
\end{aligned}
$$

Discrete Version of the Proposed ADMM

$$
\begin{aligned}
\boldsymbol{\mu}_i^{k+1} &= \operatorname{prox}^{W}_{\frac{1}{\alpha}(F_i(\boldsymbol{\mu}_i)+\langle \boldsymbol{\nu}_i^k,\boldsymbol{\mu}_i\rangle)}(\boldsymbol{\zeta}^k) \\
&= \operatorname*{arg\,inf}_{\boldsymbol{\mu}_i\in\Delta^{N-1}} \left\{ \min_{\boldsymbol{M}\in\Pi_N(\boldsymbol{\mu}_i,\boldsymbol{\zeta}^k)} \frac{1}{2}\langle \boldsymbol{C},\boldsymbol{M}\rangle + \frac{1}{\alpha}\big(F_i(\boldsymbol{\mu}_i)+\langle \boldsymbol{\nu}_i^k,\boldsymbol{\mu}_i\rangle\big) \right\} \\
\boldsymbol{\zeta}^{k+1} &= \operatorname*{arg\,inf}_{\boldsymbol{\zeta}\in\Delta^{N-1}} \left\{ \left(\sum_{i=1}^{n} \min_{\boldsymbol{M}_i\in\Pi_N(\boldsymbol{\mu}_i^{k+1},\boldsymbol{\zeta})} \frac{1}{2}\langle \boldsymbol{C},\boldsymbol{M}_i\rangle \right) - \frac{2}{\alpha}\langle \boldsymbol{\nu}_{\mathrm{sum}}^k,\boldsymbol{\zeta}\rangle \right\} \\
\boldsymbol{\nu}_i^{k+1} &= \boldsymbol{\nu}_i^k + \alpha\big(\boldsymbol{\mu}_i^{k+1}-\boldsymbol{\zeta}^{k+1}\big)
\end{aligned}
$$

With Sinkhorn regularization:

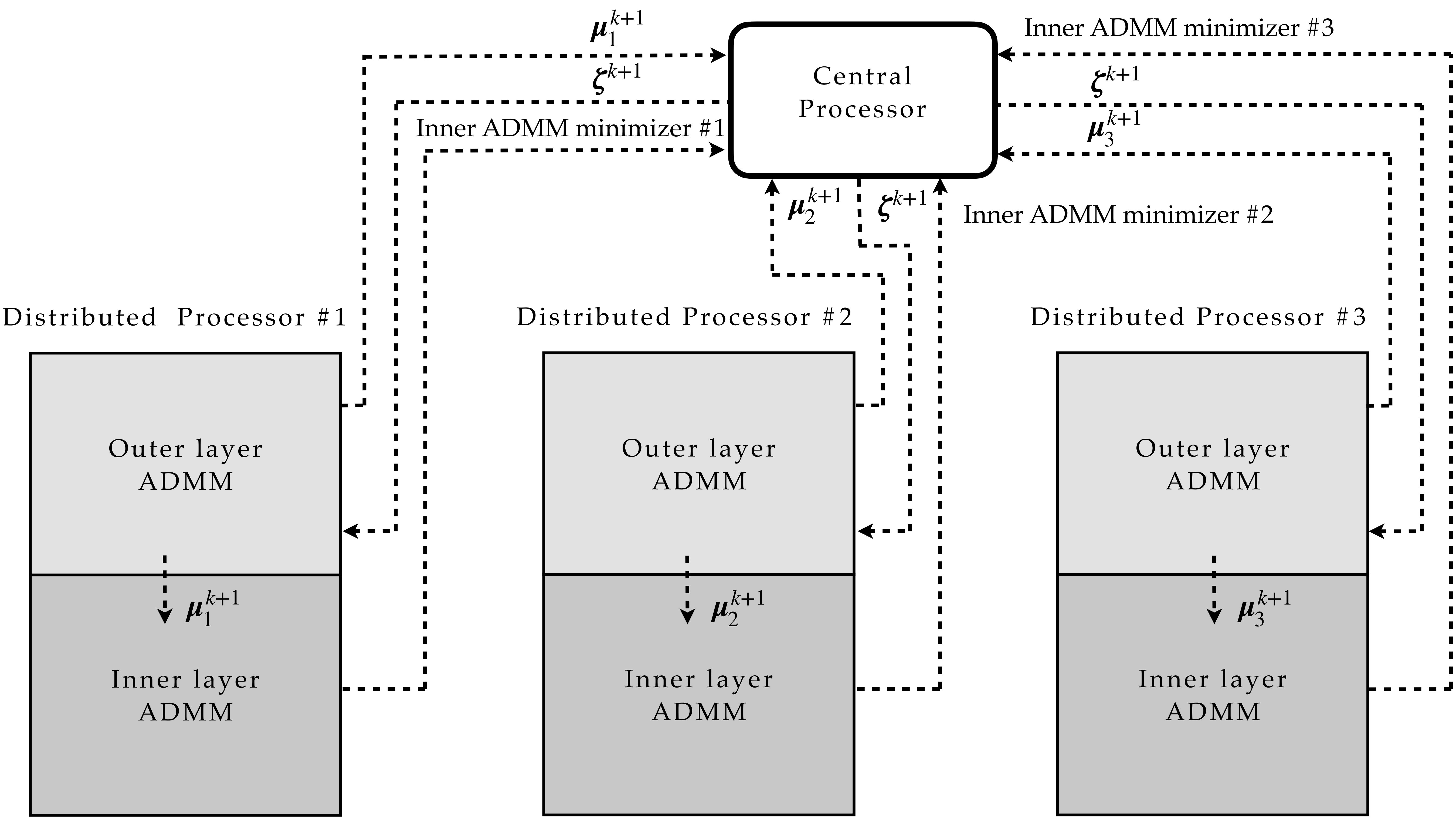
Discrete Sinkhorn divergence

Outer layer ADMM

$$
\begin{aligned}
\boldsymbol{\mu}_i^{k+1} &= \operatorname{prox}^{W_\varepsilon}_{\frac{1}{\alpha}(F_i(\boldsymbol{\mu}_i)+\langle \boldsymbol{\nu}_i^k,\boldsymbol{\mu}_i\rangle)}(\boldsymbol{\zeta}^k) \\
&= \operatorname*{arg\,inf}_{\boldsymbol{\mu}_i\in\Delta^{N-1}} \left\{ \min_{\boldsymbol{M}\in\Pi_N(\boldsymbol{\mu}_i,\boldsymbol{\zeta}^k)} \left\langle \frac{1}{2}\boldsymbol{C}+\varepsilon\log\boldsymbol{M},\boldsymbol{M}\right\rangle + \frac{1}{\alpha}\big(F_i(\boldsymbol{\mu}_i)+\langle \boldsymbol{\nu}_i^k,\boldsymbol{\mu}_i\rangle\big) \right\} \\
\boldsymbol{\zeta}^{k+1} &= \operatorname*{arg\,inf}_{\boldsymbol{\zeta}\in\Delta^{N-1}} \left\{ \left(\sum_{i=1}^{n} \min_{\boldsymbol{M}_i\in\Pi_N(\boldsymbol{\mu}_i^{k+1},\boldsymbol{\zeta})} \left\langle \frac{1}{2}\boldsymbol{C}+\varepsilon\log\boldsymbol{M}_i,\boldsymbol{M}_i\right\rangle \right) - \frac{2}{\alpha}\langle \boldsymbol{\nu}_{\mathrm{sum}}^k,\boldsymbol{\zeta}\rangle \right\} \\
\boldsymbol{\nu}_i^{k+1} &= \boldsymbol{\nu}_i^k + \alpha\big(\boldsymbol{\mu}_i^{k+1}-\boldsymbol{\zeta}^{k+1}\big)
\end{aligned}
$$

Inner layer ADMM

Overall Schematic



μ_i update $\rightsquigarrow$ Outer Consensus (Sinkhorn) ADMM

Example. $\Phi(\boldsymbol{\mu}) := \langle \boldsymbol{a}, \boldsymbol{\mu} \rangle$, $\boldsymbol{a} \in \mathbb{R}^N \backslash \{\mathbf{0}\}$, $\boldsymbol{\mu}, \boldsymbol{\zeta} \in \Delta^{N-1}$, $\boldsymbol{\Gamma} := \exp\left(-\boldsymbol{C}/2\varepsilon\right)$, $\varepsilon > 0$

$$\operatorname{prox}^{W_\varepsilon}_{\frac{1}{\alpha}\Phi}(\boldsymbol{\zeta}) = \exp\left(-\frac{1}{\alpha\varepsilon}\boldsymbol{a}\right) \odot \left(\boldsymbol{\Gamma}^\top \left(\boldsymbol{\zeta} \oslash \left(\boldsymbol{\Gamma} \exp\left(-\frac{1}{\alpha\varepsilon}\boldsymbol{a}\right)\right)\right)\right)$$

μ_i update ⇝ Outer Consensus (Sinkhorn) ADMM

Example. $\Phi(\boldsymbol{\mu}) := \langle \boldsymbol{a}, \boldsymbol{\mu} \rangle$, $\boldsymbol{a} \in \mathbb{R}^N \backslash \{\mathbf{0}\}$, $\boldsymbol{\mu}, \boldsymbol{\zeta} \in \Delta^{N-1}$, $\boldsymbol{\Gamma} := \exp\left(-\boldsymbol{C}/2\varepsilon\right)$, $\varepsilon > 0$

$$\operatorname{prox}^{W_\varepsilon}_{\frac{1}{\alpha}\Phi}(\boldsymbol{\zeta}) = \exp\left(-\frac{1}{\alpha\varepsilon}\boldsymbol{a}\right) \odot \left(\boldsymbol{\Gamma}^\top \left(\boldsymbol{\zeta} \oslash \left(\boldsymbol{\Gamma} \exp\left(-\frac{1}{\alpha\varepsilon}\boldsymbol{a}\right)\right)\right)\right)$$

Example. $G_i(\boldsymbol{\mu_i}) := F_i(\boldsymbol{\mu}_i) + \langle \boldsymbol{\nu}_i^k, \boldsymbol{\mu}_i \rangle$, $\boldsymbol{\zeta}^k \in \Delta^{N-1}$, $k \in \mathbb{N}_0$.

