A Geometric Approach for Learning Reach Sets

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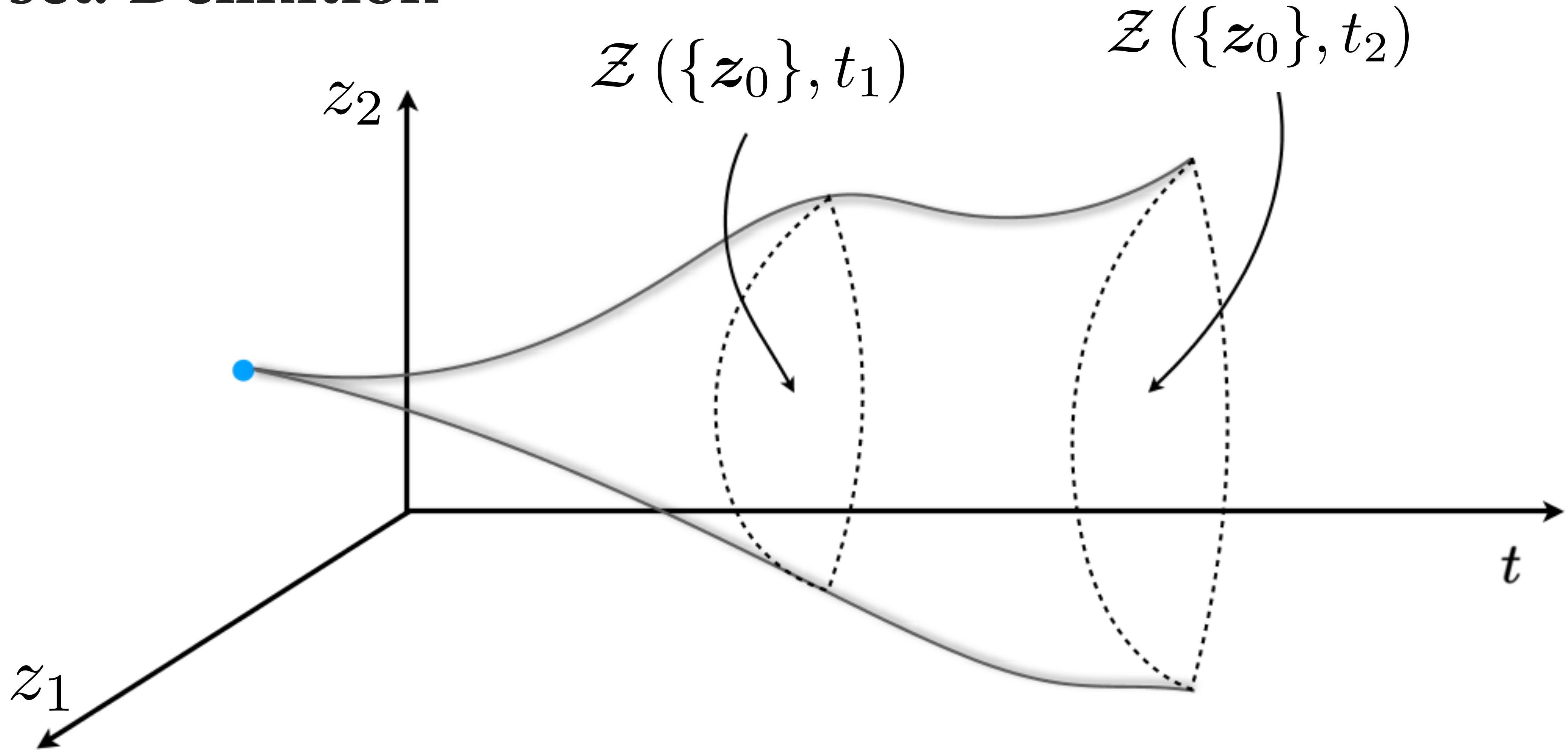
Reach set: Definition



Compact set

Controlled dynamics

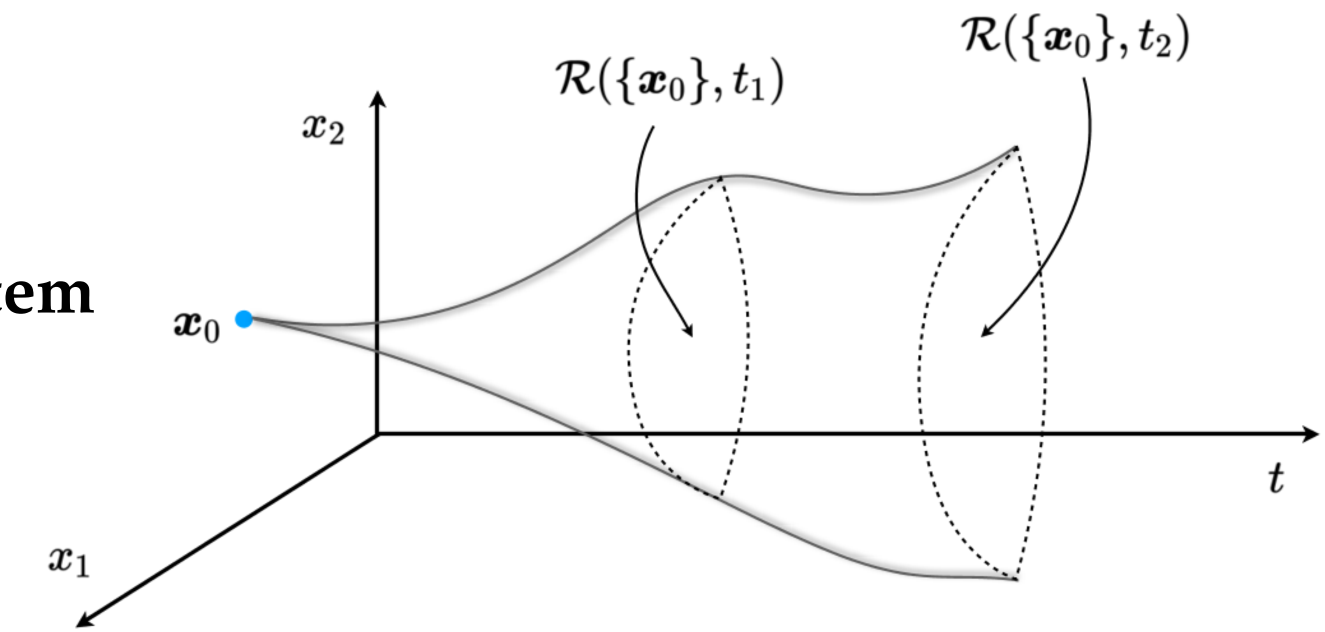
$$\dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}, \boldsymbol{v}), \quad \boldsymbol{z}(t=0) \in \mathbb{R}^{n_z}, \quad \boldsymbol{v} \in \mathcal{V} \subset \mathbb{R}^m$$

Forward reach set at time *t*

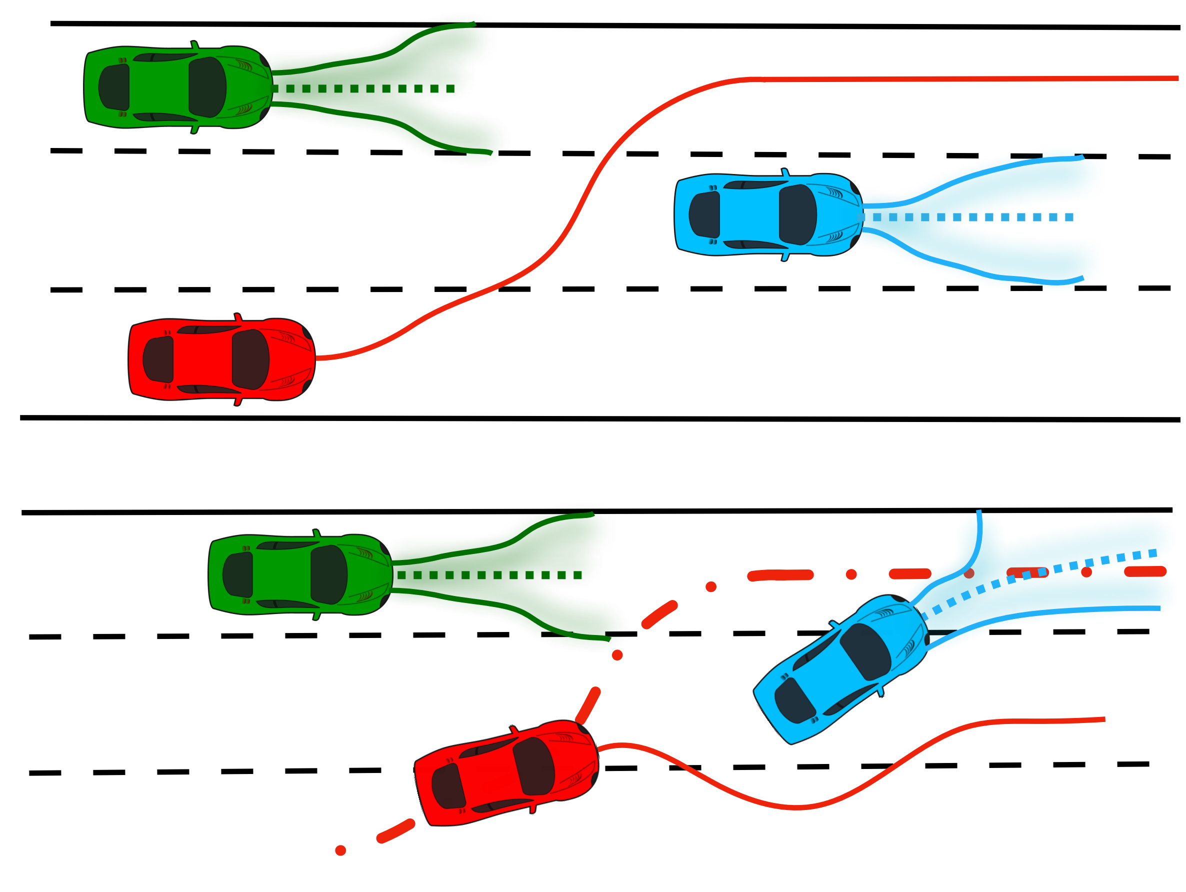
$$\mathcal{Z}_t := \{\boldsymbol{z}(t) \in \mathbb{R}^{n_z} \mid \dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}, \boldsymbol{v}), \quad \boldsymbol{z}(t=0) \in \mathbb{R}^{n_z}, \quad \boldsymbol{v} \in \mathcal{V} \subset \mathbb{R}^m\}.$$

Reach set: Applications

Predicting the states of an uncertain system

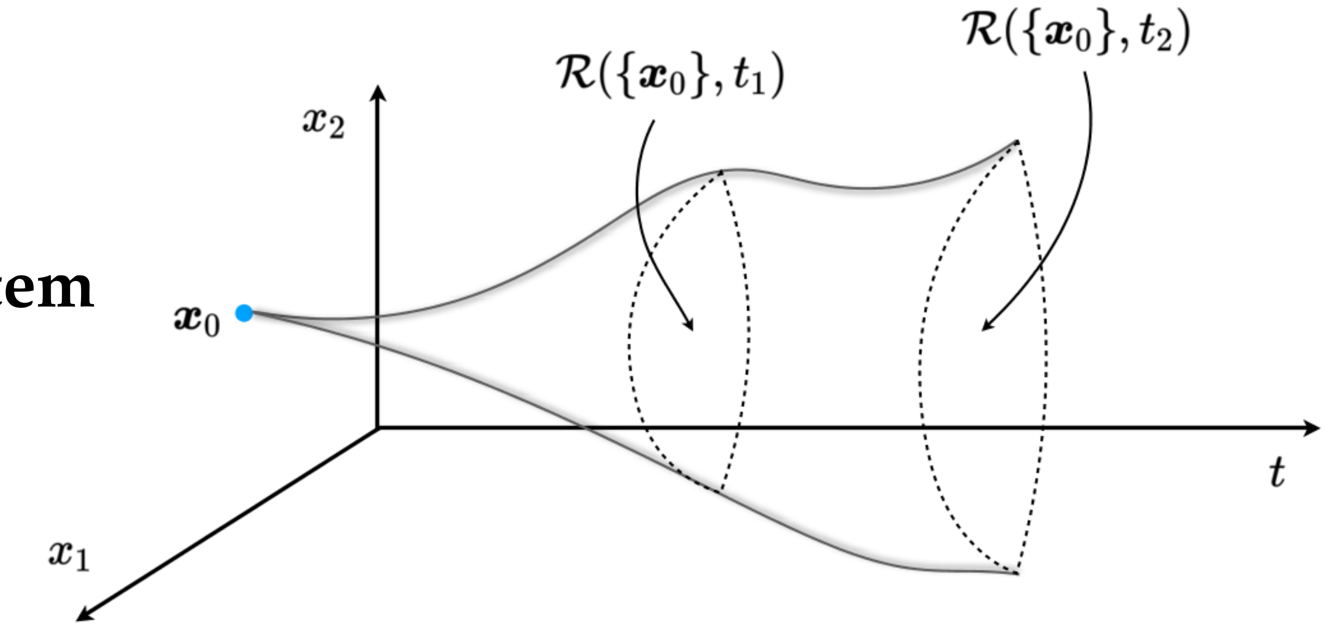


Safety critical applications such as motion planning & collision warning systems

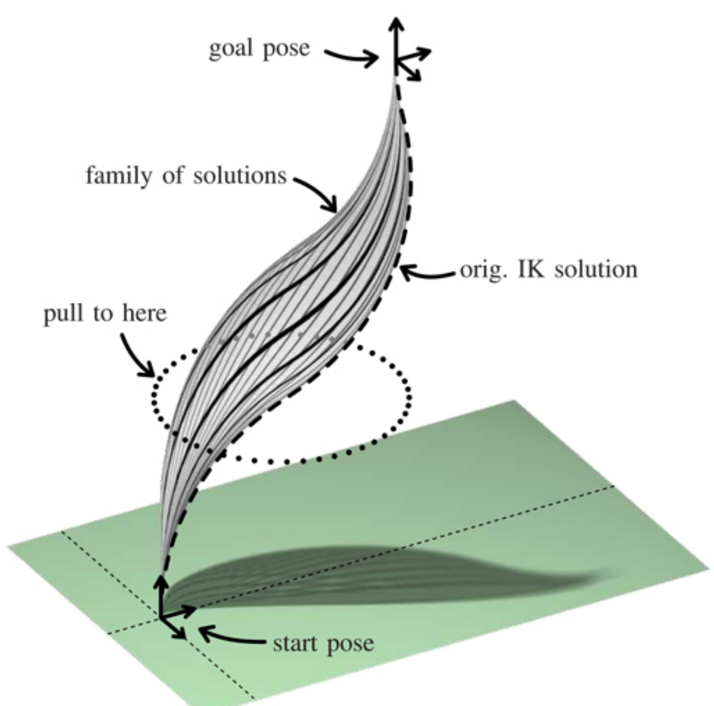


Reach set: Applications

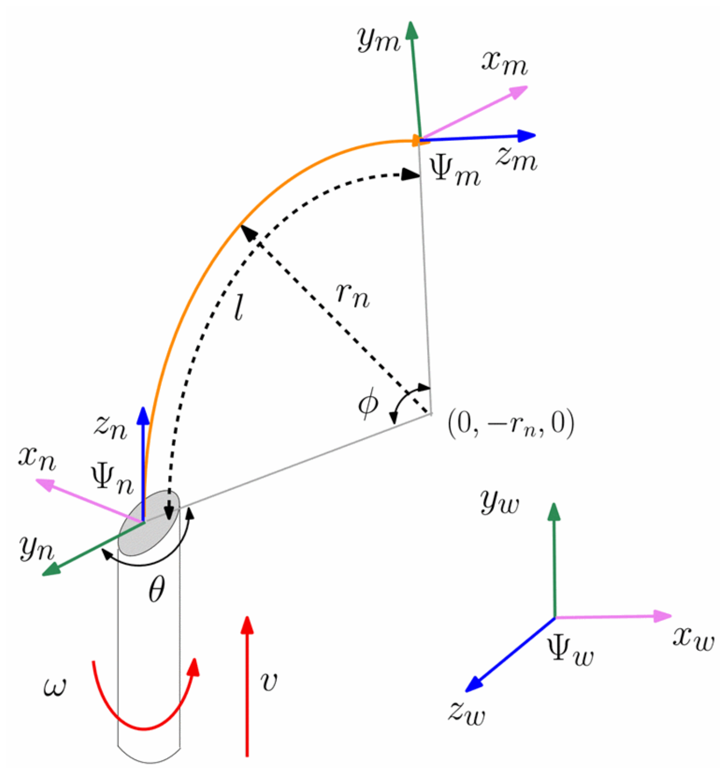
Predicting the states of an uncertain system



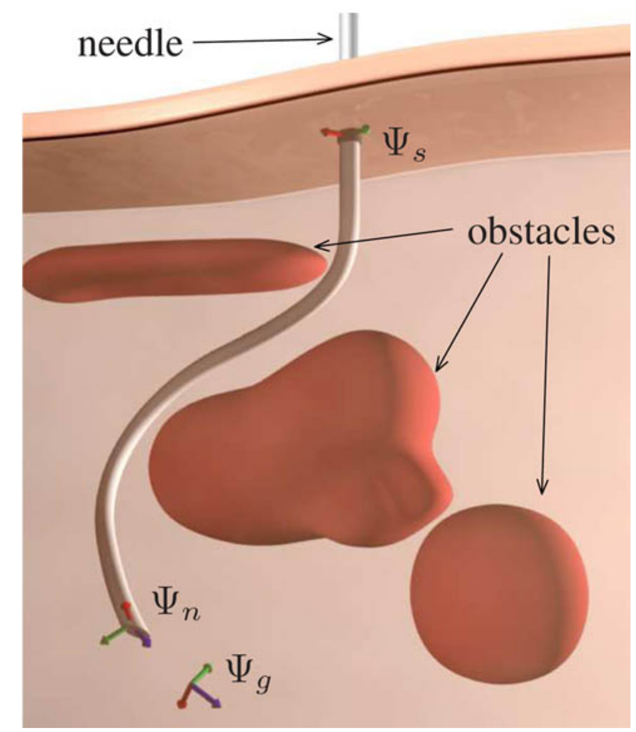
Needle steering w. input uncertainties



Credit: Duindam *et al., 2009*



Credit: Patil and Alterovitz, *2010*

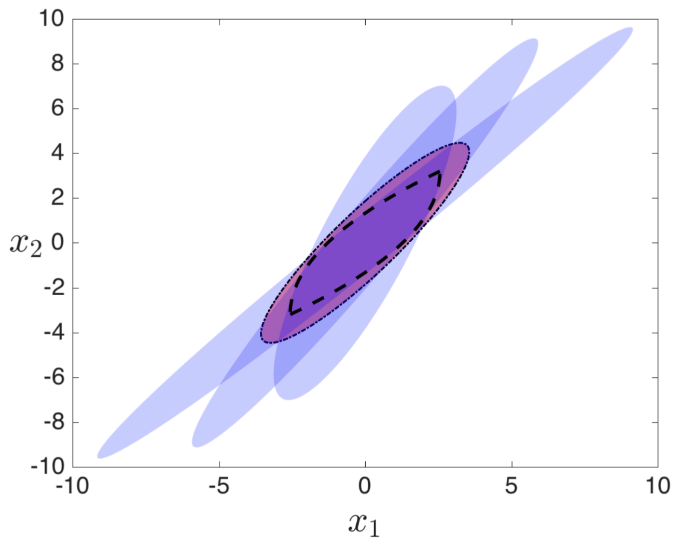


Credit: Duindam *et al., 2009*

Existing algorithms for reach set computation

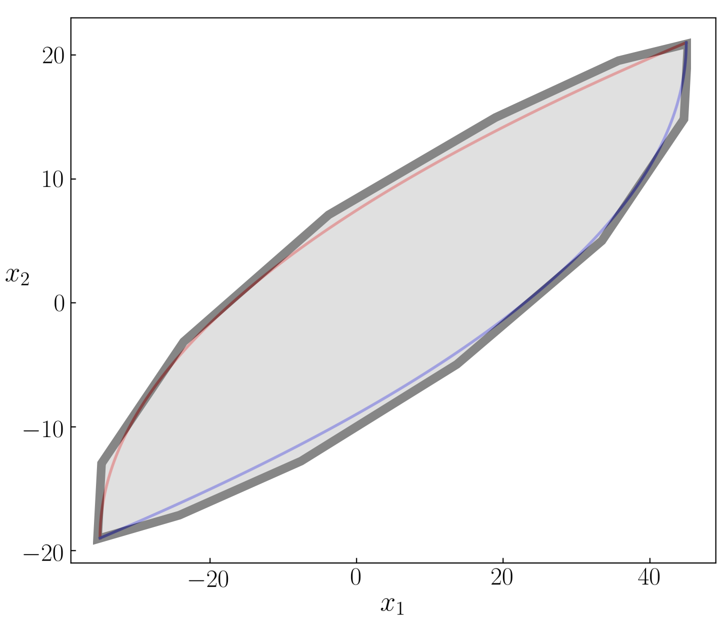
Parametric

Ellipsoidal over-approximation



Elipsoidal toolbox
[Kurzhansky et al., 2006]