Convex

$$\boldsymbol{\mu}_i^{k+1} = \operatorname{prox}^{W_\varepsilon}_{\frac{1}{\alpha}(F_i(\boldsymbol{\mu}_i) + \langle \boldsymbol{\nu}_i^k, \boldsymbol{\mu}_i \rangle)}\left(\boldsymbol{\zeta}^k\right) = \exp\left(\frac{\boldsymbol{\lambda}_{1i}^{\mathrm{opt}}}{\alpha\varepsilon}\right) \odot \left(\exp\left(-\frac{\boldsymbol{C}^\top}{2\varepsilon}\right) \exp\left(\frac{\boldsymbol{\lambda}_{0i}^{\mathrm{opt}}}{\alpha\varepsilon}\right)\right)$$

where $\boldsymbol{\lambda}_{0i}^{\mathrm{opt}}, \boldsymbol{\lambda}_{1i}^{\mathrm{opt}} \in \mathbb{R}^N$ solve

$$\exp\left(\frac{\boldsymbol{\lambda}_{0i}^{\mathrm{opt}}}{\alpha\varepsilon}\right) \odot \left(\exp\left(-\frac{\boldsymbol{C}}{2\varepsilon}\right) \exp\left(\frac{\boldsymbol{\lambda}_{1i}^{\mathrm{opt}}}{\alpha\varepsilon}\right)\right) = \boldsymbol{\zeta}_k,$$

$$\mathbf{0} \in \partial_{\boldsymbol{\lambda}_{1i}^{\mathrm{opt}}} G_i^*\left(-\boldsymbol{\lambda}_{1i}^{\mathrm{opt}}\right) - \exp\left(\frac{\boldsymbol{\lambda}_{1i}^{\mathrm{opt}}}{\alpha\varepsilon}\right) \odot \left(\exp\left(-\frac{\boldsymbol{C}^\top}{2\varepsilon}\right) \exp\left(\frac{\boldsymbol{\lambda}_{0i}^{\mathrm{opt}}}{\alpha\varepsilon}\right)\right).$$

ζ update $\rightsquigarrow$ Inner (Euclidean) ADMM

Theorem.

Consider the convex problem

$$\begin{aligned}\left(\boldsymbol{u}_1^{\mathrm{opt}}, \ldots, \boldsymbol{u}_n^{\mathrm{opt}}\right) = \underset{(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)\in\mathbb{R}^{nN}}{\arg\min} & \sum_{i=1}^{n} \left\langle \boldsymbol{\mu}_i^{k+1}, \log\left(\boldsymbol{\Gamma}\exp\left(\boldsymbol{u}_i/\varepsilon\right)\right)\right\rangle \\ \text{subject to} & \sum_{i=1}^{n} \boldsymbol{u}_i = \frac{2}{\alpha}\boldsymbol{\nu}_{\mathrm{sum}}^k.\end{aligned} \qquad (\heartsuit)$$

Then

$$\boldsymbol{\zeta}^{k+1} = \exp\left(\boldsymbol{u}_i^{\mathrm{opt}}/\varepsilon\right) \odot \left(\boldsymbol{\Gamma}\left(\boldsymbol{\mu}_i^{k+1} \oslash \left(\boldsymbol{\Gamma}\exp\left(\boldsymbol{u}_i^{\mathrm{opt}}/\varepsilon\right)\right)\right)\right) \in \Delta^{N-1} \quad \forall\, i \in [n].$$

ζ update ⇝ Inner (Euclidean) ADMM

Theorem.

Let $\quad f_i(\boldsymbol{u}_i) := \left\langle \boldsymbol{\mu}_i^{k+1}, \log\left(\boldsymbol{\Gamma} \exp\left(\boldsymbol{u}_i/\varepsilon\right)\right)\right\rangle, \quad \boldsymbol{u}_i \in \mathbb{R}^N, \quad$ for all $i \in [n]$,

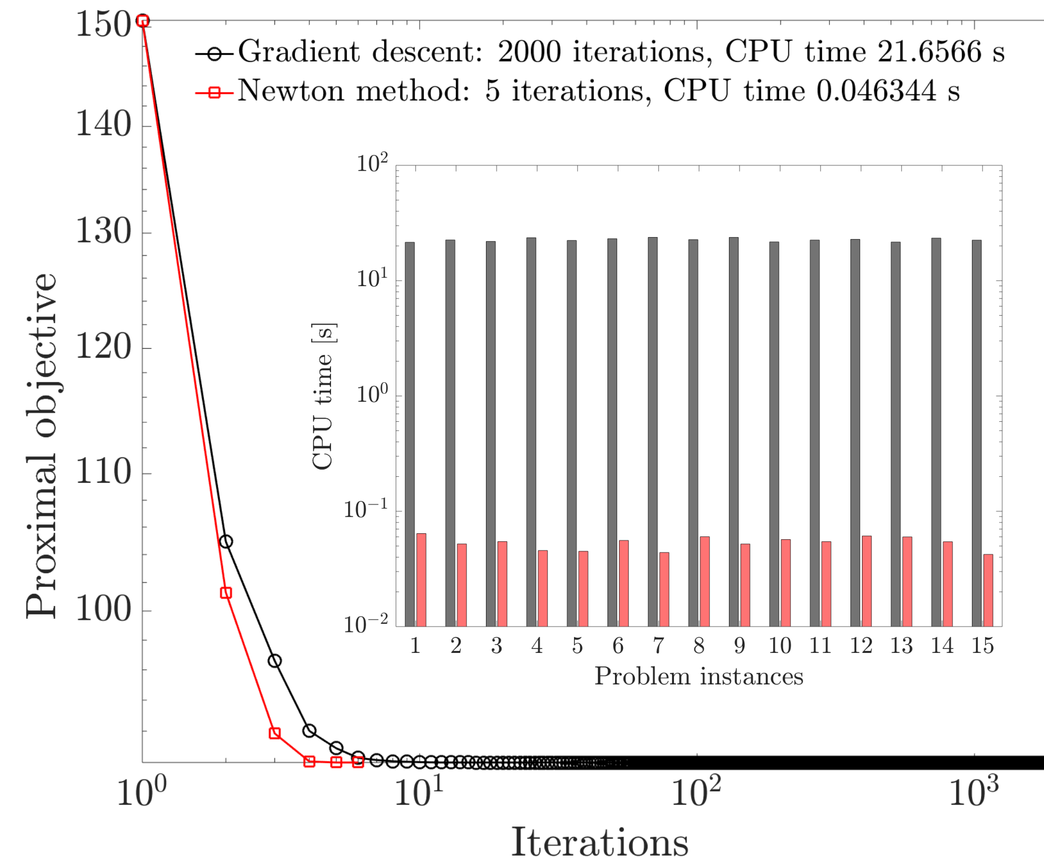
Then the following Euclidean ADMM solves (♥)

$$\boldsymbol{u}_i^{\ell+1} = \boxed{\operatorname{prox}_{\frac{1}{\tau} f_i}^{\|\cdot\|_2}\left(\boldsymbol{z}_i^{\ell} - \widetilde{\boldsymbol{\nu}}_i^{\ell}\right)}$$

No analytical solution, use e.g.,
Newton's method (has structured Hess)

$$\boldsymbol{z}_i^{\ell+1} = \left(\boldsymbol{u}_i^{\ell+1} - \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{u}_i^{\ell+1}\right) + \left(\widetilde{\boldsymbol{\nu}}_i^{\ell} - \frac{1}{n}\sum_{i=1}^{n} \widetilde{\boldsymbol{\nu}}_i^{\ell}\right) + \frac{2}{n\alpha}\boldsymbol{\nu}_{\text{sum}}^{k}$$

$$\widetilde{\boldsymbol{\nu}}_i^{\ell+1} = \widetilde{\boldsymbol{\nu}}_i^{\ell} + \left(\boldsymbol{u}_i^{\ell+1} - \boldsymbol{z}_i^{\ell+1}\right)$$



ζ update $\rightsquigarrow$ Inner (Euclidean) ADMM

Theorem.

Let $f_i(\boldsymbol{u}_i) := \left\langle \boldsymbol{\mu}_i^{k+1}, \log\left(\boldsymbol{\Gamma} \exp\left(\boldsymbol{u}_i/\varepsilon\right)\right) \right\rangle, \quad \boldsymbol{u}_i \in \mathbb{R}^N, \quad$ for all $i \in [n]$,

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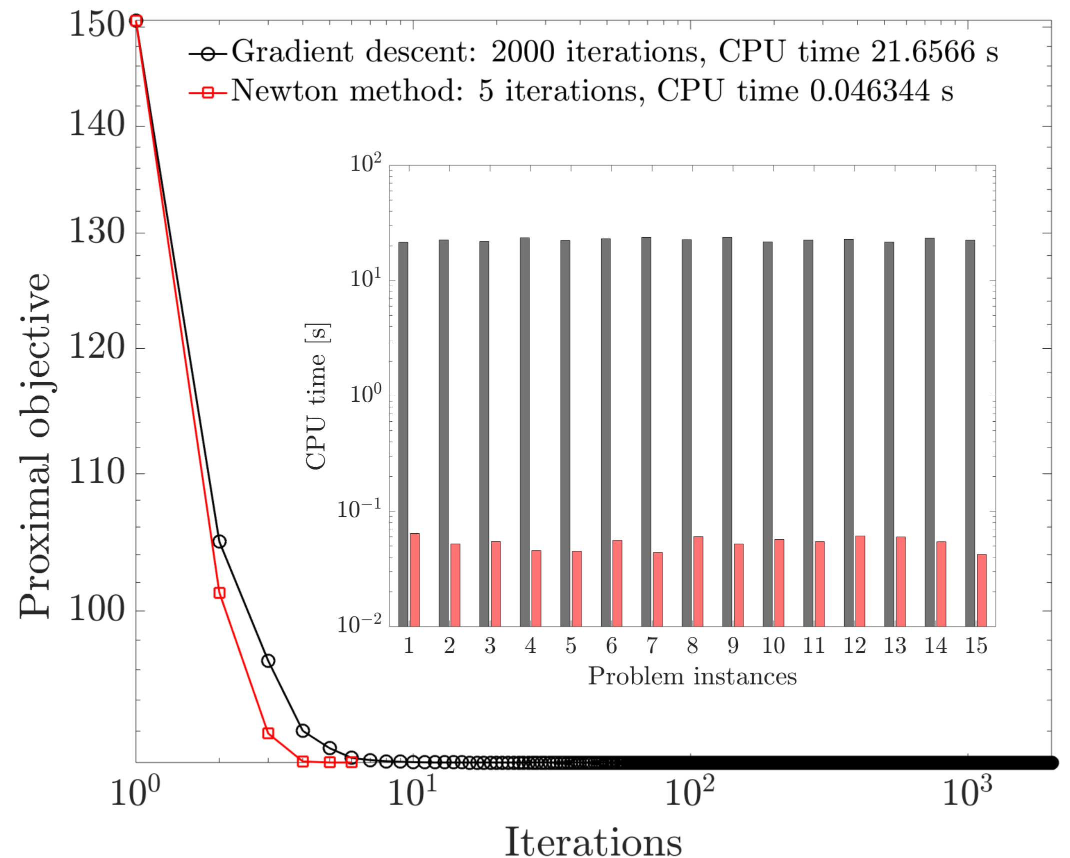
$$\widetilde{\boldsymbol{\nu}}_i^{\ell+1} = \widetilde{\boldsymbol{\nu}}_i^{\ell} + \left(\boldsymbol{u}_i^{\ell+1} - \boldsymbol{z}_i^{\ell+1}\right)$$

Theorem (informal).
Guaranteed convergence for inner layer ADMM under some constraints on hyper-parameters



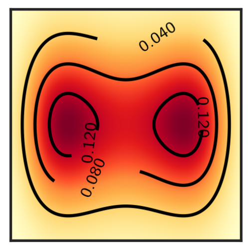
Experiment #1

Linear Fokker-Planck-Kolmogorov PDE

$$\frac{\partial \mu}{\partial t} = \nabla \cdot (\mu \nabla V) + \beta^{-1} \Delta \mu$$

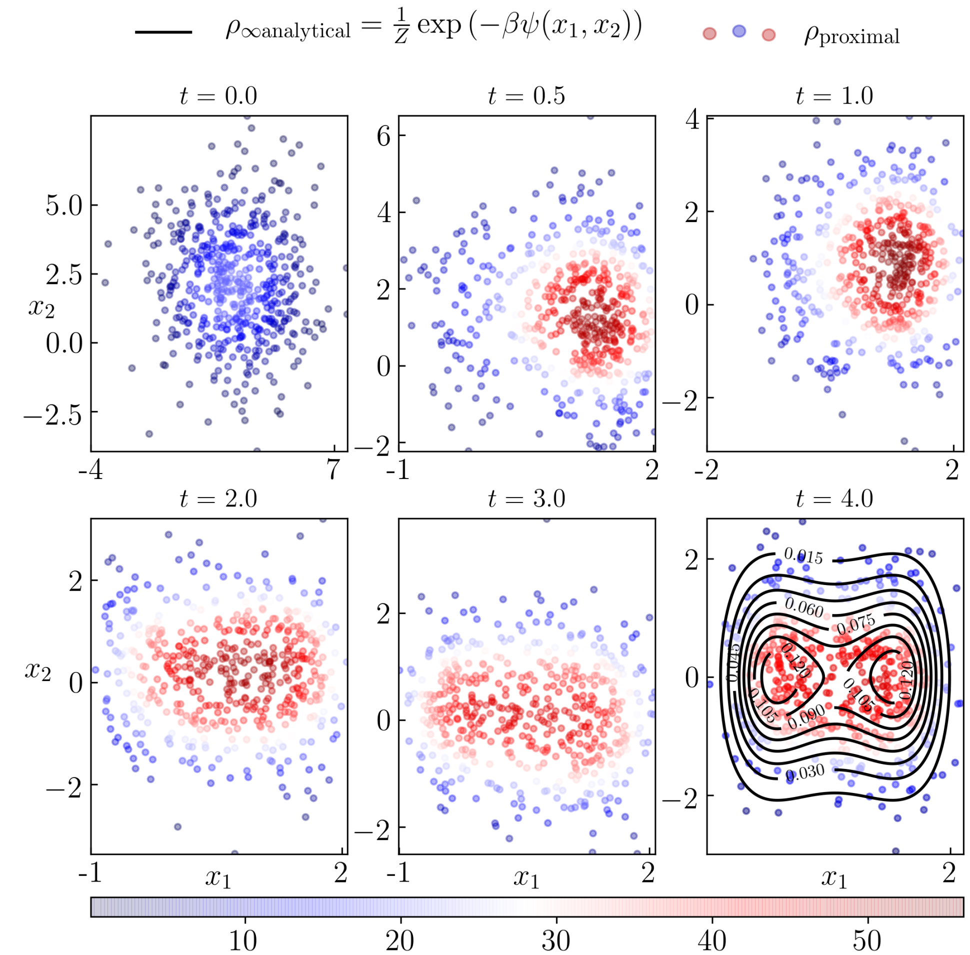
$$V(x_1, x_2) = \frac{1}{4}\left(1 + x_1^4\right) + \frac{1}{2}\left(x_2^2 - x_1^2\right)$$

$$\mu_\infty \propto \exp(-\beta V(x_1, x_2)) \mathrm{d}x_1 \mathrm{d}x_2$$



Centralized computation:

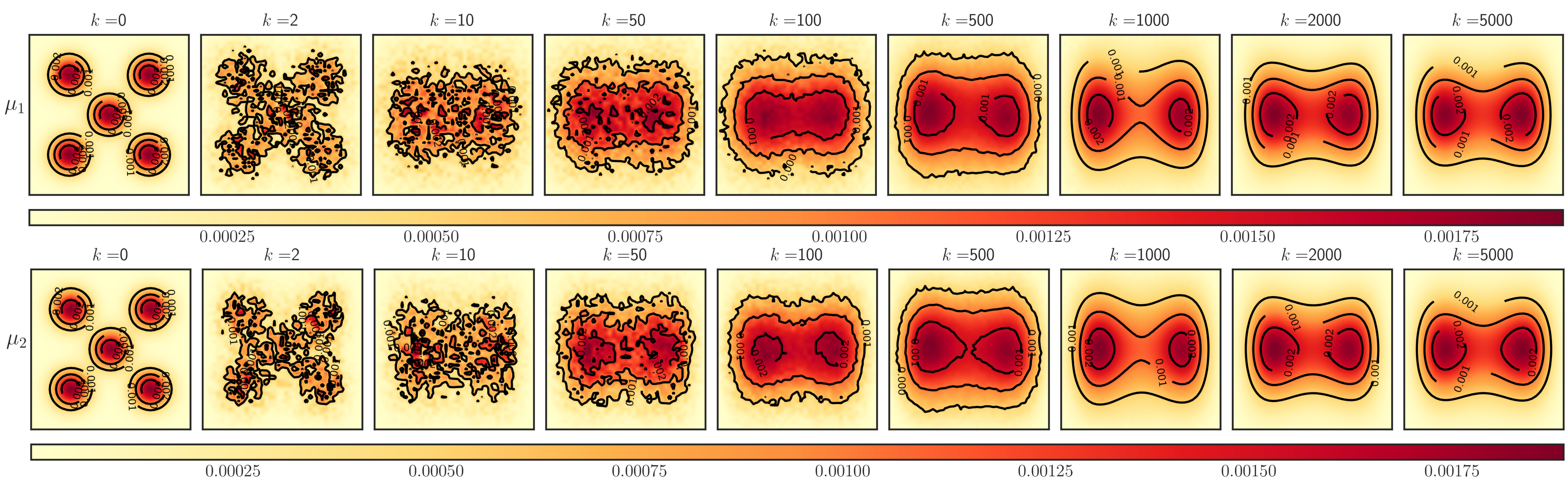
Caluya and Halder, *IEEE Trans. Automatic Control*, 2019