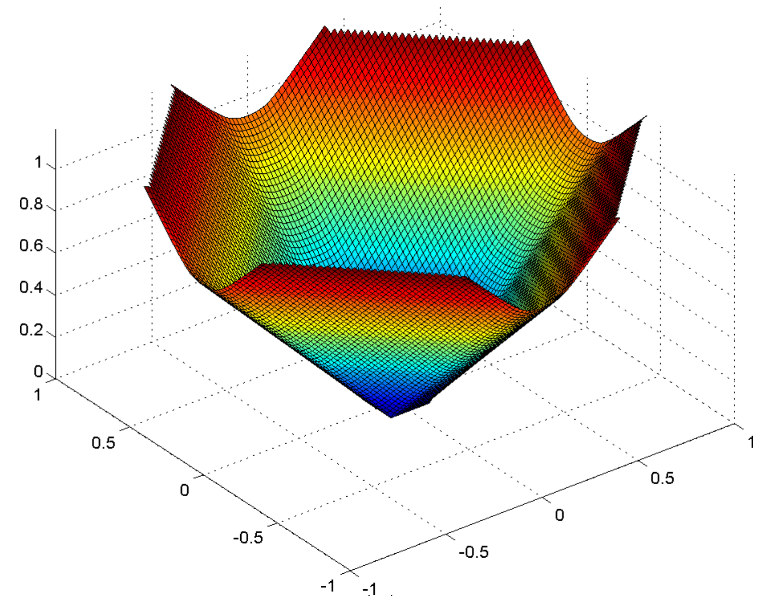
Zonotopic over-approximation

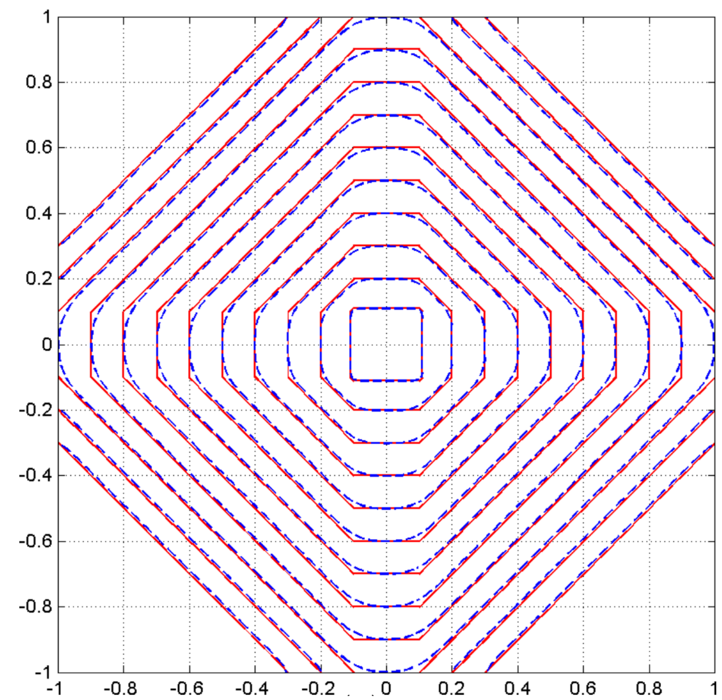


CORA toolbox
[Althoff et al., 2015]

Nonparametric

Zero sub-level set of the viscosity solution of HJB PDE

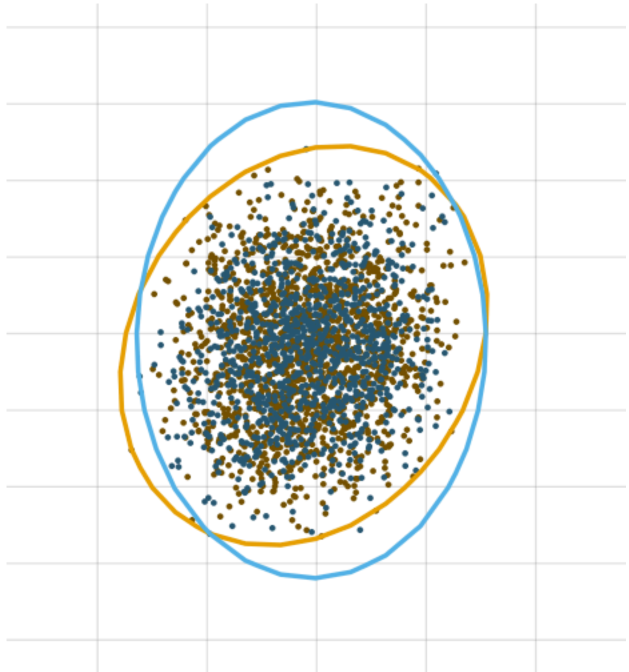


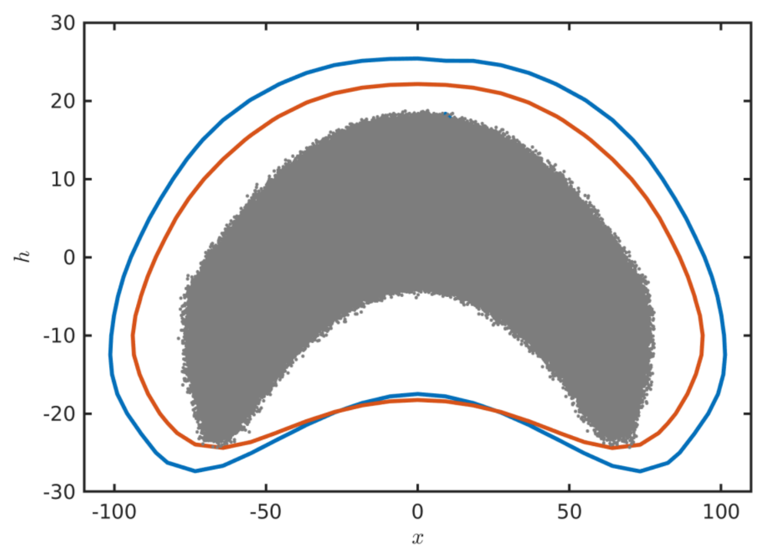


Level set toolbox
[Mitchell et al., 2008]

Semiparametric

Sample-based statistical learning





[Devonport and Arcak, 2020]

Existing algorithms for reach set computation

No specific algebraic or topological results about the ground truth

Difficult to quantitatively compare performance between two given algorithms

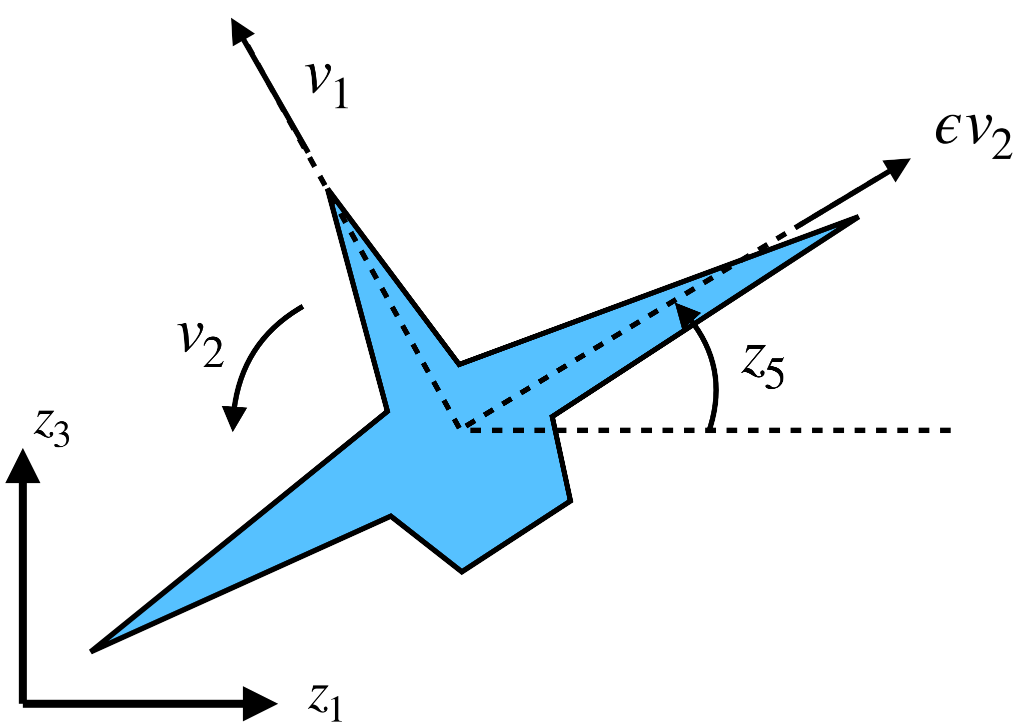
One-size-fits-all algorithms ignore the specific geometry induced by different class of systems

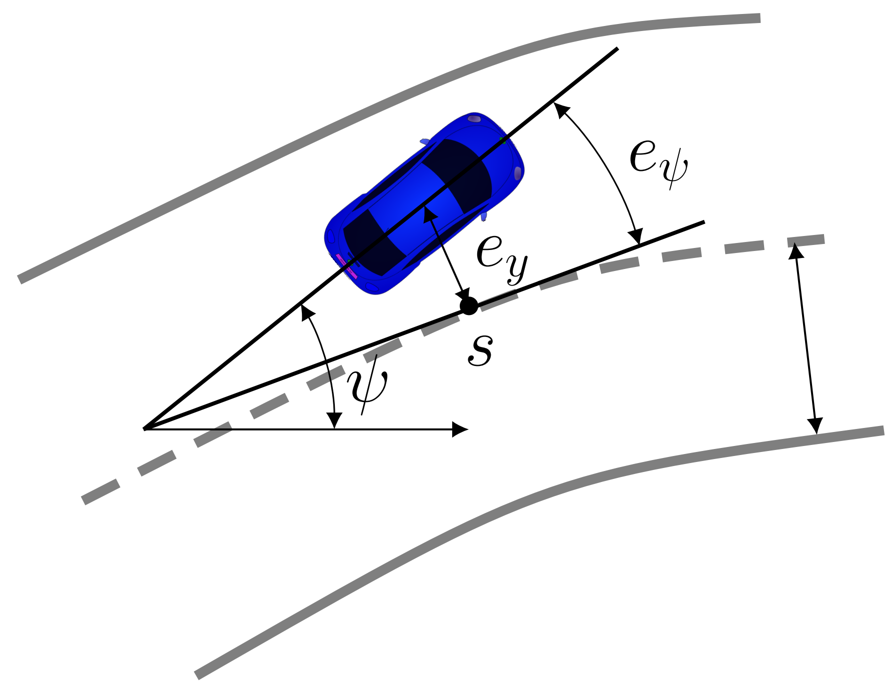
Our approach

Generic ⟶ specific algorithm exploiting geometry of the true set

Overall contribution

Algorithms for learning the reach sets of full state (static and dynamic) feedback linearizable systems





These reach sets are in general, compact and nonconvex

Background: Static state feedback linearizable systems

Control input

$$\dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}, \boldsymbol{v}), \quad \boldsymbol{z} \in \mathbb{R}^{n_z}, \quad \boldsymbol{v} \in \mathcal{V} \subseteq \mathbb{R}^{m_z}$$

There exist

a state diffeomorphism: $\boldsymbol{\tau} : \boldsymbol{z} \in \mathbb{R}^{n_z} \to \boldsymbol{x} \in \mathbb{R}^{n_x}, \; n_x = n_z$

Compact, continuous in *t*

along with the input homeomorphism: $\boldsymbol{\tau}_u : (\boldsymbol{z}, \boldsymbol{v}) \in \mathbb{R}^{n_z} \times \mathcal{V} \to \boldsymbol{u} \in \mathcal{U}(t) \subset \mathbb{R}^m$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(t) \in \mathbb{R}^{n_x}, \quad \boldsymbol{u}(t) \in \mathcal{U}(t) \subset \mathbb{R}^m,$$

$$\boldsymbol{A} := \mathrm{blkdiag}(\boldsymbol{A}_1, ..., \boldsymbol{A}_m), \quad \boldsymbol{B} := \mathrm{blkdiag}(\boldsymbol{b}_1, ..., \boldsymbol{b}_m),$$

This is known as the integrator dynamics a.k.a Brunovsky normal form

$$\boldsymbol{A}_j := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{r_j \times r_j}, \qquad \boldsymbol{b}_j := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_{r_j \times 1},$$

Relative degree vector

$$\boldsymbol{r} = (r_1, r_2, \ldots, r_m)^\top \in \mathbb{Z}_+^m,$$
$$r_1 + r_2 + \ldots r_m = n_x$$

Example: Static state feedback linearizable system

Single link manipulator dynamics with flexible joints and negligible damping: System (1)

$$\dot{z}_1 = z_2, \qquad \dot{z}_3 = z_4,$$
$$\dot{z}_2 = -\sin(z_1) - (z_1 - z_3), \qquad \dot{z}_4 = (z_1 - z_3) + v, \qquad \boldsymbol{z} \in \mathbb{R}^4 \text{ and } v \in \mathcal{V} \subset \mathbb{R}.$$

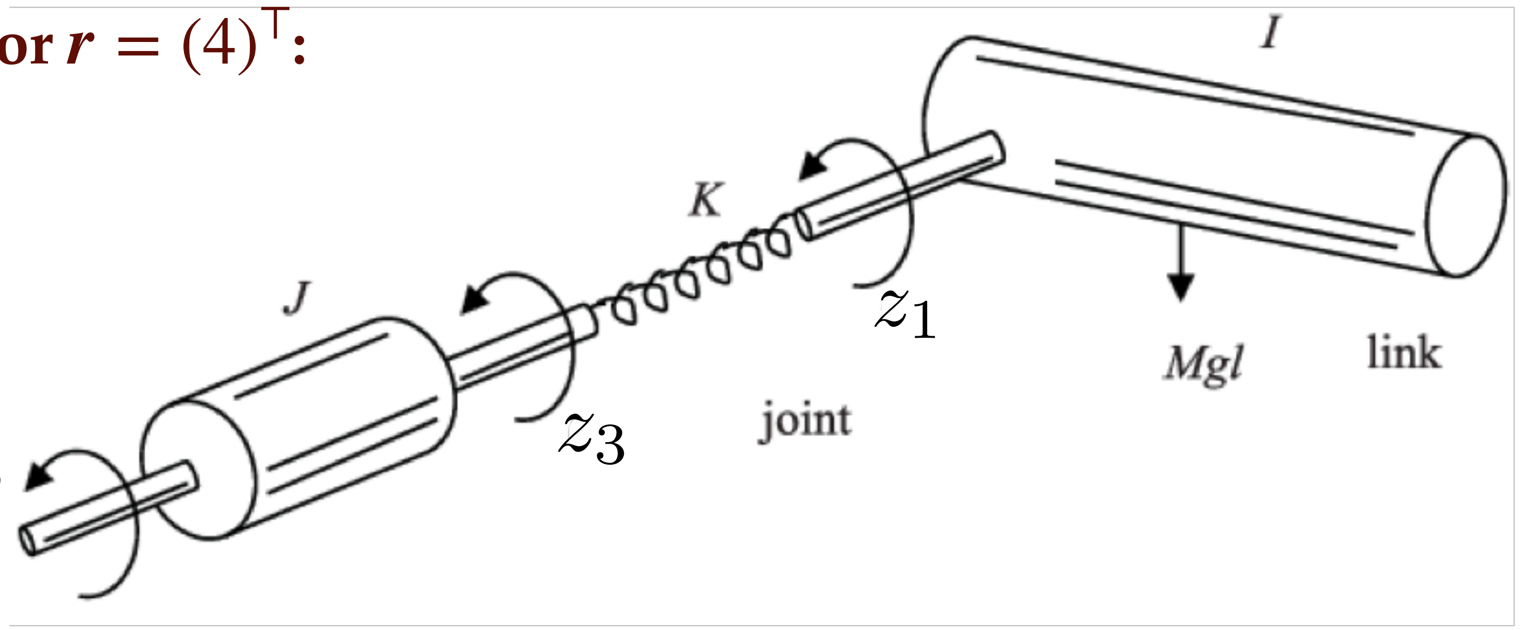
Diffeomorphism $\boldsymbol{\tau}$ and homeomorphism $\boldsymbol{\tau}_u$:

$$\boldsymbol{x} = \boldsymbol{\tau}(\boldsymbol{z}) = \begin{bmatrix} z_1 \\ z_2 \\ -\sin(z_1) - (z_1 - z_3) \\ -z_2\cos(z_1) - (z_2 - z_4) \end{bmatrix}, \qquad \boldsymbol{z} = \boldsymbol{\tau}^{-1}(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + \sin(x_1) + x_1 \\ x_4 + x_2\cos(x_1) + x_2 \end{bmatrix},$$

$$\boldsymbol{u} = \boldsymbol{\tau}_u(\boldsymbol{z}, \boldsymbol{v}) = -(\cos(z_1) + 2)(-\sin(z_1) + z_3 - z_1) + (z_2^2 - 1)\sin(z_1) + v.$$

Normal form with relative degree vector $\boldsymbol{r} = (4)^\top$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u.$$



Example: Static state feedback linearizable system

System with 5 states and 2 inputs: System (2)

$$\dot{z}_1 = z_2 + z_2^2 + v_1,$$
$$\dot{z}_2 = z_3 - z_1 z_4 + z_4 z_5, \qquad \dot{z}_4 = z_5,$$
$$\dot{z}_3 = z_2 z_4 + z_1 z_5 - z_5^2 + \cos(z_1 - z_5) v_1 + v_2, \qquad \dot{z}_5 = z_2^2 + v_2, \qquad \boldsymbol{z} \in \mathbb{R}^5 \text{ and } \boldsymbol{v} \in \mathcal{V} \subset \mathbb{R}^2.$$

Diffeomorphism $\boldsymbol{\tau}$ and homeomorphism $\boldsymbol{\tau}_u$:

$$\boldsymbol{x} = \boldsymbol{\tau}(\boldsymbol{z}) = \begin{bmatrix} z_1 - z_5 \\ z_2 \\ z_3 - z_1 z_4 + z_4 z_5 \\ z_4 \\ z_5 \end{bmatrix}, \quad \boldsymbol{z} = \boldsymbol{\tau}^{-1}(\boldsymbol{x}) = \begin{bmatrix} x_1 + x_5 \\ x_2 \\ x_3 + (x1 + x_5) x_4 - x_4 x_5 \\ x_4 \\ x_5 \end{bmatrix},$$

$$\boldsymbol{u} = \boldsymbol{\tau}_u(\boldsymbol{z}, \boldsymbol{v}) = \begin{bmatrix} \cos(z_1 - z_5) v_1 + v_2 \\ z_2^2 + v_2 \end{bmatrix}.$$

Normal form with relative degree vector $\boldsymbol{r} = (3,2)^\top$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Background: Dynamic state feedback linearizable systems

Control input

$$\dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}, \boldsymbol{v}), \quad \boldsymbol{z} \in \mathbb{R}^{n_z}, \quad \boldsymbol{v} \in \mathcal{V} \subseteq \mathbb{R}^m$$