Runtime 99.89 s on Macbook Air 1.1 GHz intel i5 8GB RAM

Distributed computation:

$$F_1(\boldsymbol{\mu}) = \langle \boldsymbol{V}_k, \boldsymbol{\mu} \rangle \qquad F_2(\boldsymbol{\mu}) = \left\langle \beta^{-1} \log \boldsymbol{\mu}, \boldsymbol{\mu} \right\rangle$$



Experiment # 2

Aggregation-drift-diffusion nonlinear PDE

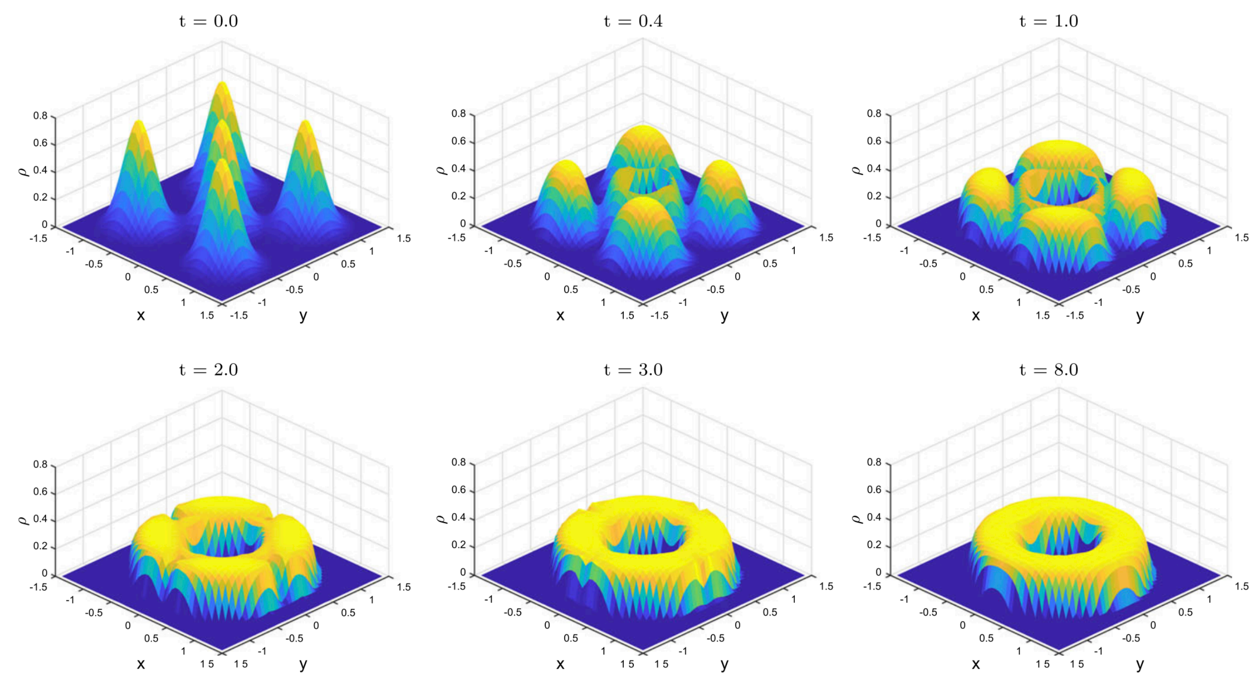
$$\frac{\partial \mu}{\partial t} = \underbrace{\nabla \cdot (\mu \nabla (U * \mu))}_{i=1} + \underbrace{\nabla \cdot (\mu \nabla V) + \beta^{-1} \Delta \mu^2}_{i=2}$$

$$U(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}\|_2^2 - \ln \|\boldsymbol{x}\|_2$$

$$V(\boldsymbol{x}) = -\frac{1}{4} \ln \|\boldsymbol{x}\|_2$$

Centralized computation:

Carrillo, Craig, Wang and Wei, *FOCM*, 2021

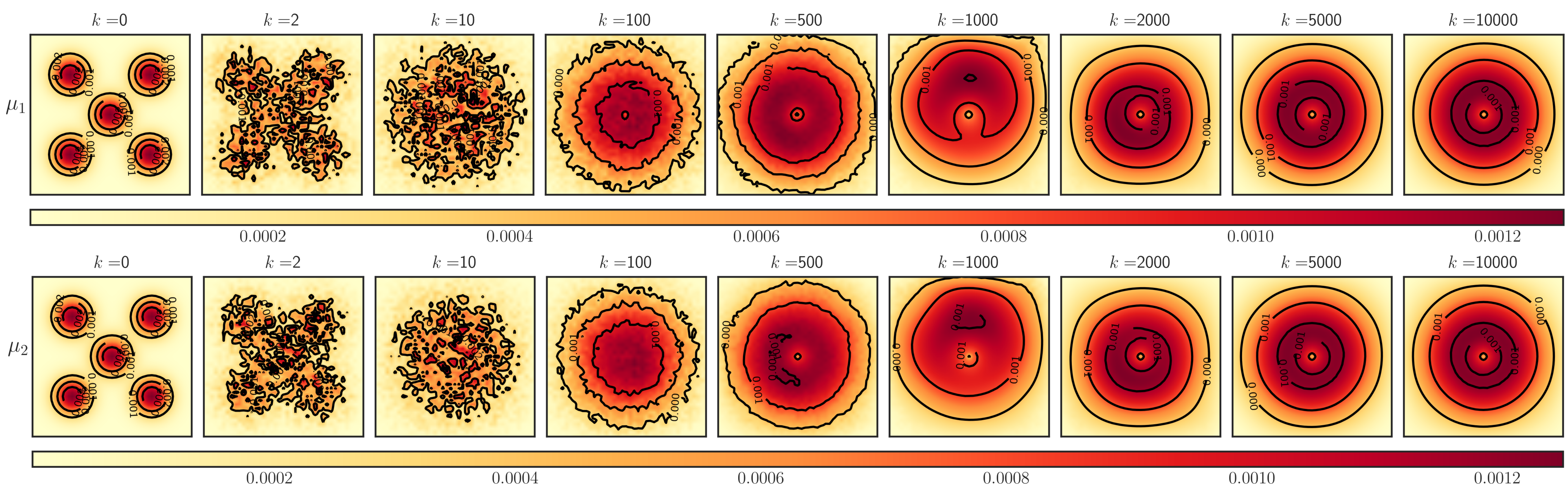


$$\lim_{\beta^{-1} \downarrow 0} \mu_\infty = \mathrm{Unif}\,(\mathscr{A})$$

Annulus with inner radius $1/2$ and outer radius $\sqrt{5}/2$

Distributed computation:

$$F_1(\boldsymbol{\mu}) = \langle \boldsymbol{U}_k \boldsymbol{\mu}, \boldsymbol{\mu} \rangle \qquad F_2(\boldsymbol{\mu}) = \langle \boldsymbol{V}_k + \beta^{-1} \log \boldsymbol{\mu}, \boldsymbol{\mu} \rangle$$



Experiment #2 (contd.)

Aggregation-drift-diffusion nonlinear PDE

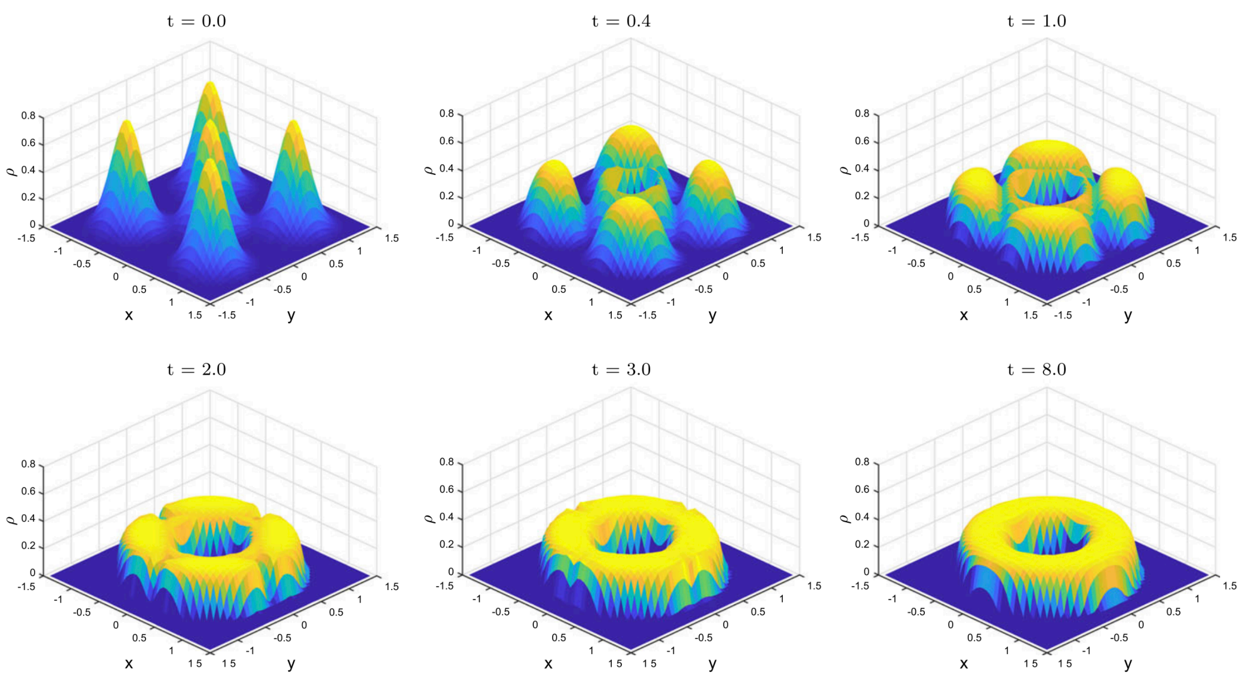
$$\frac{\partial \mu}{\partial t} = \underbrace{\nabla \cdot (\mu \nabla (U * \mu))}_{i=1} + \underbrace{\nabla \cdot (\mu \nabla V) + \beta^{-1} \Delta \mu^2}_{i=2}$$

$$U(\boldsymbol{x}) = \frac{1}{2} \|\boldsymbol{x}\|_2^2 - \ln \|\boldsymbol{x}\|_2$$

$$V(\boldsymbol{x}) = -\frac{1}{4} \ln \|\boldsymbol{x}\|_2$$

Centralized computation:

Carrillo, Craig, Wang and Wei, *FOCM*, 2021

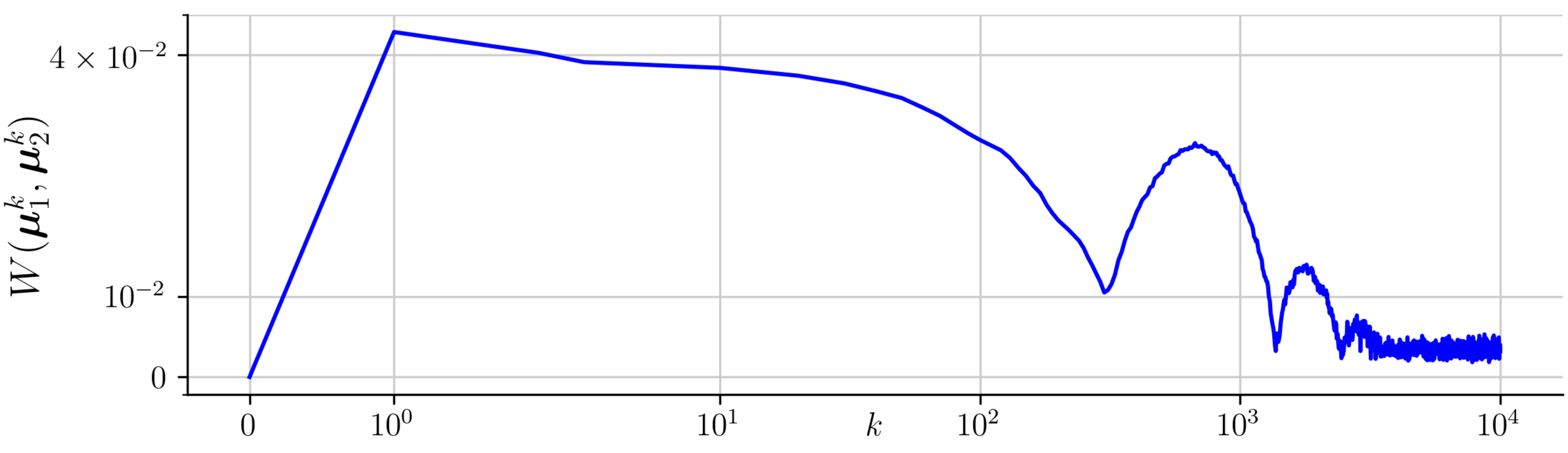


$$\lim_{\beta^{-1} \downarrow 0} \mu_\infty = \mathrm{Unif}\,(\mathscr{A})$$

Annulus with inner radius $1/2$ and outer radius $\sqrt{5}/2$

Distributed computation:

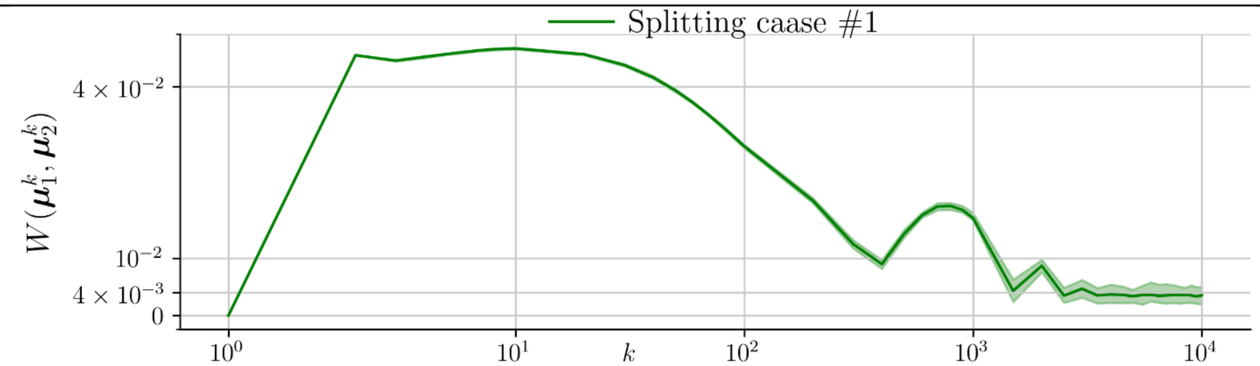
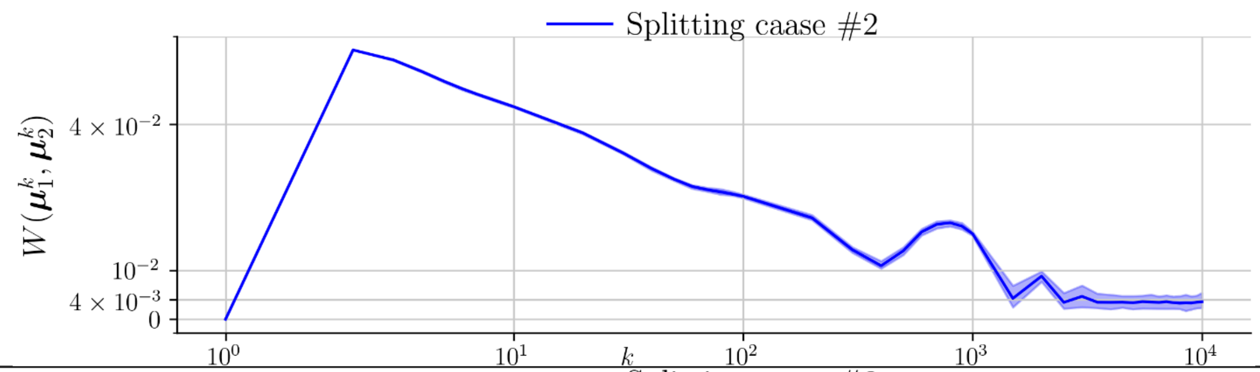
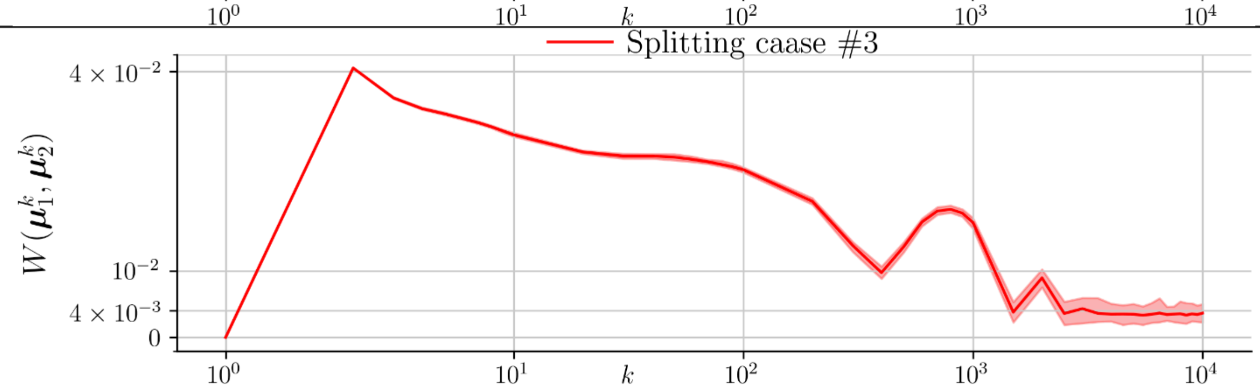
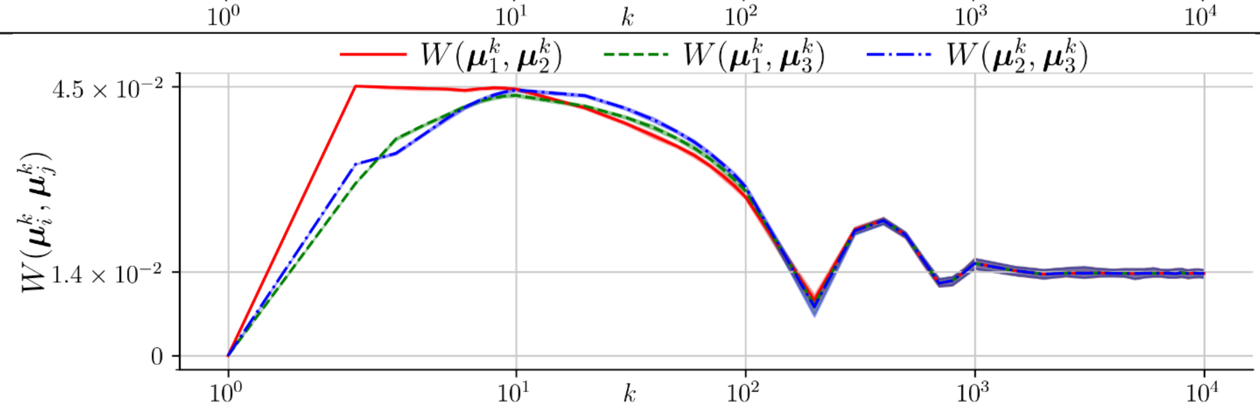
$$F_1(\boldsymbol{\mu}) = \langle \boldsymbol{U}_k \boldsymbol{\mu}, \boldsymbol{\mu} \rangle \qquad F_2(\boldsymbol{\mu}) = \langle \boldsymbol{V}_k + \beta^{-1} \log \boldsymbol{\mu}, \boldsymbol{\mu} \rangle$$



Experiment #2 (contd.)

B_n is nth Bell number, e.g.,
$B_2 = 2, B_3 = 5, B_4 = 15, B_5 = 52, \ldots$

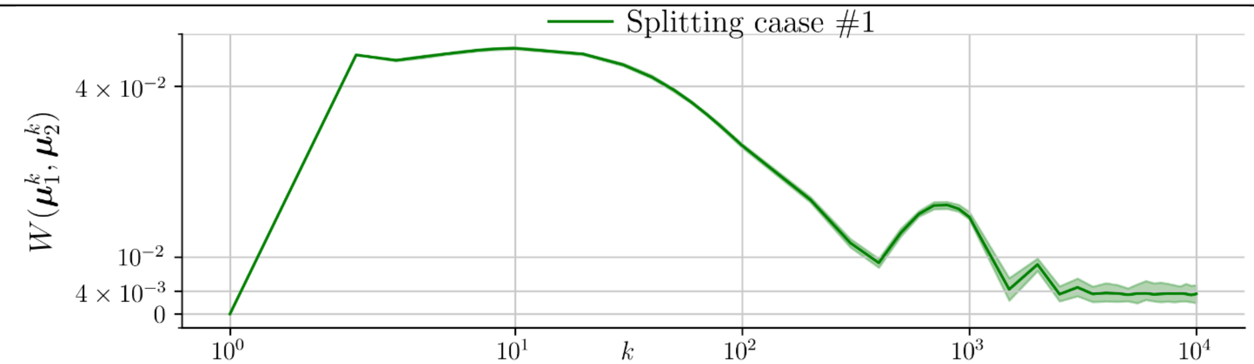
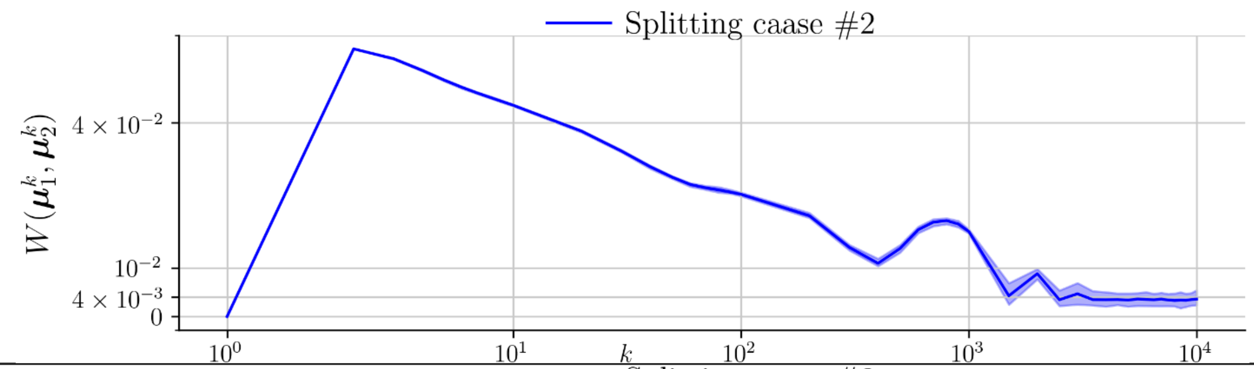
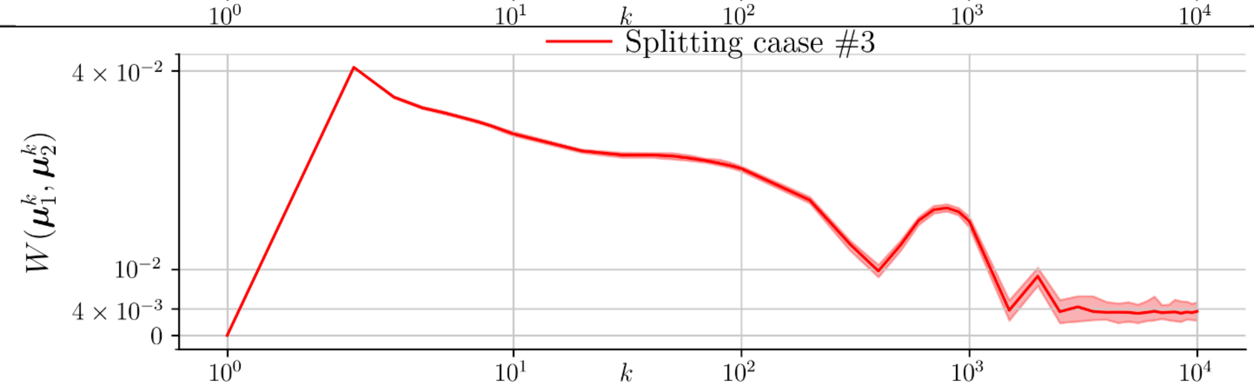
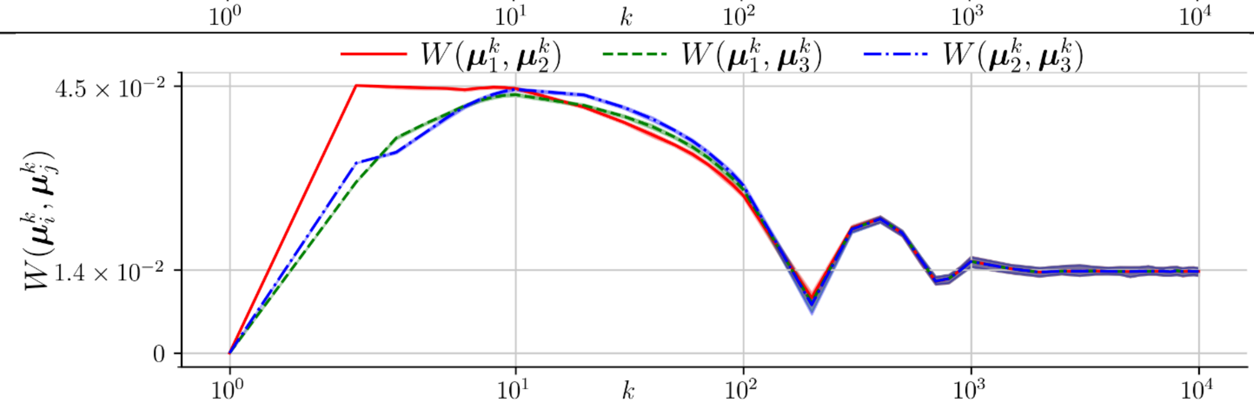
100 run statistics for each of the 4 ways of splitting: ($B_n - 1$ ways in general)

<table>
<tr><th>Splitting case</th><th>Functionals</th><th>Wasserstein distance</th></tr>
<tr><td>#1</td><td>$F_1(\boldsymbol{\mu}) = \left\langle \boldsymbol{V}_k + \beta^{-1}\boldsymbol{\mu}, \boldsymbol{\mu} \right\rangle,$
$F_2(\boldsymbol{\mu}) = \left\langle \boldsymbol{U}_k \boldsymbol{\mu}^k, \boldsymbol{\mu} \right\rangle$</td><td>
</td></tr>
<tr><td>#2</td><td>$F_1(\boldsymbol{\mu}) = \langle \boldsymbol{U}_k \boldsymbol{\mu}^k + \beta^{-1}\boldsymbol{\mu}, \boldsymbol{\mu} \rangle,$
$F_2(\boldsymbol{\mu}) = \langle \boldsymbol{V}_k, \boldsymbol{\mu} \rangle$</td><td>
</td></tr>
<tr><td>#3</td><td>$F_1(\boldsymbol{\mu}) = \langle \boldsymbol{U}_k \boldsymbol{\mu}^k + \boldsymbol{V}_k, \boldsymbol{\mu} \rangle,$
$F_2(\boldsymbol{\mu}) = \left\langle \beta^{-1}\boldsymbol{\mu}, \boldsymbol{\mu} \right\rangle$</td><td>
</td></tr>
<tr><td>#4</td><td>$F_1(\boldsymbol{\mu}) = \langle \boldsymbol{V}_k, \boldsymbol{\mu} \rangle,$
$F_2(\boldsymbol{\mu}) = \left\langle \boldsymbol{U}_k \boldsymbol{\mu}^k \right\rangle,$
$F_3(\boldsymbol{\mu}) = \left\langle \beta^{-1}\boldsymbol{\mu}, \boldsymbol{\mu} \right\rangle$</td><td>
</td></tr>
</table>

Experiment #2 (contd.)