Augmented state vector

There exist compensator states $\boldsymbol{w} \in \mathbb{R}^{n_w}$ such that

a state diffeomorphism: $\boldsymbol{\tau} : \boldsymbol{\rho} \in \mathbb{R}^{n_z + n_w} \to \boldsymbol{x} \in \mathbb{R}^{n_x}, \; n_x = n_z + n_w, \; \boldsymbol{\rho} := (\boldsymbol{z}, \boldsymbol{w})$

along with the input homeomorphism: $\boldsymbol{\tau}_u : (\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{w}, \dot{\boldsymbol{w}}, \ddot{\boldsymbol{w}}, \cdots) \to \boldsymbol{u} \in \mathcal{U}(t) \subset \mathbb{R}^m$

$$\boldsymbol{u}(t) = \boldsymbol{C}\left(\boldsymbol{z}(t), \boldsymbol{w}(t), \dot{\boldsymbol{w}}, \ddot{\boldsymbol{w}}, \cdots\right) \boldsymbol{v}(t) + \boldsymbol{d}\left(\boldsymbol{z}(t), \boldsymbol{w}(t), \dot{\boldsymbol{w}}, \ddot{\boldsymbol{w}}, \cdots\right),$$

$$\dot{\boldsymbol{w}}(t) = \boldsymbol{\phi}\left(\boldsymbol{z}, \boldsymbol{w}, \boldsymbol{v}, \dot{\boldsymbol{v}}, \ddot{\boldsymbol{v}}, \cdots\right), \quad \forall t \geq 0$$

In today's talk: **Compensator states are affine in control, and independent of time derivatives of the control**

Projection

Note: $\boldsymbol{x} \xrightarrow{\boldsymbol{\tau}^{-1}} \boldsymbol{\rho} \xrightarrow{\Pi_z} \boldsymbol{z}.$

Example: Dynamic state feedback linearizable system

System with 4 states and 2 inputs: System (3)

$$\dot{z}_1 = z_2 - v_1, \qquad \dot{z}_3 = v_1,$$
$$\dot{z}_2 = z_4 v_1, \qquad \dot{z}_4 = v_2, \qquad \boldsymbol{z} \in \mathbb{R}^4 \text{ and } \boldsymbol{v} \in \mathcal{V} \subset \mathbb{R}^2.$$

Compensator variable: $w = z_2 - v_1.$

Diffeomorphism $\boldsymbol{\tau}$ and homeomorphism $\boldsymbol{\tau}_u$:

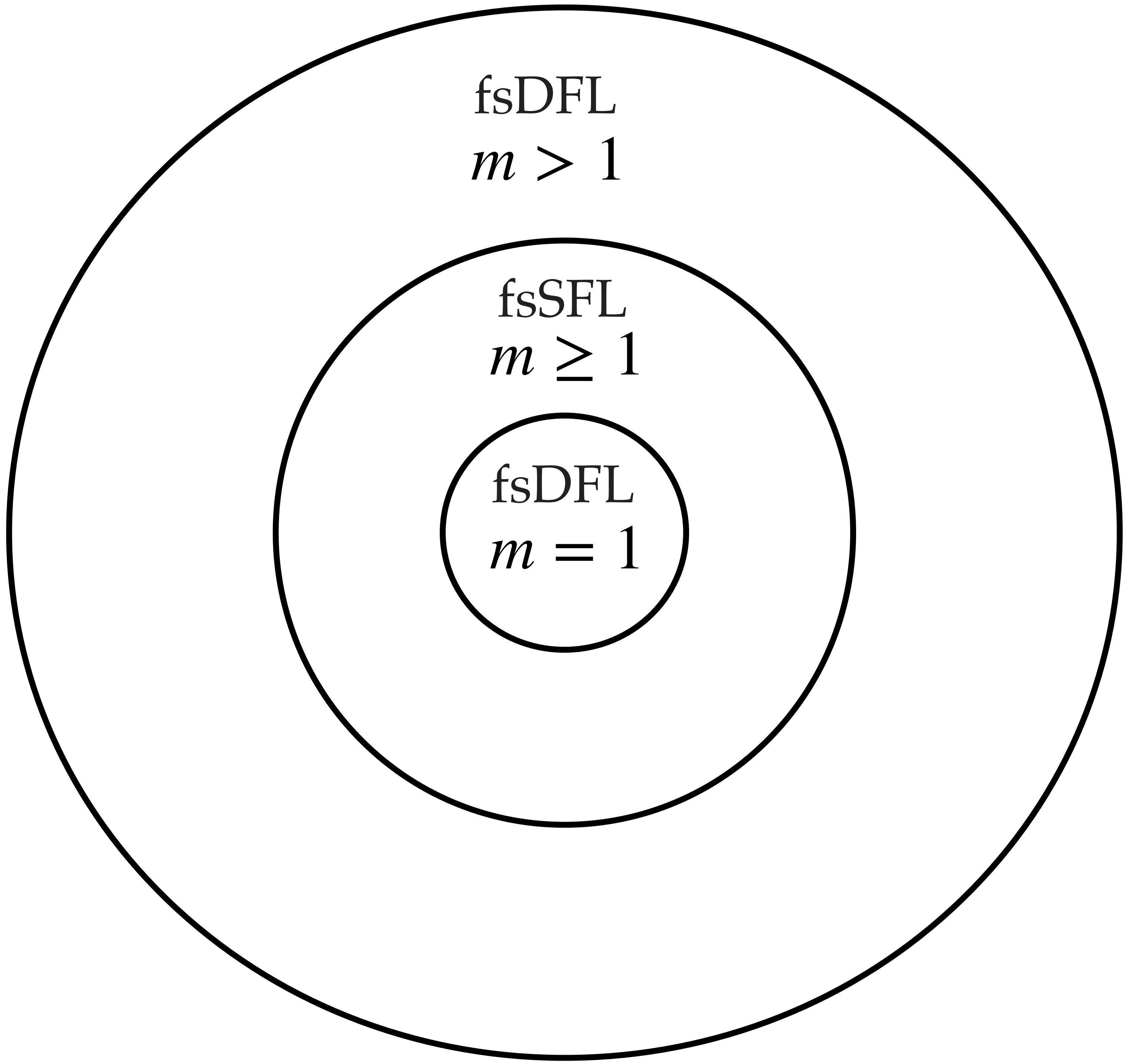
$$\boldsymbol{x} = \boldsymbol{\tau}(\boldsymbol{z}, w) = \begin{bmatrix} z_1 \\ w \\ z_1 + z_3 \\ z_2 \\ z_4(z_2 - w) \end{bmatrix}, \qquad \boldsymbol{z} = \boldsymbol{\tau}^{-1}(\boldsymbol{x}, w) = \begin{bmatrix} x_1 \\ x_4 \\ x_3 - x_1 \\ x_5/(x_4 - w) \\ w \end{bmatrix},$$

$$\boldsymbol{u} = \boldsymbol{\tau}_u(\boldsymbol{z}, \boldsymbol{w}, \boldsymbol{v}) = \begin{bmatrix} \dot{w} \\ v_2(z_2 - w) + z_4(z_4 v_1 - \dot{w}) \end{bmatrix}.$$

Normal form with relative degree vector $\boldsymbol{r} = (2,3)^\top$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

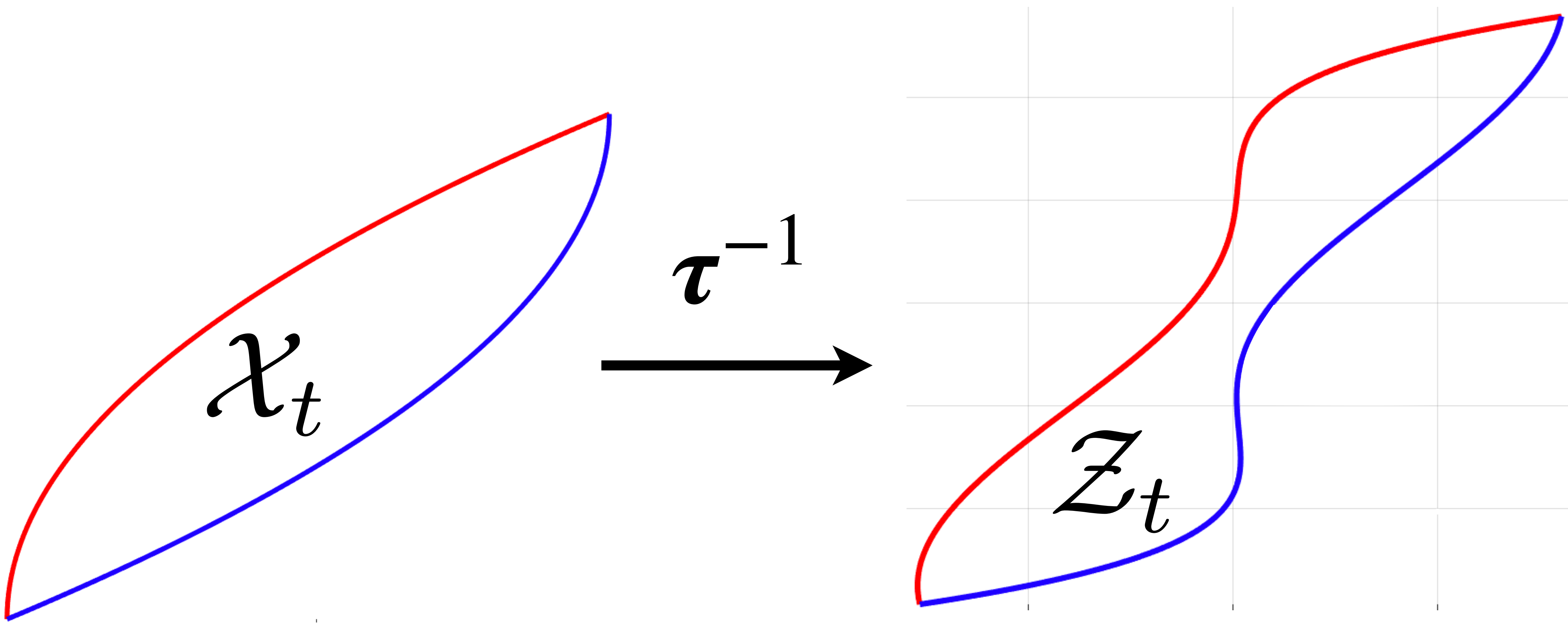
Venn diagram for full state (fs) static/dynamic (S/D) feedback linearizable (FL) systems



Main idea

Compute the reach set and its functionals in normal coordinate *x*

Map them back to original coordinate *z* via known diffeomorphism



Outline of this talk

1. Integrator reach sets with time invariant set-valued input uncertainties

2. Integrator reach sets with time varying set-valued input uncertainties

3. Intersection detection

4. Learning the reach set for full state feedback linearizable systems

5. Parallelization

6. Future plans

Integrator Reach Sets with Time Invariant Set-valued Input Uncertainties

Support function of $\mathcal{X}_t$ with $\mathcal{U} \subset \mathbb{R}^m$

$$\mathcal{X}\left(\mathcal{X}_0, t\right) := \left\{\boldsymbol{x}(t) \in \mathbb{R}^{n_x} \,|\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(t=0) \in \mathcal{X}_0, \boldsymbol{u}(t) \in \mathcal{U}\right\}$$

Integrator reach set

Single input integrator dynamics: $\mathcal{X}_j\left(\mathcal{X}_0, t\right) \subset \mathbb{R}^{r_j}, \quad j = 1, \ldots, m$

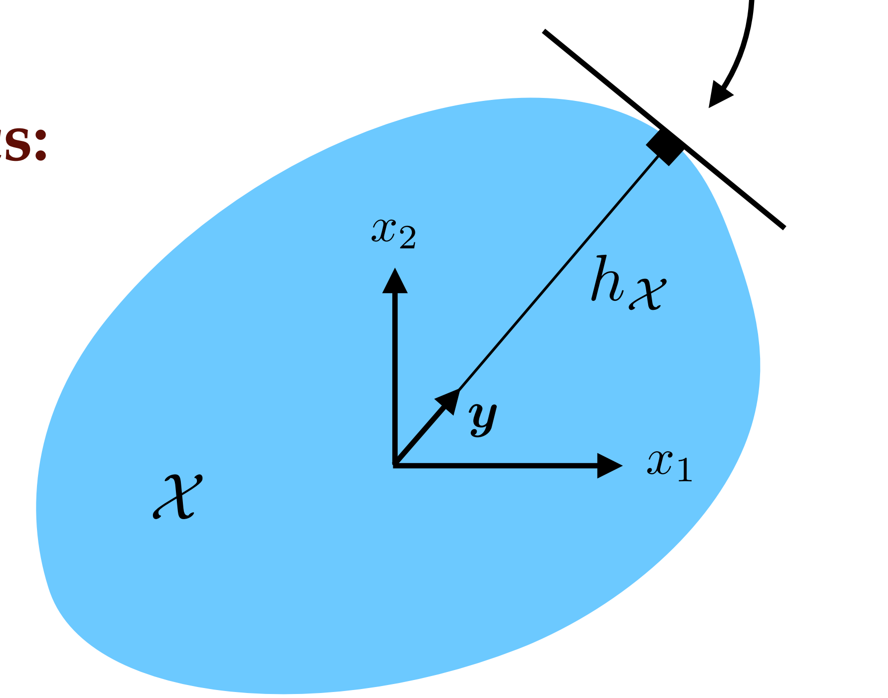
$$\begin{aligned}\mathcal{X}\left(\mathcal{X}_0, t\right) &= \mathcal{X}_1\left(\mathcal{X}_{10}, t\right) \times \mathcal{X}_2\left(\mathcal{X}_{20}, t\right) \times \ldots \times \mathcal{X}_m\left(\mathcal{X}_{m0}, t\right) \\ &= \mathcal{X}_1\left(\mathcal{X}_{10}, t\right) \dot{+} \mathcal{X}_2\left(\mathcal{X}_{20}, t\right) \dot{+} \ldots \dot{+} \mathcal{X}_m\left(\mathcal{X}_{m0}, t\right)\end{aligned}$$

Minkowski sum

Supporting hyperplane

Support function of single input integrator dynamics:

$$\begin{aligned}h_{\mathcal{X}(\mathcal{X}_0,t)}(\boldsymbol{y}) &:= \sup_{\boldsymbol{x}\in\mathcal{X}} \left\{\langle \boldsymbol{x}, \boldsymbol{y}\rangle \;\middle|\; \boldsymbol{y} \in \mathbb{S}^{n_x - 1}\right\} \\ &= \sum_{j=1}^{m} h_{\mathcal{X}_j(\mathcal{X}_{j0},t)}(\boldsymbol{y}_j)\end{aligned}$$



Support function of $\mathcal{X}_t$ with $\mathcal{U} \subset \mathbb{R}^m$

Theorem 1. For compact convex $\mathcal{X}_0 \in \mathbb{R}^{n_x}$ and compact $\mathcal{U} \in \mathbb{R}^m$, the support function of the integrator reach set is

$$h_{\mathcal{X}(\mathcal{X}_0,t)}(\boldsymbol{y}) = \sum_{j=1}^{m} \left\{ \sup_{\boldsymbol{x}_{j0} \in \mathcal{X}_{j0}} \langle \boldsymbol{y}_j, \exp\left(t\boldsymbol{A}\right)\boldsymbol{x}_{j0}\rangle + \nu_j \langle \boldsymbol{y}_j, \boldsymbol{\zeta}_j(t)\rangle + \mu_j \int_0^t |\langle \boldsymbol{y}_j, \boldsymbol{\xi}_j(s)\rangle| \, \mathrm{d}s \right\}$$

where

$$\mu_j := \frac{\beta_j - \alpha_j}{2}, \quad \nu_j := \frac{\beta_j + \alpha_j}{2}, \quad \alpha_j := \min_{\boldsymbol{u} \in \mathcal{U}} u_j, \quad \beta_j := \max_{\boldsymbol{u} \in \mathcal{U}} u_j, \quad j = 1, \cdots, m,$$

$$\boldsymbol{\xi}(s) := \begin{pmatrix} \mu_1 \boldsymbol{\xi}_1(s) \\ \vdots \\ \mu_m \boldsymbol{\xi}_m(s) \end{pmatrix}, \ \boldsymbol{\xi}_j(s) := \begin{pmatrix} s^{r_j - 1}/(r_j - 1)! \\ s^{r_j - 2}/(r_j - 2)! \\ \vdots \\ s \\ 1 \end{pmatrix}, \ \boldsymbol{\zeta}_j(t_0, t) := \int_{t_0}^t \boldsymbol{\xi}_j(s) \, \mathrm{d}s \in \mathbb{R}^{r_j}$$

Parametric formula of boundary $\partial\mathcal{X}_t$ for $\mathcal{U} \in \mathbb{R}^m$

Theorem. Assume $\mathcal{X}_0 \equiv \{\boldsymbol{x}_0\}$. Then

Components of the boundary

$$\boldsymbol{x}_j^{\text{bdy}}(k) = \sum_{\ell=1}^{r_j} \mathbf{1}_{k\leq\ell} \frac{t^{\ell-k}}{(\ell-k)!} \boldsymbol{x}_{j0}(\ell) + \frac{\nu_j\, t^{r_j-k+1}}{(r_j-k+1)!}$$
$$\pm \frac{\mu_j}{(r_j-k+1)!}\left\{(-1)^{r_j-1}\, t^{r_j-k+1} + 2\sum_{q=1}^{r_j-1}(-1)^{q+1}\, s_q^{r_j-k+1}\right\},$$

Parameters: $0 \leq s_1 \leq s_2 \leq \ldots \leq s_{r_j-1} \leq t, \quad j = 1, \ldots, m$

Each single input integrator reach set has two bounding surfaces:

$$\mathcal{X}_j(\{\boldsymbol{x}_0\}, t) = \{\boldsymbol{x} \in \mathbb{R}^{r_j} \mid p_j^{\text{upper}}(\boldsymbol{x}) \leq 0,\ p_j^{\text{lower}}(\boldsymbol{x}) \leq 0\},$$

with boundary:

$$\partial\mathcal{X}_j(\{\boldsymbol{x}_0\}, t) = \{\boldsymbol{x} \in \mathbb{R}^{r_j} \mid p_j^{\text{upper}}(\boldsymbol{x}) = 0\} \cup \{\boldsymbol{x} \in \mathbb{R}^{r_j} \mid p_j^{\text{lower}}(\boldsymbol{x}) = 0\}.$$

Implicit formula of boundary $\partial\mathcal{X}_t$ for $\mathcal{U} \in \mathbb{R}^m$

Generating function of the parametric form:

$$F(\tau) = \sum_{k\geq 0} A_k \tau^k = \frac{(1-s_1\tau)(1-s_3\tau)\cdots}{(1-s_2\tau)(1-s_4\tau)\cdots}, \qquad \textbf{(1)}$$

Taking the logarithmic derivative for $q = 1, \cdots, n_x - 1$

$$\frac{F'(\tau)}{F(\tau)} = -s_1 \sum_{k\geq 0} (s_1\tau)^k + s_2 \sum_{k\geq 0} (s_2\tau)^k - s_3 \sum_{k\geq 0} (s_3\tau)^k + \ldots,$$

Integrating with respect to τ:

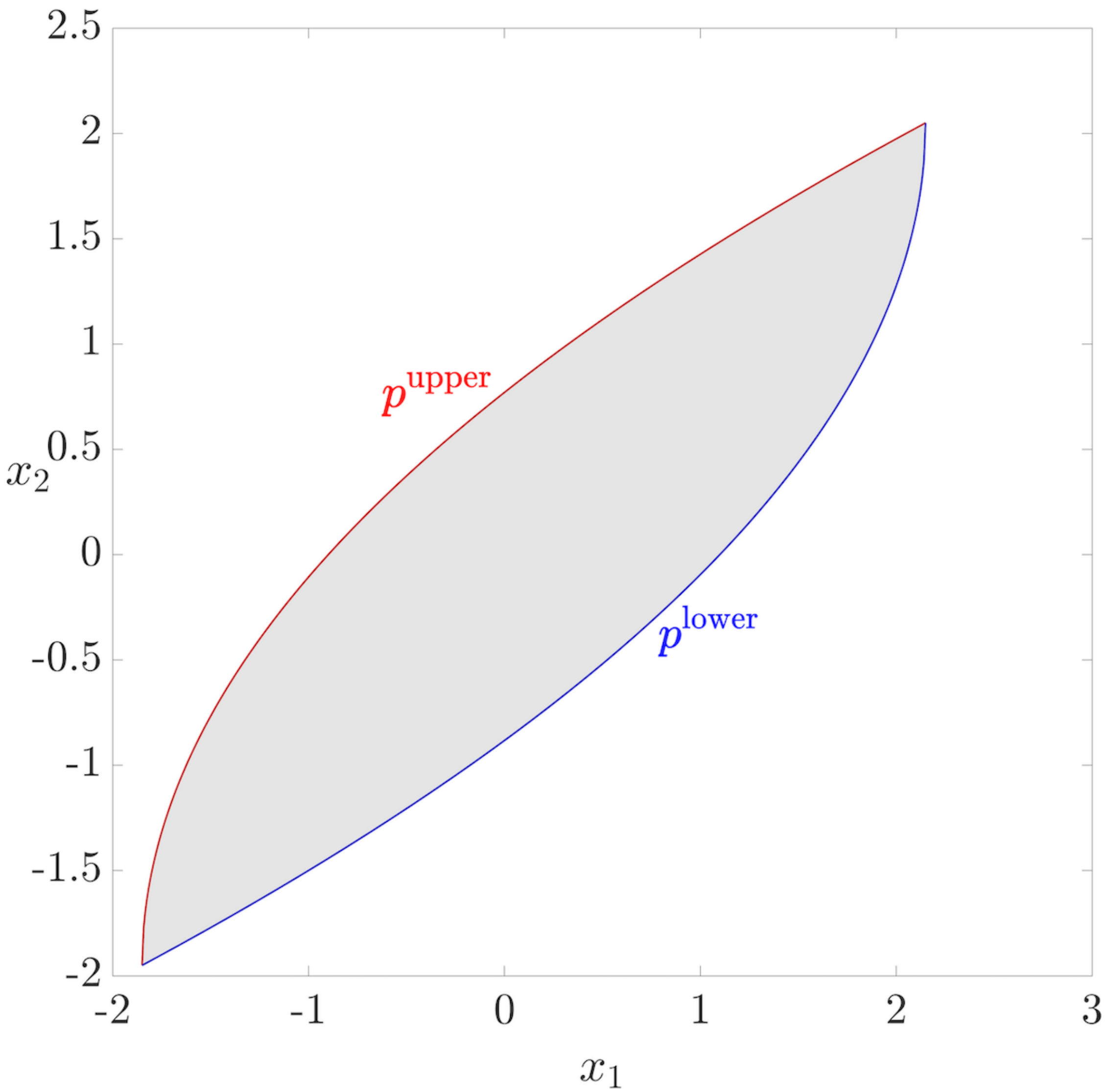
$$F(\tau) = \exp\left(-\sum_{k=1}^{n_x} \frac{\lambda_k}{k} \tau^k\right), \qquad \textbf{(2)}$$

Equating (1) and (2), the following Hankel determinant gives implicit formula

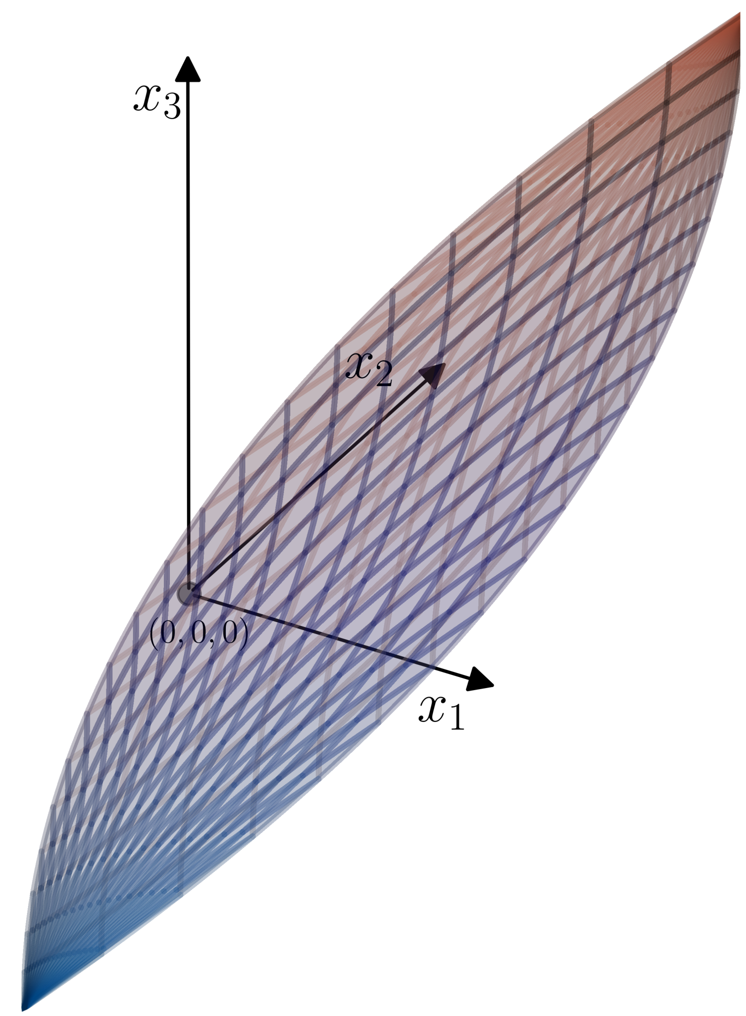
$$\boxed{\det[A_{n_x - 2\delta + i + j}]_{i,j=0}^{\delta} = 0.}$$

Taxonomy of $\mathcal{X}_t$ for $\mathcal{U} \in \mathbb{R}^m$

Theorem. The set $\mathcal{X}_t$ with $\mathcal{X}_0 \equiv \{\boldsymbol{x}_0\}$ is $\boxed{\text{semialgebraic}}$



The single input double integrator reach set



The single input triple integrator reach set

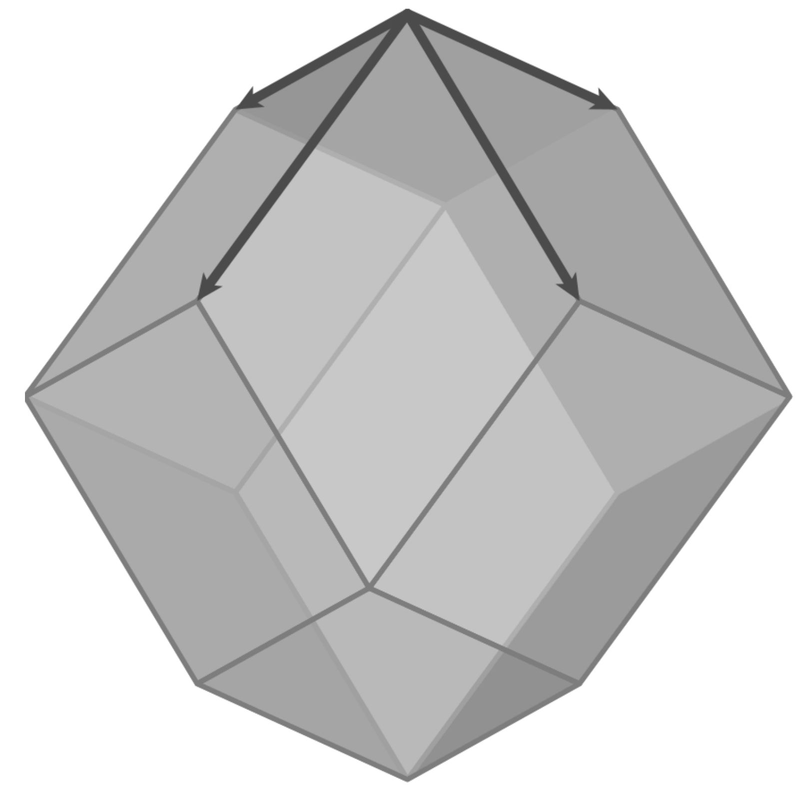
Taxonomy of $\mathcal{X}_t$ for $\mathcal{U} \in \mathbb{R}^m$

Zonotope of dimension d

Generators

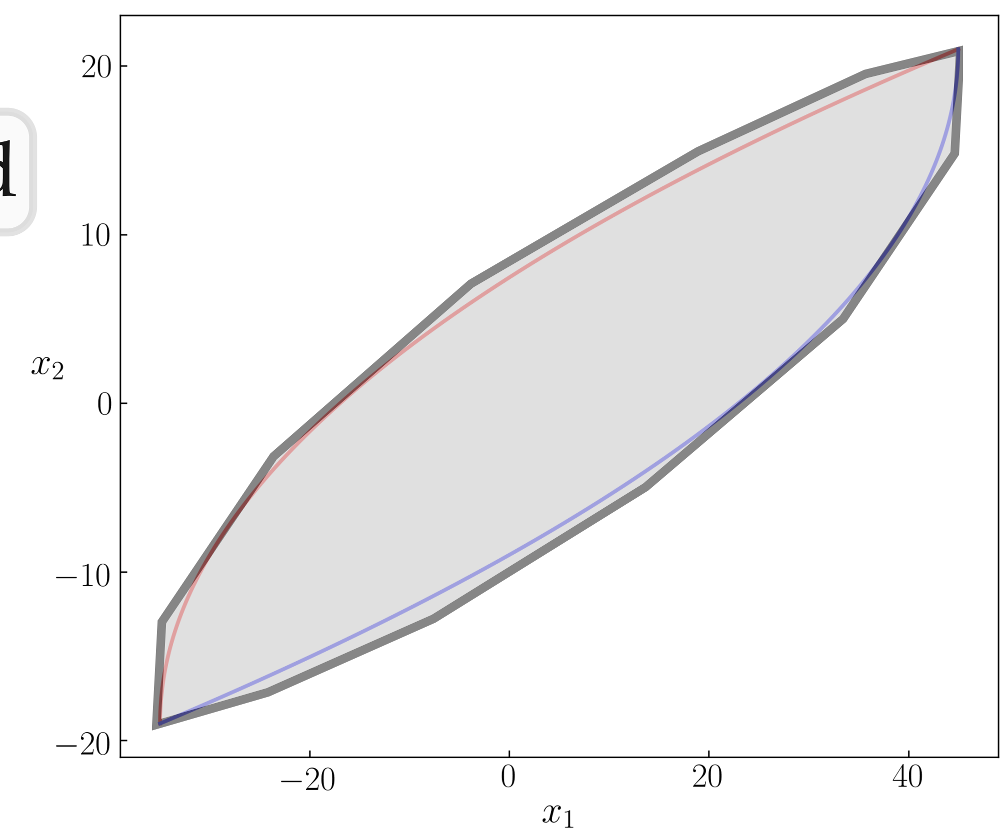
$$\mathcal{Z}_n := \left\{ \sum_{j=1}^{n} \gamma_j \boldsymbol{v}_j \,|\, \gamma_j \in [-1,1], \boldsymbol{v}_j \in \mathbb{R}^d, j = 1, \ldots, n \right\}$$

$$h_{\mathcal{Z}_n}(\boldsymbol{y}) = \sum_{j=1}^{n} |\langle \boldsymbol{y}, \boldsymbol{v}_j \rangle|, \quad \boldsymbol{y} \in \mathbb{R}^d$$



Zonoid: Limiting set of the Minkowski sum of line segments

Theorem. The set $\mathcal{X}_t$ with $\mathcal{X}_0 \equiv \{\boldsymbol{x}_0\}$ is zonoid

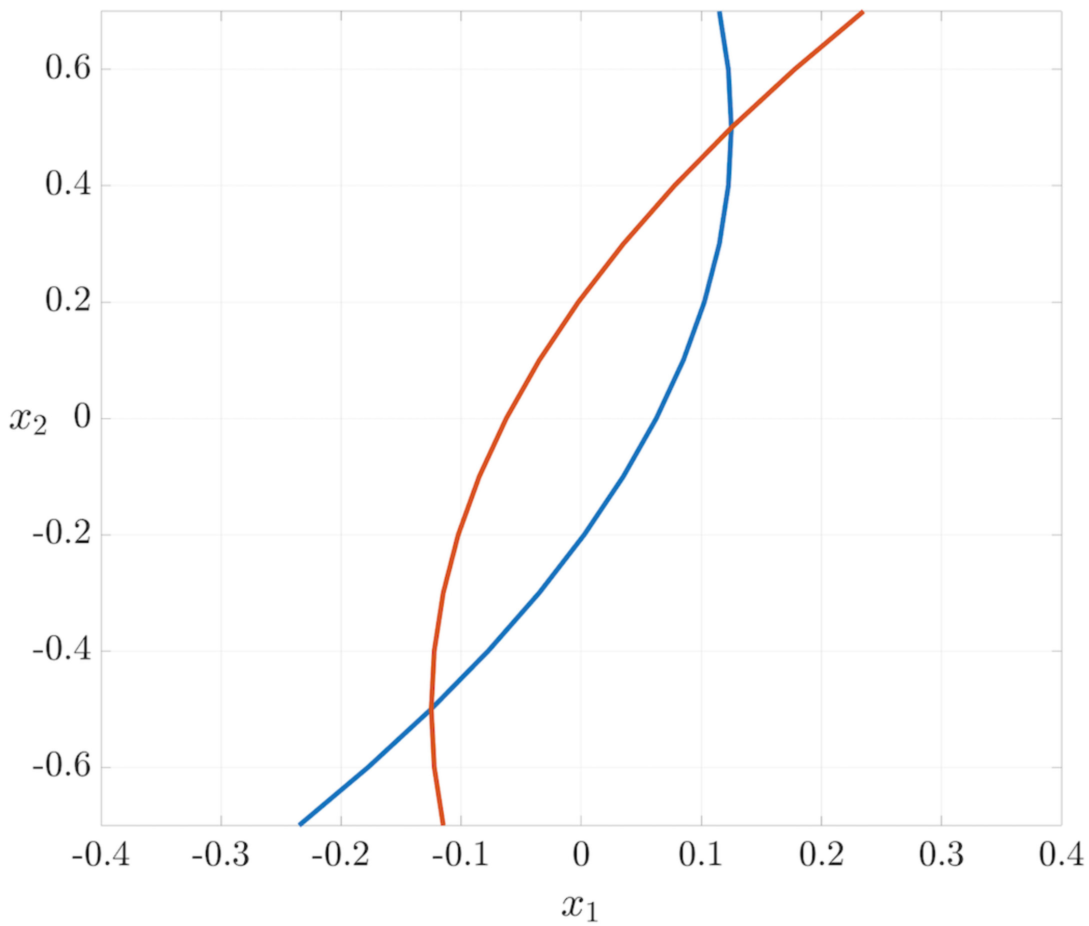


Taxonomy of $\mathcal{X}_t$ for $\mathcal{U} \in \mathbb{R}^m$

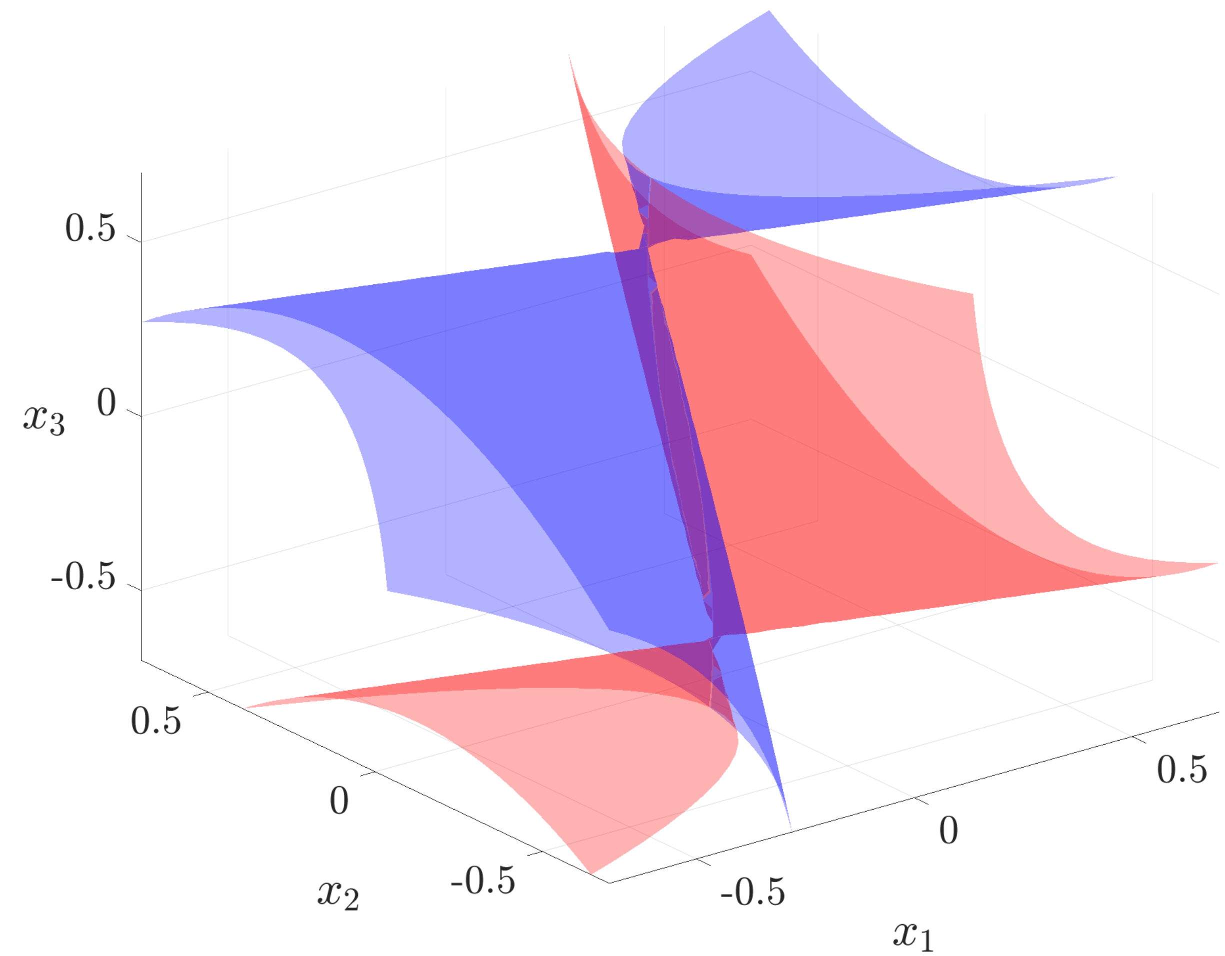
Integrator Reach Set Is Not Spectrahedron

Polynomial degree of n_x dimensional integrator reach set surface:

$$\left(\left\lfloor \frac{n_x-1}{2} \right\rfloor + 1\right)\left(n_x - \left\lfloor \frac{n_x-1}{2} \right\rfloor\right)$$