Centralized av. runtime = 310.21 s

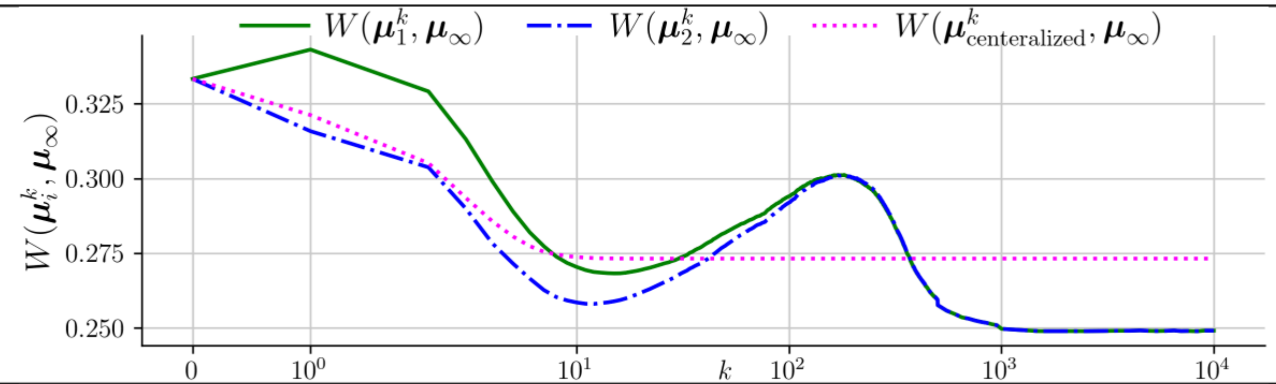
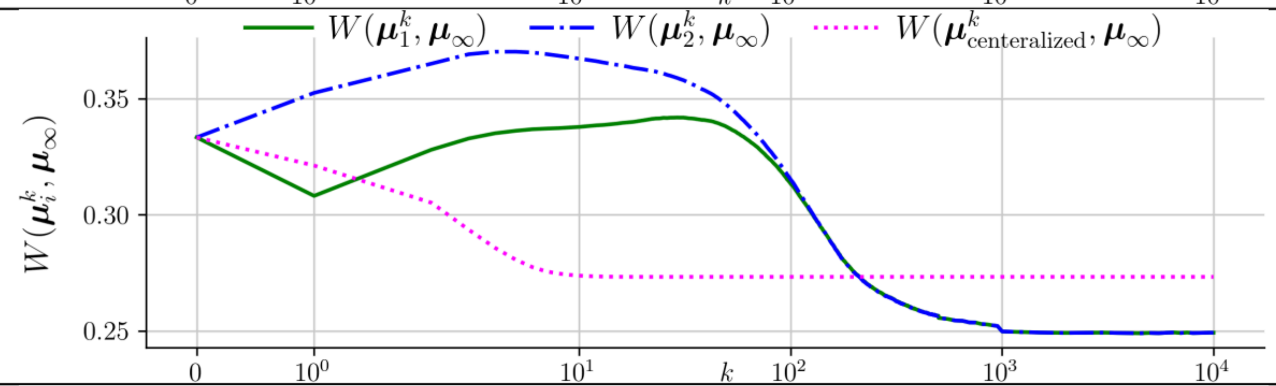
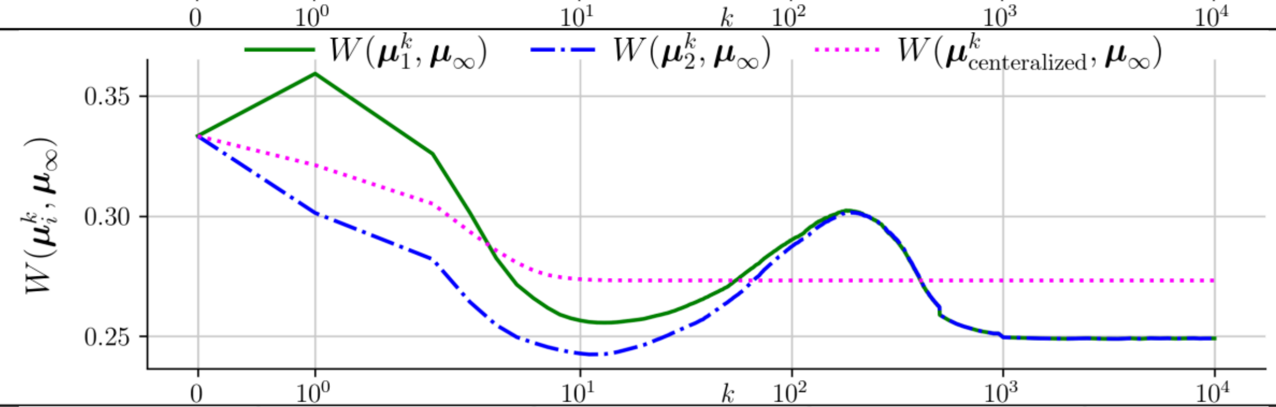
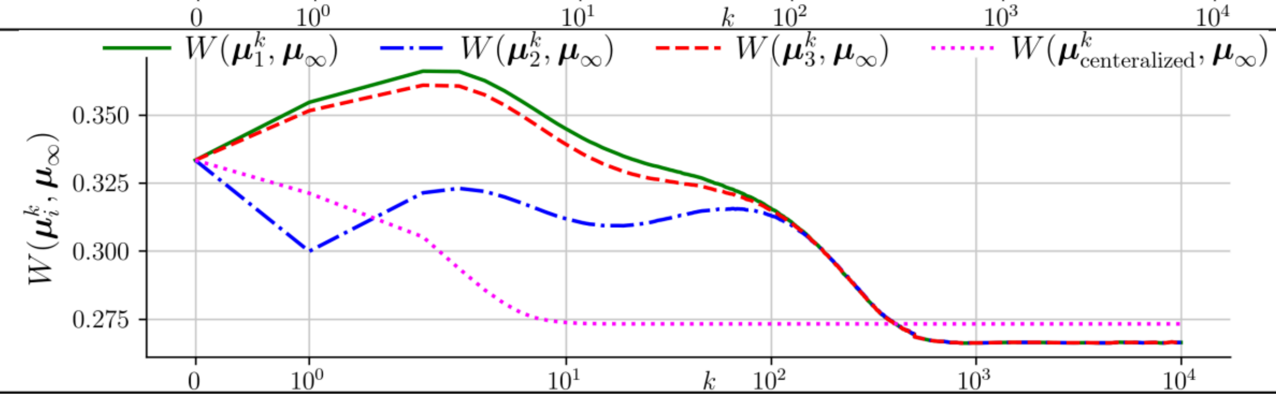
100 run statistics for each of the 4 ways of splitting: ($B_n - 1$ ways in general)

Splitting case	Functionals	Wasserstein distance
#1	$F_1(\boldsymbol{\mu}) = \left\langle \boldsymbol{V}_k + \beta^{-1}\boldsymbol{\mu}, \boldsymbol{\mu} \right\rangle,$ $F_2(\boldsymbol{\mu}) = \left\langle \boldsymbol{U}_k \boldsymbol{\mu}^k, \boldsymbol{\mu} \right\rangle$ av. runtime = 294.06 s	
#2	$F_1(\boldsymbol{\mu}) = \langle \boldsymbol{U}_k \boldsymbol{\mu}^k + \beta^{-1}\boldsymbol{\mu}, \boldsymbol{\mu} \rangle,$ $F_2(\boldsymbol{\mu}) = \langle \boldsymbol{V}_k, \boldsymbol{\mu} \rangle$ av. runtime = 285.32 s	
#3	$F_1(\boldsymbol{\mu}) = \langle \boldsymbol{U}_k \boldsymbol{\mu}^k + \boldsymbol{V}_k, \boldsymbol{\mu} \rangle,$ $F_2(\boldsymbol{\mu}) = \left\langle \beta^{-1}\boldsymbol{\mu}, \boldsymbol{\mu} \right\rangle$ av. runtime = 289.87 s	
#4	$F_1(\boldsymbol{\mu}) = \langle \boldsymbol{V}_k, \boldsymbol{\mu} \rangle,$ $F_2(\boldsymbol{\mu}) = \left\langle \boldsymbol{U}_k \boldsymbol{\mu}^k \right\rangle,$ $F_3(\boldsymbol{\mu}) = \left\langle \beta^{-1}\boldsymbol{\mu}, \boldsymbol{\mu} \right\rangle$ av. runtime = 108.99 s	

Experiment #2 (contd.)

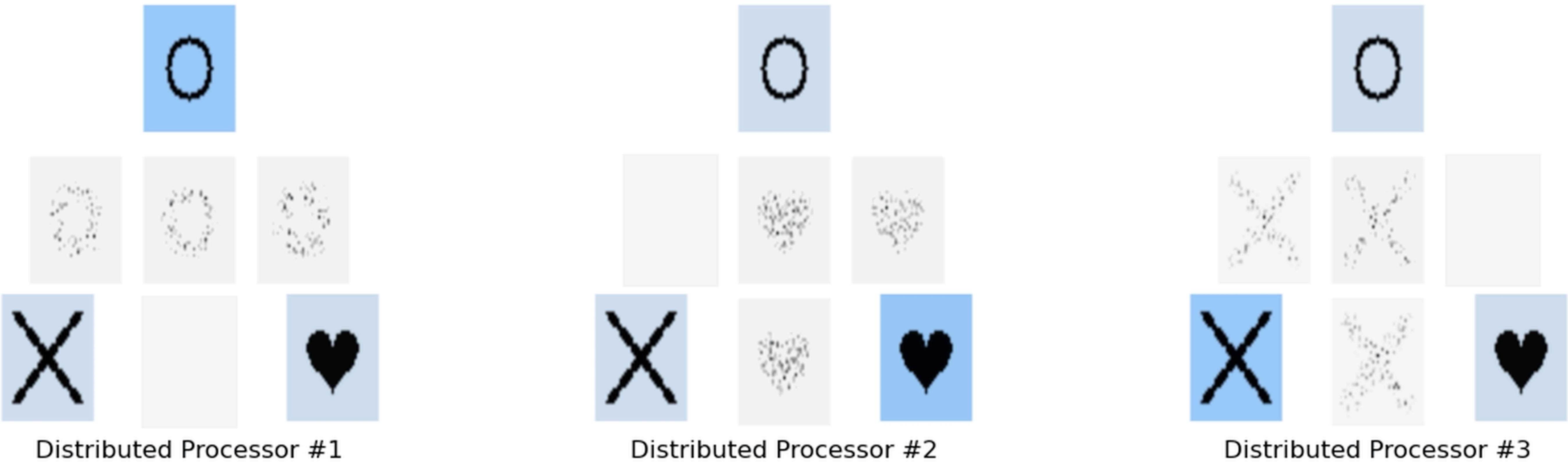
Centralized is pink dotted (repeated in subplots)

100 run statistics for each of the 4 ways of splitting: ($B_n - 1$ ways in general)

Case	Functionals	Wasserstein distances
#1	$F_1(\boldsymbol{\mu}) = \left\langle \boldsymbol{V}_k + \beta^{-1}\boldsymbol{\mu}, \boldsymbol{\mu} \right\rangle$, $F_2(\boldsymbol{\mu}) = \left\langle \boldsymbol{U}_k \boldsymbol{\mu}^k, \boldsymbol{\mu} \right\rangle$	
#2	$F_1(\boldsymbol{\mu}) = \langle \boldsymbol{U}_k \boldsymbol{\mu}^k + \beta^{-1}\boldsymbol{\mu}, \boldsymbol{\mu} \rangle$, $F_2(\boldsymbol{\mu}) = \langle \boldsymbol{V}_k, \boldsymbol{\mu} \rangle$	
#3	$F_1(\boldsymbol{\mu}) = \langle \boldsymbol{U}_k \boldsymbol{\mu}^k + \boldsymbol{V}_k, \boldsymbol{\mu} \rangle$, $F_2(\boldsymbol{\mu}) = \left\langle \beta^{-1}\boldsymbol{\mu}, \boldsymbol{\mu} \right\rangle$	
#4	$F_1(\boldsymbol{\mu}) = \left\langle \boldsymbol{V}_k, \boldsymbol{\mu} \right\rangle$, $F_2(\boldsymbol{\mu}) = \left\langle \boldsymbol{U}_k \boldsymbol{\mu}^k, \boldsymbol{\mu} \right\rangle$, $F_3(\boldsymbol{\mu}) = \left\langle \beta^{-1}\boldsymbol{\mu}, \boldsymbol{\mu} \right\rangle$	

Experiment #3

Sinkhorn regularized barycenter

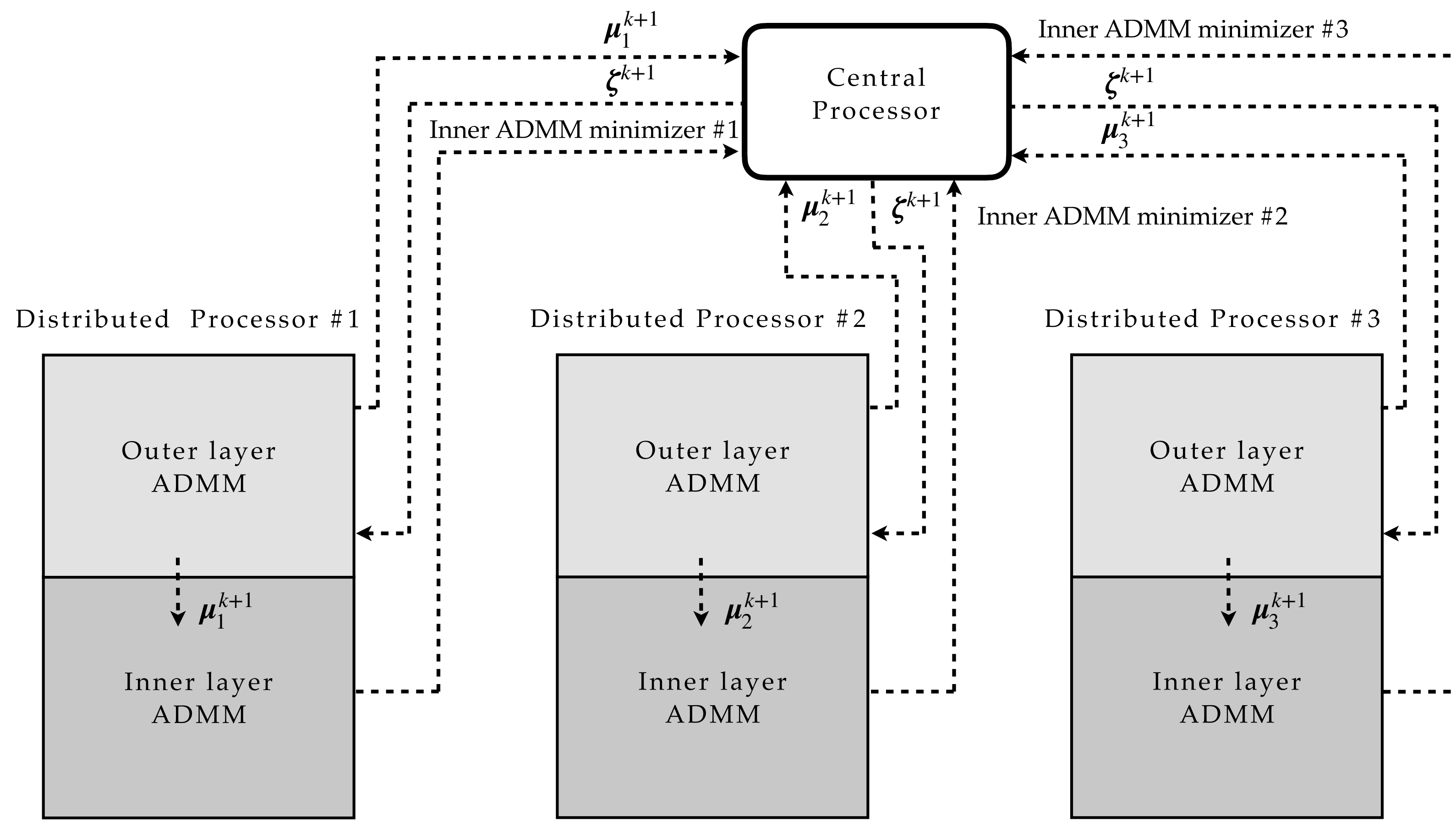


Summary

Distributed computation for measure-valued optimization

Realizes measure-valued operator splitting

Takes advantage of the existing proximal and JKO type algorithms



preprint arXiv:2309.07351

Ongoing

Convergence guarantees for the outer layer ADMM (technically challenging)

Is there an optimal way to split?

Open Postdoc Positions on Stochastic Control, Learning, Optimal Transport, Schrödinger Bridge

URL: https://isu.wd1.myworkdayjobs.com/IowaStateJobs/job/Ames-IA/Postdoctoral-Research-Associate_R13304

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Thank You

Support: 1923278, 2112755, 2111688

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More Results for Experiment #2

Effect of Varying the Outer Layer ADMM Barrier Parameter α

α	10	10.5	11	11.5	12	12.5	13	13.5	14	14.5	15
F^{10000}, case #1	10.8945	10.9153	10.9058	10.9224	10.8978	10.9064	10.8922	10.9203	10.9124	10.9203	10.9139
F^{10000}, case #2	11.0544	11.0586	11.0624	11.0598	11.0618	11.0578	11.0694	11.0692	11.0591	11.0570	11.0561
F^{10000}, case #3	11.0282	11.0344	11.0296	11.0325	11.0275	11.0312	11.0338	11.0301	11.0395	11.0351	11.0305
F^{10000}, case #4	16.5034	16.5051	16.5087	16.5012	16.5106	16.5080	16.5049	16.5029	16.5030	16.5018	16.5057

Effect of Varying the Inner Layer ADMM Iteration Number

Inner layer ADMM iter. #	3	4	5	6	7	8	9	10
F^{10000}, case #1	10.9263	10.8981	10.9165	10.8997	10.9124	10.9157	10.8813	10.9009
F^{10000}, case #2	11.0638	11.0546	11.0643	11.0625	11.0632	11.0583	11.0701	11.0678
F^{10000}, case #3	11.0368	11.0457	11.0374	11.0381	11.0363	11.0359	11.0318	11.0322
F^{10000}, case #4	16.5072	16.5023	16.5046	16.5001	16.5123	16.5039	16.5045	16.5034