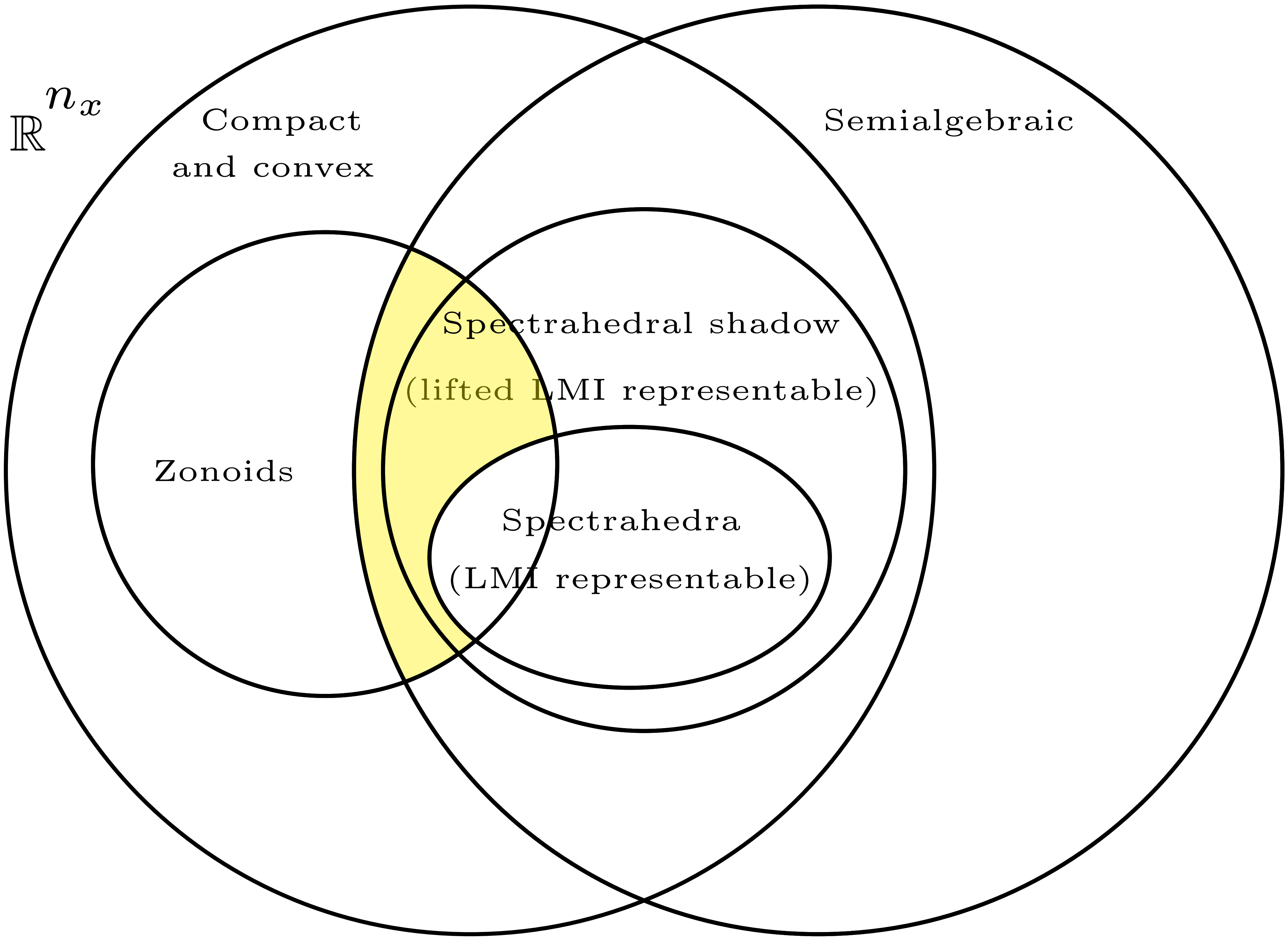


Degree of $\partial \mathcal{X}_t$ is 2
Number of intersections by generic line is 4



Degree of $\partial \mathcal{X}_t$ is 4
Number of intersections by generic line is 6

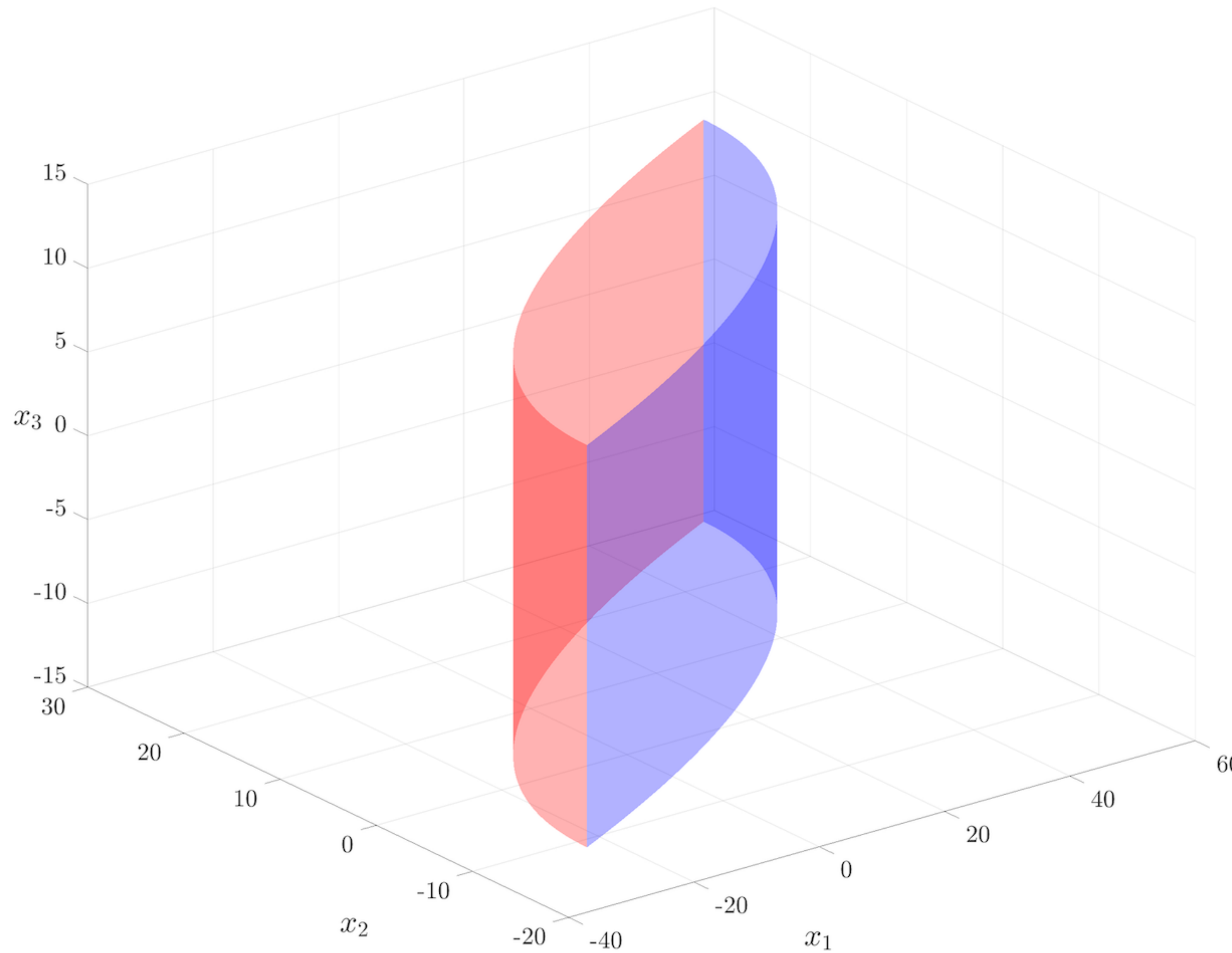
Taxonomy of $\mathcal{X}_t$ for $\mathcal{U} \in \mathbb{R}^m$



Volume of $\mathcal{X}_t$ for $\mathcal{U} \in \mathbb{R}^m$

Theorem. $$\mathrm{vol}\left(\mathcal{X}\left(\{\boldsymbol{x}_0\}, t\right)\right) = 2^{n_x} \prod_{j=1}^{m} \left\{ \mu_j^{r_j} t^{r_j(r_j+1)/2} \prod_{k=1}^{r_j - 1} \frac{k!}{(2k+1)!} \right\}.$$

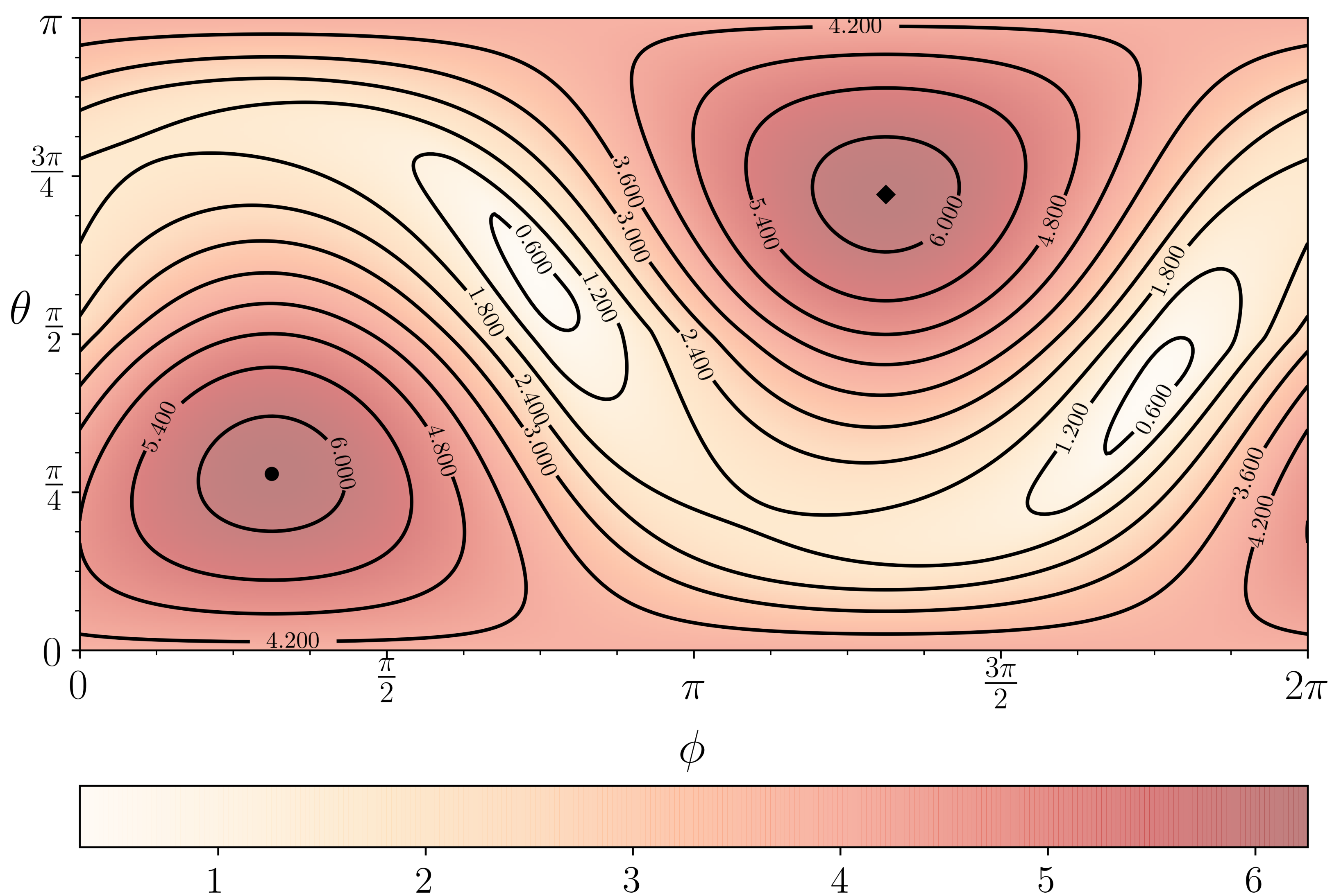
$$\mathrm{vol}\left(\mathcal{X}\right) = \frac{4}{3}\mu_1^2 \mu_2 t^4$$



The integrator reach set at $t = 4$, $\boldsymbol{r} = (2,1)^\top$

Diameter of $\mathcal{X}_t$ for $\mathcal{U} \in \mathbb{R}^m$

Theorem. $$\operatorname{diam}\left(\mathcal{X}\left(\{\boldsymbol{x}_0\}, t\right)\right) = 2 \parallel \boldsymbol{\zeta}(t) \parallel_2 = 2 \left(\sum_{j=1}^{m} \mu_j^2 \|\boldsymbol{\zeta}_j\|^2\right)^{1/2}$$

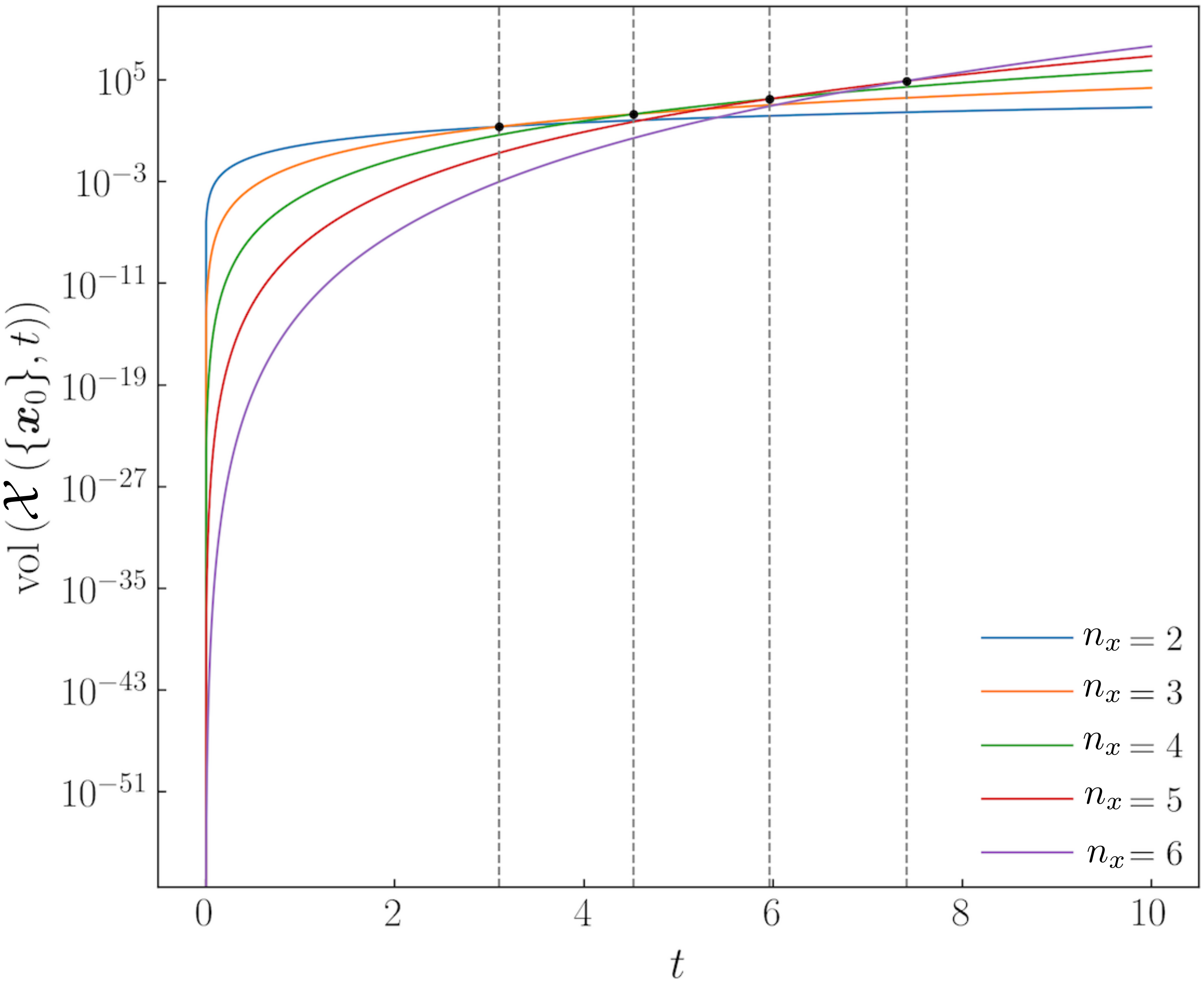


● $(\arctan(3/t), \arccos(6/\sqrt{t^4 + 9t^2 + 36}))$

◆ $(\pi + \arctan(3/t), \arccos(-6/\sqrt{t^4 + 9t^2 + 36}))$

Scaling Laws

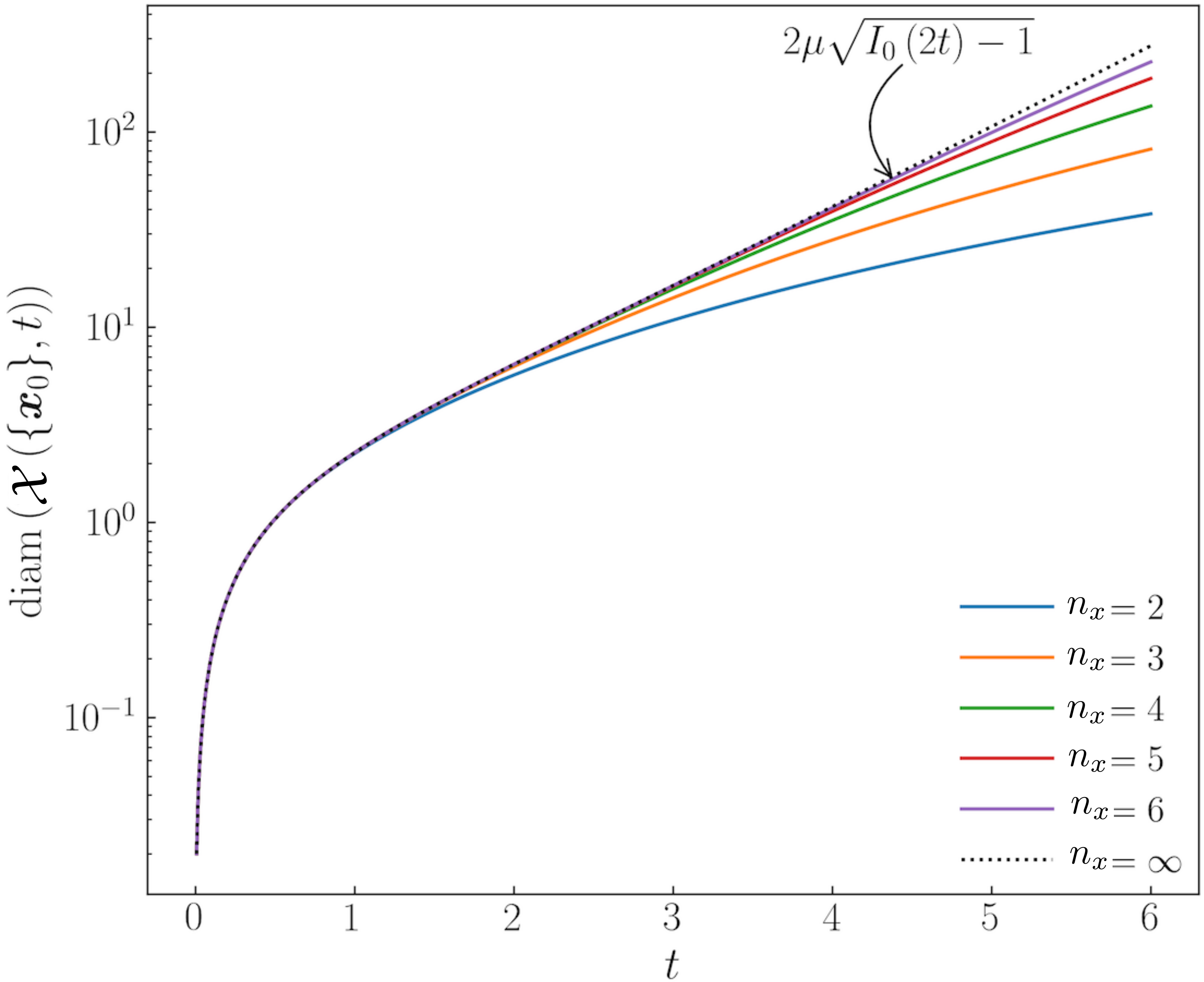
$\mathcal{X}_0 \in \mathbb{R}^{n_x}; \mathcal{X}_0 \equiv \{\boldsymbol{x}_0\}$



Volume of integrator reach set vs time

Scaling Laws

$\mathcal{X}_0 \in \mathbb{R}^{n_x}$; $\mathcal{X}_0 \equiv \{\boldsymbol{x}_0\}$

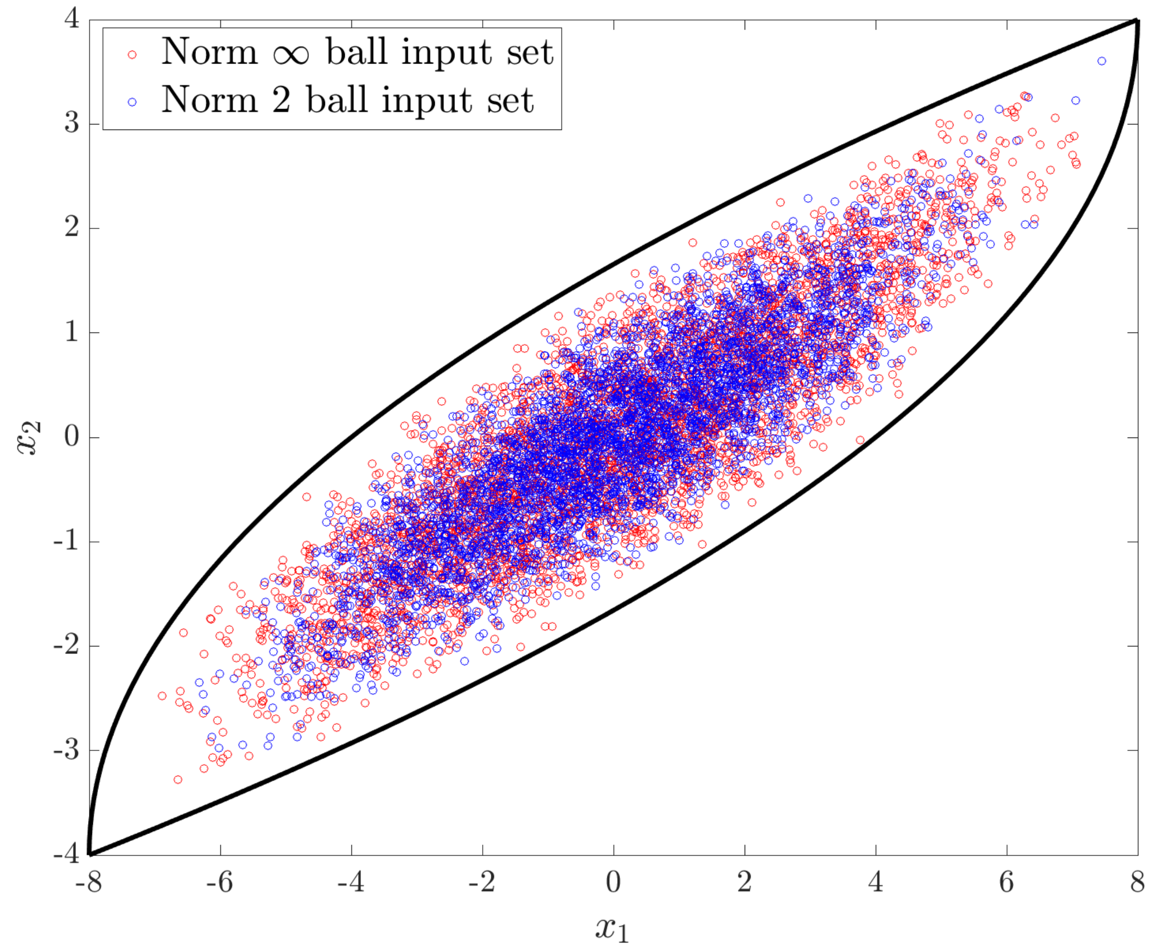


Diameter of integrator reach set vs time

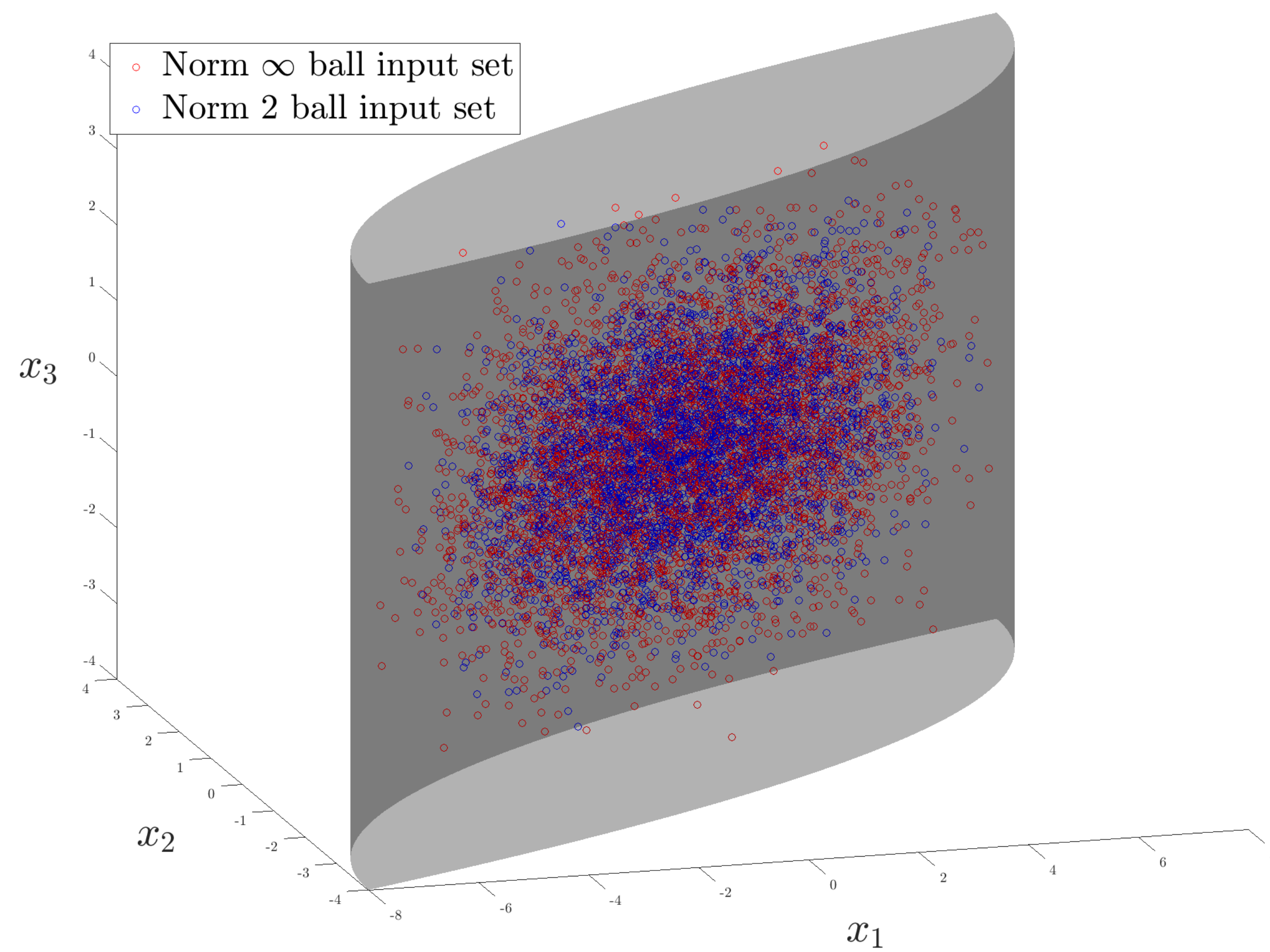
Dependence of $\mathcal{X}_t$ on the geometry of $\mathcal{U}$

$\mathcal{X}_t \subset \mathbb{R}^{n_x}$ has non-unique dependence on the geometry of $\mathcal{U} \in \mathbb{R}^m$.

Accounting for all possible combinations of worst-cases of all the input components



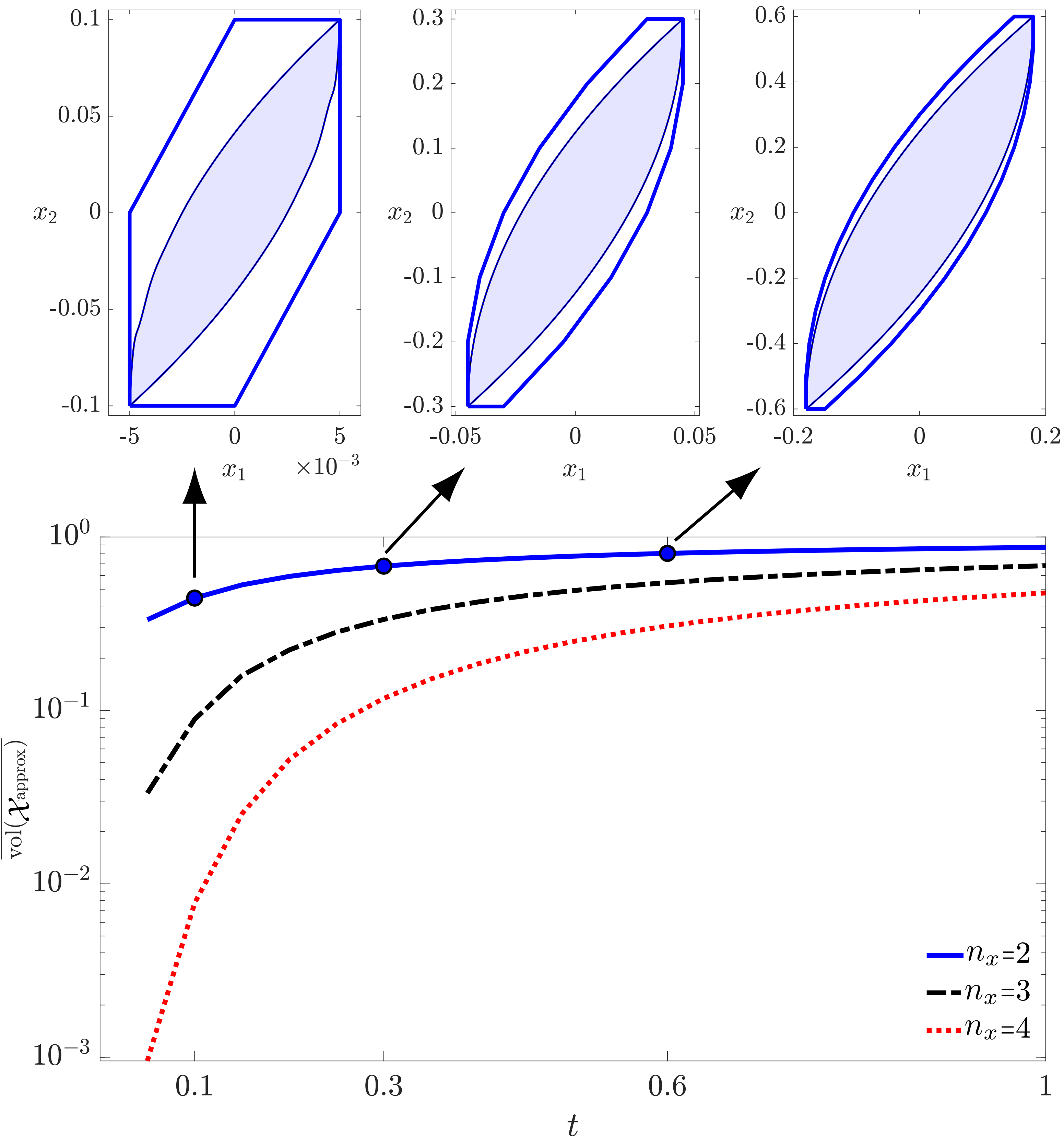
The single input double integrator reach set at $t = 4$



The integrator reach set at $t = 4$, $\boldsymbol{r} = (2,1)^\top$

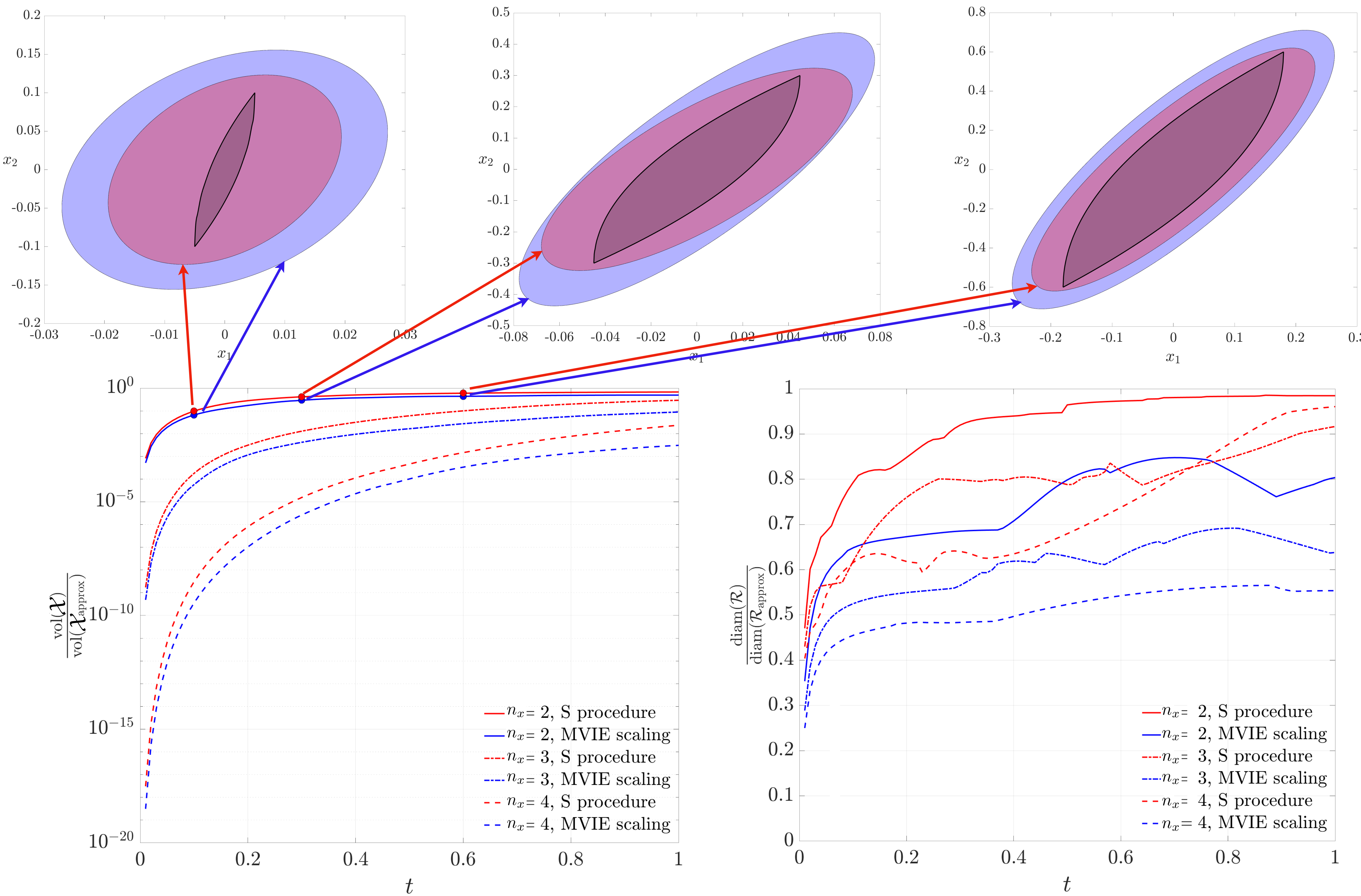
Benchmarking over-approximations of $\mathcal{X}_t$

From the CORA toolbox



Benchmarking over-approximation of $\mathcal{X}_t$

From the Ellipsoidal toolbox



Integrator Reach Sets with Time Varying Set-Valued Input Uncertainties

Support function of $\mathcal{X}_t$ with $\mathcal{U}(t) \subset \mathbb{R}^m$

Theorem.

$$h_{\mathcal{X}(\mathcal{X}_0,t)} = \sum_{j=1}^{m} \left\{ h_{\mathcal{X}_{j0}} \left(\exp\!\left(t\boldsymbol{A}_j^\top\right) \boldsymbol{y} \right) + \int_0^t \left[\nu_j(s) \langle \boldsymbol{y}_j, \boldsymbol{\xi}(s) \rangle + \mu_j(s) \,|\langle \boldsymbol{y}_j, \boldsymbol{\xi_j}(s) \rangle|\, \right] \right\} \mathrm{d}s$$

where

$$\mu_j(t) := (\beta_j(t) - \alpha_j(t))/2, \;\; \nu_j(t) := (\beta_j(t) + \alpha_j(t))/2,$$

$$\alpha_j(t) := \min_{\boldsymbol{u}(t) \in \mathcal{U}(t)} u_j(t), \;\; \boldsymbol{\beta_j(t) := \max_{u(t) \in \mathcal{U}(t)} u_j(t)}, \quad \boldsymbol{j = 1, \cdots, m},$$

$$\boldsymbol{\xi}(s) := \begin{pmatrix} \mu_1 \boldsymbol{\xi}_1(s) \\ \vdots \\ \mu_m \boldsymbol{\xi}_m(s) \end{pmatrix}, \;\; \boldsymbol{\xi}_j(s) := \begin{pmatrix} s^{r_j - 1}/(r_j - 1)! \\ s^{r_j - 2}/(r_j - 2)! \\ \vdots \\ s \\ 1 \end{pmatrix}.$$

Parametric equations of $\partial\mathcal{X}_t$ for $\mathcal{U}(t) \subset \mathbb{R}^m$

Theorem. Assume $\mathcal{X}_0 \equiv \{\boldsymbol{x}_0\}$. Then

Components of the boundary

$$\boldsymbol{x}_j^{\text{bdy}} = \exp\left(t\boldsymbol{A}_j\right)\boldsymbol{x}_{j0} + \int_0^t \nu_j(s)\boldsymbol{\xi}_j(s)\mathrm{d}s \pm \int_0^{s_1} \mu_j(s)\boldsymbol{\xi}_j(s)\mathrm{d}s$$
$$\mp \int_{s_1}^{s_2} \mu_j(s)\boldsymbol{\xi}_j(s)\mathrm{d}s \pm \cdots \pm (-1)^{r_j} \int_{s_{r_j-1}}^{t} \mu_j(s)\boldsymbol{\xi}_j(s)\mathrm{d}s.$$

Parameter vector of the jth block: $\boldsymbol{s}_j = \left(s_1, s_2, \ldots, s_{r_j-1}\right)$, $j = 1, \ldots, m$

Parameter space of the jth block: $\mathcal{S}_j := \left\{\boldsymbol{s}_j \mid 0 \leq s_1 \leq s_2 \leq \ldots \leq s_{r_j-1} \leq t\right\} \subset \mathbb{R}^{r_j-1}$,

Each single input integrator reach set has two bounding surfaces:

$$\partial\mathcal{X}_j\left(\boldsymbol{x}_0, \boldsymbol{s}_j\right) := \partial\mathcal{X}_j^{\text{upper}}(\boldsymbol{x}_0, \boldsymbol{s}_j) \cup \partial\mathcal{X}_j^{\text{lower}}(\boldsymbol{x}_0, \boldsymbol{s}_j), \quad \boldsymbol{s}_j \in \mathcal{S}_j \subset \mathbb{R}^{r_j-1}.$$

Taxonomy of $\mathcal{X}_t$ for $\mathcal{U}(t) \subset \mathbb{R}^m$

Equations of the boundary only depend on the extremal curves $\alpha(\tau),\ \beta(\tau),\ 0 \leq \tau \leq t$

Semialgebraic set iff $\boldsymbol{\alpha}(\tau), \boldsymbol{\beta}(\tau)$ are polynomial in time τ

Still zonoid for singleton initial condition

$\mathrm{vol}\left(\mathcal{Z}_t\right)$ for full state static feedback linearizable systems

$$\mathrm{vol}(\mathcal{Z}_t) = \int_{\mathcal{Z}_t} \mathrm{d}\boldsymbol{z} = \int_{\mathcal{S}_1\times\ldots\times\mathcal{S}_m} \int_{[0,1]^m} \det\left(\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{x}}\right) \prod_{j=1}^{m} \det\left(\frac{\partial \boldsymbol{x}_j}{\partial \boldsymbol{s}_j}\,\frac{\partial \boldsymbol{x}_j}{\partial \lambda_j}\right) \mathrm{d}\boldsymbol{s}_j\,\mathrm{d}\lambda_j$$

$$\boldsymbol{x}_j(\boldsymbol{s}_j,\lambda_j) = \lambda_j \mathcal{X}_j^{\text{upper}}(\boldsymbol{s}_j) + (1-\lambda_j)\mathcal{X}_j^{\text{lower}}(\boldsymbol{s}_j), \quad \boldsymbol{s}_j \in \boldsymbol{\mathcal{S}}_j, \quad 0 \le \lambda_j \le 1, \quad \forall j \in [m]$$

where

$$\left(\frac{\partial \boldsymbol{x}_j}{\partial \boldsymbol{s}_j} \quad \frac{\partial \boldsymbol{x}_j}{\partial \lambda_j}\right) = \begin{pmatrix} 2\mu_j(s_1)\xi_1(s_1) & -2\mu_j(s_2)\xi_1(s_2) & \cdots & (-1)^{r_j} 2\mu_j\left(s_{r_j-1}\right)\xi_1\left(s_{r_j-1}\right) & f_1(\boldsymbol{s}_j) \\ 2\mu_j(s_1)\xi_2(s_1) & -2\mu_j(s_2)\xi_2(s_2) & \cdots & (-1)^{r_j} 2\mu_j\left(s_{r_j-1}\right)\xi_2\left(s_{r_j-1}\right) & f_2(\boldsymbol{s}_j) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\mu_j(s_1)\xi_{r_j}(s_1) & -2\mu_j(s_2)\xi_{r_j}(s_2) & \cdots & (-1)^{r_j} 2\mu_j\left(s_{r_j-1}\right)\xi_{r_j}\left(s_{r_j-1}\right) & f_{r_j}(\boldsymbol{s}_j) \end{pmatrix} \in \mathbb{R}^{r_j \times r_j}$$

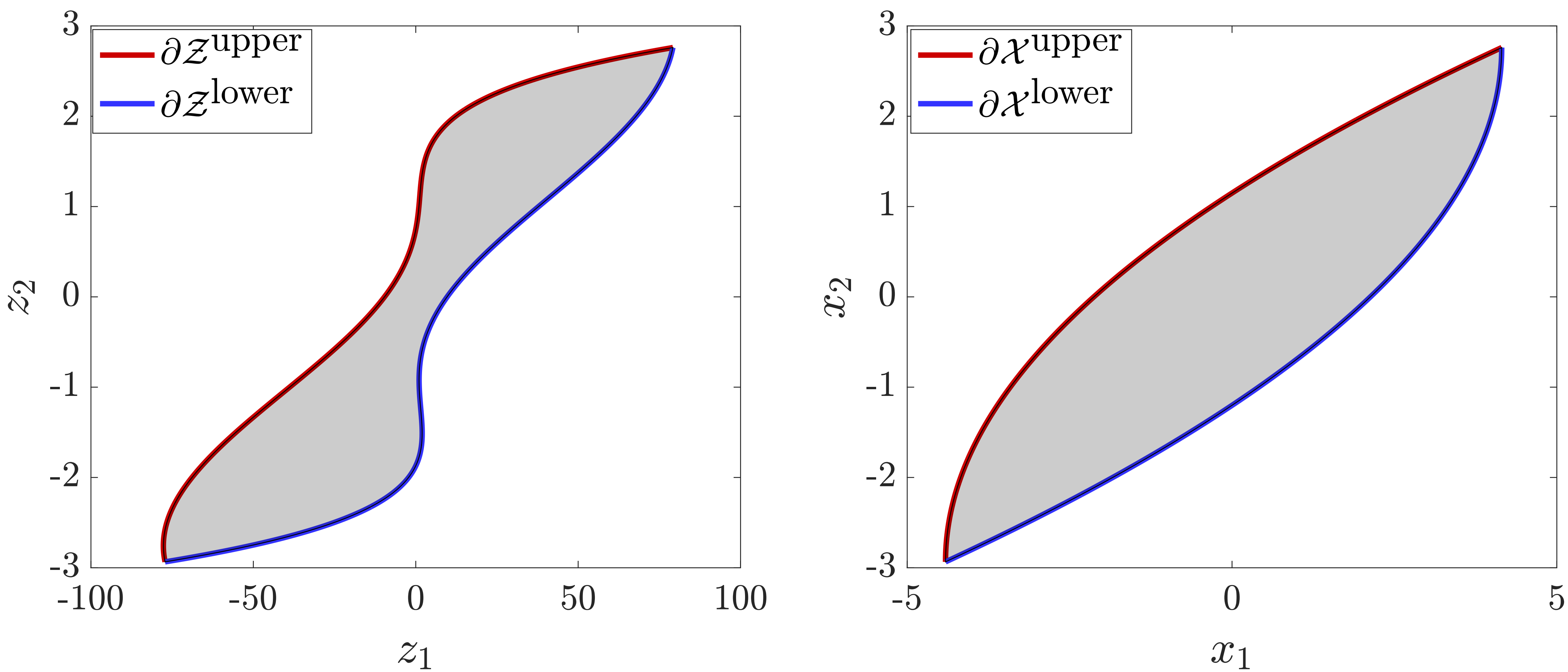
$$f_i(\boldsymbol{s}_j) = 2\sum_{i=1}^{r_j-1} (-1)^{i+1} \left[\int_0^{s_i} \mu_j(\tau)\xi_i(\tau)\mathrm{d}\tau + \int_0^{t-s_{r_j-i}} \mu_j(\tau)\xi_i(\tau)\mathrm{d}\tau\right] + 2(-1)^{r_j+1} \int_0^{t} \mu_j(\tau)\xi_i(\tau)\mathrm{d}\tau, \quad \forall (i,j) \in [r_j]\times[m]$$

$$\det\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{x}} = 1/\det(\boldsymbol{\tau}).$$

$\mathrm{vol}\left(\mathcal{Z}_t\right)$ for full state static feedback linearizable systems

Special case: volume of integrator reach set w. time-varying set-valued input uncertainties

$$\mathrm{vol}(\mathcal{X}_t) = \int_{\mathcal{X}_t} \mathrm{d}\boldsymbol{x} = \prod_{j=1}^{m} \left(\int_{\mathcal{S}_j} \det\left(\frac{\partial \boldsymbol{x}_j}{\partial \boldsymbol{s}_j} \frac{\partial \boldsymbol{x}_j}{\partial \lambda_j} \right) \mathrm{d}\boldsymbol{s}_j \right)$$



$\mathrm{vol}\left(\mathcal{Z}_t\right) =$ **206.7362**

$\mathrm{vol}\left(\mathcal{X}_t\right) =$ **15.4292**

Intersection Detection

Intersection detection

Static feedback linearizable agents A and B:

$$\dot{\boldsymbol{z}}^{\mathsf{A}} = \boldsymbol{f}\left(\boldsymbol{z}^{\mathsf{A}}, \boldsymbol{v}^{\mathsf{A}}\right), \quad \dot{\boldsymbol{z}}^{\mathsf{B}} = \boldsymbol{f}\left(\boldsymbol{z}^{\mathsf{B}}, \boldsymbol{v}^{\mathsf{B}}\right), \quad \boldsymbol{z}_0^{\mathsf{A}}, \boldsymbol{z}_0^{\mathsf{B}} \in \mathbb{R}^{n_x}, \quad \mathcal{V}^{\mathsf{A}}(s), \mathcal{V}^{\mathsf{B}}(s) \subset \mathbb{R}^m,$$

Compact input sets

Corresponding zonoids in integrator coordinates:

$$\mathcal{X}_t^{\mathsf{A}} = \boldsymbol{\tau}\left(\mathcal{Z}_t^{\mathsf{A}}\right), \; \mathcal{X}_t^{\mathsf{B}} = \boldsymbol{\tau}\left(\mathcal{Z}_t^{\mathsf{B}}\right), \quad \mathcal{U}^{\mathsf{A}}(s), \mathcal{U}^{\mathsf{B}}(s) \subset \mathbb{R}^m, \qquad 0 \leq s \leq t, \qquad \boldsymbol{r} = (r_1, \ldots, r_m)^\top.$$

Since $\boldsymbol{\tau}$ is injective

$$\mathcal{X}_t^{\mathsf{A}} \cap \mathcal{X}_t^{\mathsf{B}} \neq (=) \varnothing \;\Leftrightarrow\; \mathcal{Z}_t^{\mathsf{A}} \cap \mathcal{Z}_t^{\mathsf{B}} \neq (=) \varnothing.$$

Dynamic feedback linearizable agents A and B:

$$\begin{aligned} \mathcal{X}_t^{\mathsf{A}} \cap \mathcal{X}_t^{\mathsf{B}} \neq \varnothing \quad &\Longrightarrow \quad \mathcal{Z}_t^{\mathsf{A}} \cap \mathcal{Z}_t^{\mathsf{B}} \neq \varnothing, \\ \mathcal{X}_t^{\mathsf{A}} \cap \mathcal{X}_t^{\mathsf{B}} = \varnothing \quad &\Longleftarrow \quad \mathcal{Z}_t^{\mathsf{A}} \cap \mathcal{Z}_t^{\mathsf{B}} = \varnothing, \quad \boldsymbol{z} = \Pi_z(\boldsymbol{\rho}). \end{aligned}$$

Augmented states

Intersection detection: certifying $\mathcal{X}_t^{\mathtt{A}} \cap \mathcal{X}_t^{\mathtt{B}} \neq (=) \varnothing$

✕ Direct computation of distance is unwieldy:

$$\mathrm{dist}\,(\mathtt{A},\mathtt{B}) := \min_{\boldsymbol{x}^{\mathtt{A}} \in \mathcal{X}_t^{\mathtt{A}}, \boldsymbol{x}^{\mathtt{B}} \in \mathcal{X}_t^{\mathtt{B}}} \|\boldsymbol{x}^{\mathtt{A}} - \boldsymbol{x}^{\mathtt{B}}\|_2^2 = (>)\ 0 \iff \mathcal{X}_t^{\mathtt{A}} \cap \mathcal{X}_t^{\mathtt{B}} \neq (=)\ \varnothing.$$

New idea:

$$\min_{\boldsymbol{y} \in \mathbb{S}^{n_x - 1}} h_{\mathcal{X}_t^{\mathtt{A}} \dot{-} \mathcal{X}_t^{\mathtt{B}}}(\boldsymbol{y}) \geq (<)\ 0 \iff \mathcal{X}_t^{\mathtt{A}} \cap \mathcal{X}_t^{\mathtt{B}} \neq (=)\ \varnothing.$$

Minkowski difference

$$\min_{\boldsymbol{y} \in \mathbb{S}^{n_x - 1}} h_{\mathcal{X}_t^{\mathtt{A}} \dot{-} \mathcal{X}_t^{\mathtt{B}}}(\boldsymbol{y}) = \min_{\|\boldsymbol{y}\|_2 = 1} h_{\mathcal{X}_t^{\mathtt{A}}}(\boldsymbol{y}) + h_{\mathcal{X}_t^{\mathtt{B}}}(-\boldsymbol{y}).$$

Distributed certification:

Intersection iff $\forall j \in [m]$ we have $\displaystyle \min_{\boldsymbol{y}_j \in \mathbb{R}^{r_j}, \|\boldsymbol{y}_j\|_2 = 1} h_{\mathcal{X}_{jt}^{\mathtt{A}}}(\boldsymbol{y}_j) + h_{\mathcal{X}_{jt}^{\mathtt{B}}}(-\boldsymbol{y}_j) \geq 0.$

No intersection iff $\exists j \in [m]$ s.t. < 0

Intersection detection: lossless convexification

Relaxing the unit norm constraint:

$$\min_{\boldsymbol{y}_j \in \mathbb{R}^{r_j}, \|\boldsymbol{y}_j\|_2 \leq 1} \langle \boldsymbol{c}_j(t), \boldsymbol{y}_j \rangle + \int_0^t |\langle \boldsymbol{\gamma}_j(s), \boldsymbol{y}_j \rangle| \, \mathrm{d}s,$$

$$\boldsymbol{c}_j(t) := \exp(t\boldsymbol{A}_j)\big(\boldsymbol{x}_{j0}^{\mathtt{A}} - \boldsymbol{x}_{j0}^{\mathtt{B}}\big) + \int_0^t \big(\nu_j^{\mathtt{A}}(s) - \nu_j^{\mathtt{B}}(s)\big)\, \boldsymbol{\xi}_j(s)\mathrm{d}s,$$
$$\boldsymbol{\gamma}_j(s) := \big(\mu_j^{\mathtt{A}}(s) + \mu_j^{\mathtt{B}}(s)\big)\, \boldsymbol{\xi}_j(s).$$

Discretizing $[0,\, t]$ into $K \in \mathbb{N}$ intervals:

$$\int_0^t |\langle \boldsymbol{\gamma}_j(s), \boldsymbol{y}_j \rangle| \mathrm{d}s \approx \frac{\Delta s}{2} \sum_{k=1}^{K} \left(|\langle \boldsymbol{\gamma}_j(s_{k-1}), \boldsymbol{y}_j \rangle| + |\langle \boldsymbol{\gamma}_j(s_k), \boldsymbol{y}_j \rangle| \right).$$

Intersection detection: lossless convexification

For $j \in [m]$, let $\theta_j := \left(\theta_{j0}, \cdots, \theta_{jK}\right)^\top \in \mathbb{R}^{K+1}$, and

$$\boldsymbol{\eta}_j := \begin{pmatrix} \boldsymbol{y}_j \\ \boldsymbol{\theta}_j \end{pmatrix} \in \mathbb{R}^{r_j+K+1}, \qquad \boldsymbol{\omega}_j(t) := \Delta s \begin{pmatrix} 1/2 \\ \mathbf{1}_{K-1} \\ 1/2 \end{pmatrix} \in \mathbb{R}^{K+1}_{>0},$$

$$\boldsymbol{\ell}_j(t) := \begin{pmatrix} \boldsymbol{c}_j(t) \\ \boldsymbol{\omega}_j(t) \end{pmatrix} \in \mathbb{R}^{r_j+K+1}, \qquad \boldsymbol{M}_j := \left(\begin{array}{c|c} \boldsymbol{\Gamma}_j & -\boldsymbol{I}_{K+1} \otimes \mathbf{1}_2 \\ \hline \mathbf{0}_{(K+1)\times r_j} & -\boldsymbol{I}_{K+1} \end{array}\right) \in \mathbb{R}^{3(K+1)\times(r_j+K+1)},$$

$$\boldsymbol{\Gamma}_j := \begin{pmatrix} \boldsymbol{\gamma}_j^\top(s_0) \\ -\boldsymbol{\gamma}_j^\top(s_0) \\ \boldsymbol{\gamma}_j^\top(s_1) \\ -\boldsymbol{\gamma}_j^\top(s_1) \\ \vdots \\ \boldsymbol{\gamma}_j^\top(s_K) \\ -\boldsymbol{\gamma}_j^\top(s_K) \end{pmatrix} \in \mathbb{R}^{2(K+1)\times r_j}, \qquad \boldsymbol{N}_j := \left(\begin{array}{c|c} \boldsymbol{I}_{r_j} & \mathbf{0}_{r_j\times(K+1)} \end{array}\right) \in \mathbb{R}^{r_j\times(r_j+K+1)}$$

Intersection detection: lossless convexification

Second order cone program (SOCP)

$$\begin{aligned} \min_{\boldsymbol{\eta}_j \in \mathbb{R}^{r_j+K+1}} \quad & \langle \boldsymbol{\ell}_j(t), \boldsymbol{\eta}_j \rangle \\ \text{subject to} \quad & \boldsymbol{M}_j \boldsymbol{\eta}_j \leq \mathbf{0}, \quad \|\boldsymbol{N}_j \boldsymbol{\eta}_j\|_2 \leq 1. \end{aligned}$$

The convexification is lossless, it can certify: $\mathcal{X}_{jt}^{\mathtt{A}} \cap \mathcal{X}_{jt}^{\mathtt{B}} \neq (=) \varnothing, \quad j \in [m].$

Theorem. Let $\tilde{p}_j^*$ be the optimal value of SOCP and p_j^* the optimal value of the nonconvex problem, i.e., $\|\mathbf{y}_j\|_2 = 1$, then

(i) $\tilde{p}_j^* \leq 0.$

(ii) $\tilde{p}_j^* = 0 \implies 0 \leq p_j^* \iff \mathcal{X}_{jt}^{\mathtt{A}} \cap \mathcal{X}_{jt}^{\mathtt{B}} \neq \varnothing, \quad j \in [m].$

(iii) $\tilde{p}_j^* < 0 \implies \tilde{p}_j^* = p_j^* < 0 \iff \mathcal{X}_{jt}^{\mathtt{A}} \cap \mathcal{X}_{jt}^{\mathtt{B}} = \varnothing, \quad j \in [m].$

Intersection detection: example

Compact input sets

Static feedback linearizable agents A and B:

$$\dot{\boldsymbol{z}}^{\mathtt{A}} = \boldsymbol{f}\left(\boldsymbol{z}^{\mathtt{A}}, \boldsymbol{v}^{\mathtt{A}}\right), \quad \dot{\boldsymbol{z}}^{\mathtt{B}} = \boldsymbol{f}\left(\boldsymbol{z}^{\mathtt{B}}, \boldsymbol{v}^{\mathtt{B}}\right), \quad \boldsymbol{z}_0^{\mathtt{A}}, \boldsymbol{z}_0^{\mathtt{B}} \in \mathbb{R}^{n_x}, \quad \mathcal{V}^{\mathtt{A}}(s), \mathcal{V}^{\mathtt{B}}(s) \subset \mathbb{R}^m,$$

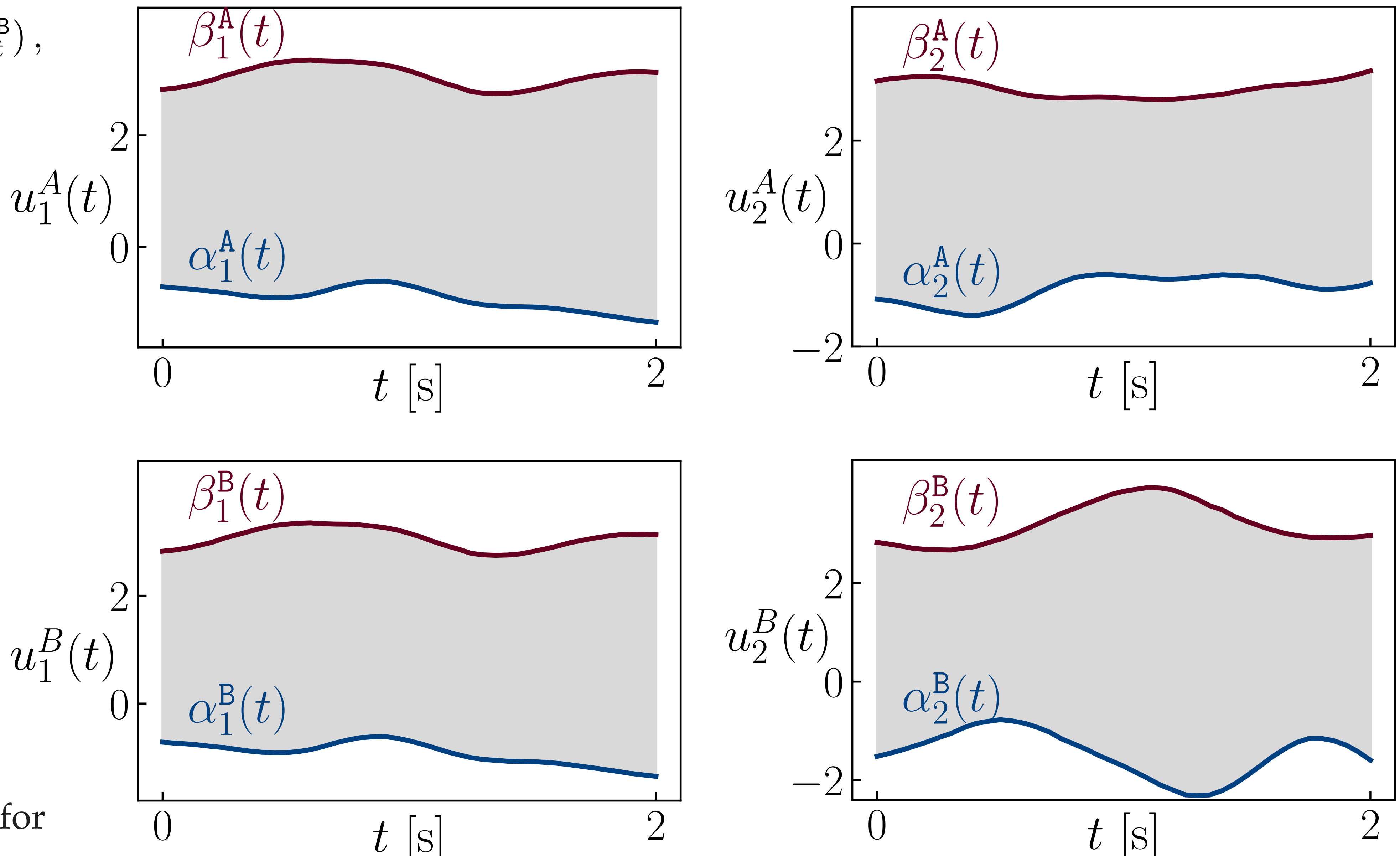
Corresponding zonoids in normal coordinates:

$$\mathcal{X}_t^{\mathtt{A}} = \boldsymbol{\tau}\left(\mathcal{Z}_t^{\mathtt{A}}\right), \; \mathcal{X}_t^{\mathtt{B}} = \boldsymbol{\tau}\left(\mathcal{Z}_t^{\mathtt{B}}\right),$$

$$\boldsymbol{r} = (3, 2)^\top,$$

$$\boldsymbol{x}_0^{\mathtt{A}} = [0.5, \mathbf{0}_{1\times 4}],$$

$$\boldsymbol{x}_0^{\mathtt{B}} = [\mathbf{0}_{1\times 3}, 5, 0]$$



Runtimes: 0.38 s and 0.37 s for $j = 1$ and $j = 2$, respectively

CVX solves SOCP for $ds = 0.5$

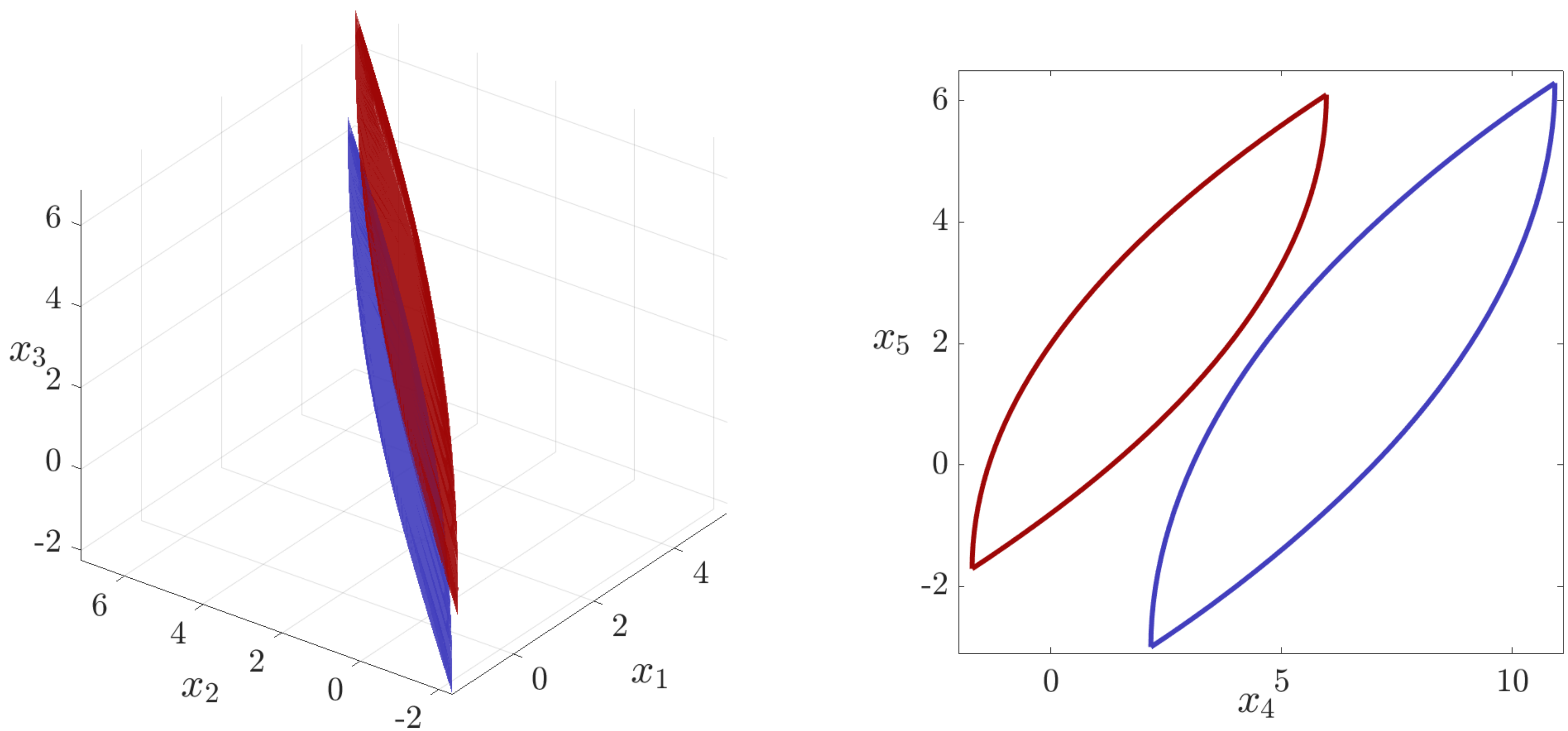
The input trajectories for agents A and B.

Intersection detection: example (continued)

The optimal values of SOCP are:

$$(\tilde{p}_1^*, \tilde{p}_2^*) = (0, -0.54) \iff \begin{cases} \mathcal{X}_{1T}^{\mathtt{A}} \cap \mathcal{X}_{1T}^{\mathtt{B}} \neq \varnothing \\ \mathcal{X}_{2T}^{\mathtt{A}} \cap \mathcal{X}_{2T}^{\mathtt{B}} = \varnothing \end{cases} \iff \mathcal{X}_{T}^{\mathtt{A}} \cap \mathcal{X}_{T}^{\mathtt{B}} = \varnothing \iff \mathcal{Z}_{T}^{\mathtt{A}} \cap \mathcal{Z}_{T}^{\mathtt{B}} = \varnothing.$$

The convexification is lossless:



Intersection detection between two integrator reach sets corresponding to agents A (red) and B (blue).

Learning the Reach Sets for Full State Feedback Linearizable Systems

Learning $\mathcal{Z}_t$

Consider fsDFL system

Augmented state vector

State diffeomorphism: $\boldsymbol{\tau} : \boldsymbol{\rho} \in \mathbb{R}^{n_z + n_w} \to \boldsymbol{x} \in \mathbb{R}^{n_x},\ n_x = n_z + n_w,\ \boldsymbol{\rho} := (\boldsymbol{z}, \boldsymbol{w})$

Input homeomorphism: $\boldsymbol{\tau}_u : (\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{w}, \dot{\boldsymbol{w}}, \ddot{\boldsymbol{w}}, \cdots) \mapsto \boldsymbol{u} \in \mathcal{U}(t) \subset \mathbb{R}^m$

Step 1 **Step 2** **Step 3**

$$\underbrace{\mathcal{V} \longrightarrow \mathcal{U}(t) \longrightarrow \{\boldsymbol{\alpha}(\tau), \boldsymbol{\beta}(\tau)\}_{\tau=0}^{t}}_{\text{Step 1}} \underbrace{\longrightarrow \partial\mathcal{X}_t}_{\text{Step 2}} \underbrace{\longrightarrow \partial\mathcal{R}_t \xrightarrow{\Pi_z} \partial\mathcal{Z}_t}_{\text{Step 3}}$$

Boundary of the reach set in augmented state

Step 1

Find the extremal trajectories $\{\alpha_j(\tau)\}_{\tau=0}^{t},\ \{\beta_j(\tau)\}_{\tau=0}^{t}$ **for time-varying** $\mathcal{U}(\tau) \subset \mathbb{R}^m,\ \tau \in [0, t]$

$$\alpha_j(\tau) := \min_{u_j(\tau) \in \mathcal{U}(\tau)} u_j(\tau), \quad \beta_j(\tau) := \max_{u_j(\tau) \in \mathcal{U}(\tau)} u_j(\tau), \quad j \in [m]$$

Step 2

Compute the reach set $\mathcal{X}_t$ **in normal coordinates**

Step 3

Numerically map $\partial\mathcal{X}_t$ **back to** $\partial\mathcal{Z}_t$

Learning $\mathcal{Z}_t$

Step 1 involves the fixed point equations:

$$\boldsymbol{\alpha}(t) = \boldsymbol{T}_{\min}\left(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)\right) := \min_{\boldsymbol{x} \in \mathcal{X}_t\left((\boldsymbol{\alpha}(\tau))_{\tau=0}^{t}, (\boldsymbol{\beta}(\tau))_{\tau=0}^{t}\right)} \overbrace{\boldsymbol{C}(\boldsymbol{x})\boldsymbol{v} + \boldsymbol{d}}^{\text{The input homeomorphism } \boldsymbol{\tau}_u}(\boldsymbol{x}),$$

$$\boldsymbol{\beta}(t) = \boldsymbol{T}_{\max}\left(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)\right) := \max_{\boldsymbol{x} \in \mathcal{X}_t\left((\boldsymbol{\alpha}(\tau))_{\tau=0}^{t}, (\boldsymbol{\beta}(\tau))_{\tau=0}^{t}\right)} \boldsymbol{C}(\boldsymbol{x})\boldsymbol{v} + \boldsymbol{d}(\boldsymbol{x}).$$

These fixed point equations are not in general contractions

Idea: Learn $\{\widehat{\boldsymbol{\alpha}}(\tau), \widehat{\boldsymbol{\beta}}(\tau)\}_{\tau=0}^{t}$ from data with guarantees

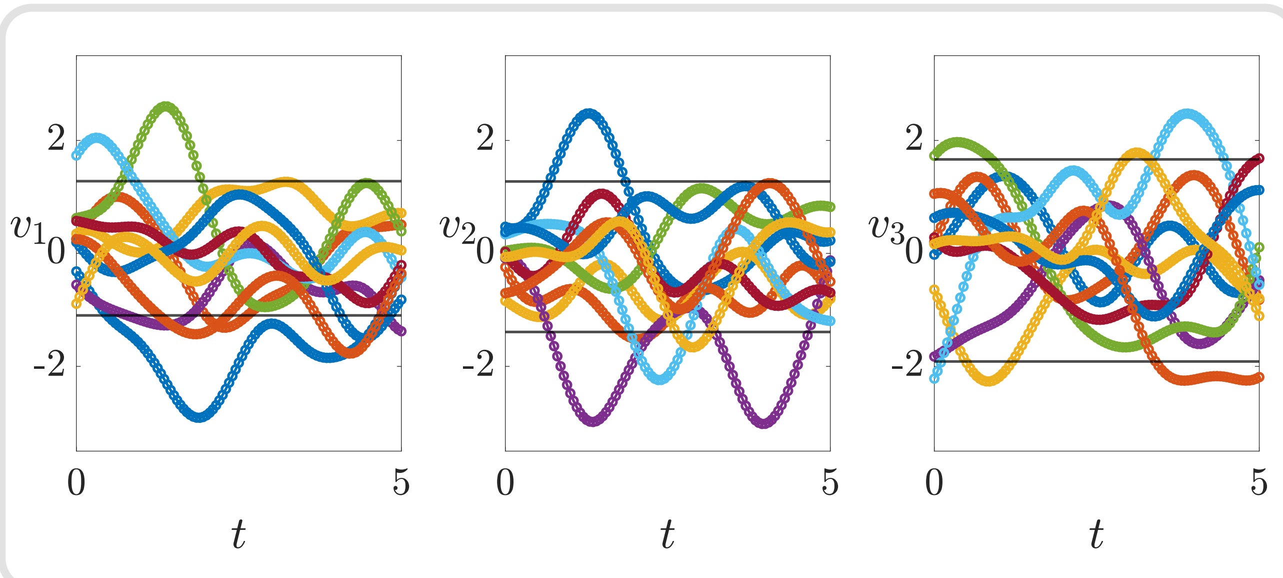
(Next slide)

Learning $\{\widehat{\boldsymbol{\alpha}}(\tau), \widehat{\boldsymbol{\beta}}(\tau)\}_{\tau=0}^{t}$

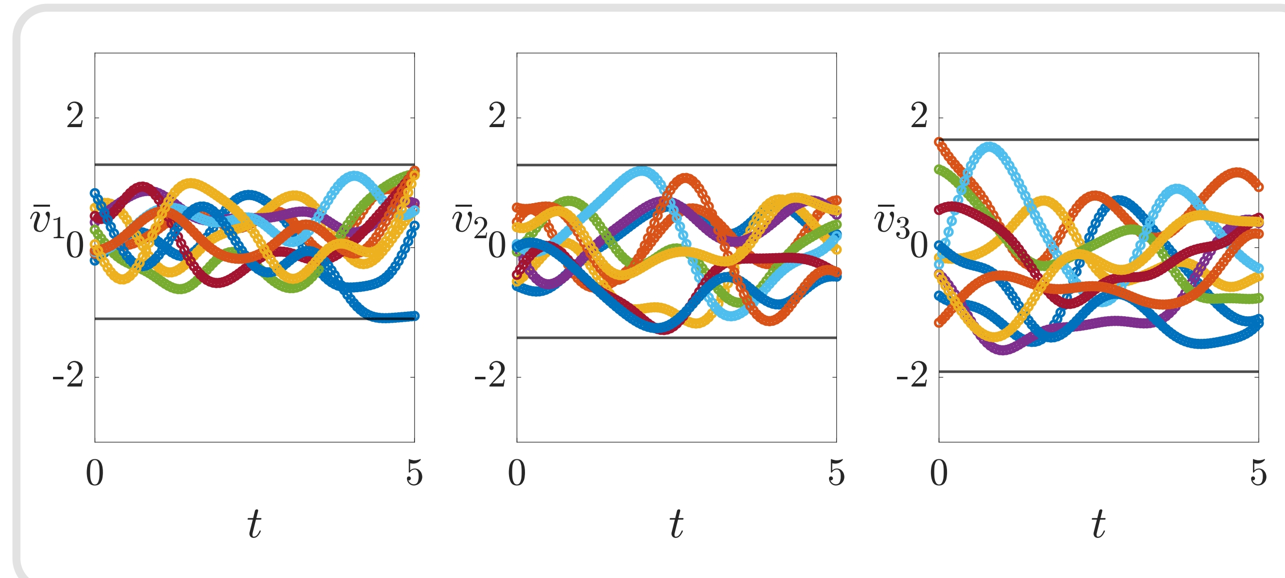
Assume $\mathcal{V}$ is convex compact set

Generate trajectory samples $\{\boldsymbol{v}^{(i)}(t)\}_{i=1}^{N}$ from $\mathcal{V} \subset \mathbb{R}^m$ via constrained Gaussian Process (GP)

Unconstrained GP sampling



Constrained GP sampling



Using statistical learning theory: $N = \left\lceil \dfrac{e}{\varepsilon_{\hat{u}}(e-1)} \left(\log \dfrac{1}{\delta_{\hat{u}}} + 2m \right) \right\rceil$

Sample complexity

Performance guarantee: $\mathbb{P}\Big(\mathrm{vol}([\boldsymbol{\alpha}(\tau), \boldsymbol{\beta}(\tau)]) - \mathrm{vol}([\widehat{\boldsymbol{\alpha}}(\tau), \widehat{\boldsymbol{\beta}}(\tau)]) \leq \varepsilon_{\hat{u}}\Big) \geq 1 - \delta_{\hat{u}}$

Accuracy $\in (0,1)$ **Confidence** $\in (0,1)$

Learning $\mathcal{Z}_t$

Step 1 **Step 2** **Step 3**

$$\mathcal{V} \longrightarrow \{\widehat{\boldsymbol{\alpha}}(\tau), \widehat{\boldsymbol{\beta}}(\tau)\}_{\tau=0}^{t} \longrightarrow \partial\widehat{\mathcal{X}}_t \longrightarrow \partial\widehat{\mathcal{R}}_t \xrightarrow{\Pi_z} \partial\widehat{\mathcal{Z}}_t$$

Inclusion guarantee (deterministic): $\widehat{\mathcal{Z}}_t \subseteq \mathcal{Z}_t$

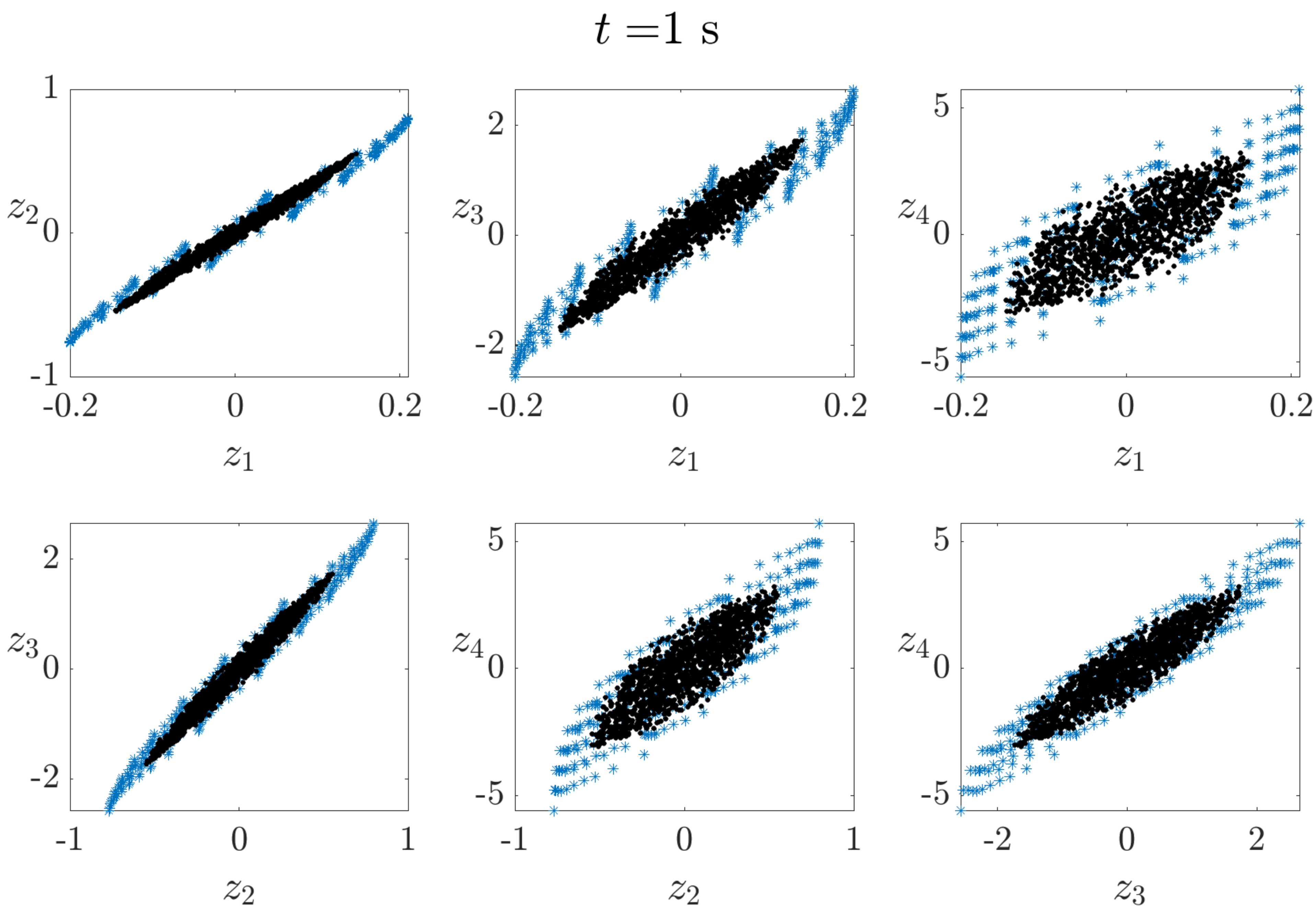
The probabilistic inclusion during the transformation

$$\overbrace{(\varepsilon_{\hat{u}}, \delta_{\hat{u}})}^{\hat{\mathcal{U}}} \longrightarrow \overbrace{(\varepsilon_{x}, \delta_{x})}^{\hat{\mathcal{X}}_t} \longrightarrow \overbrace{(\varepsilon_{\hat{\rho}}, \delta_{\hat{\rho}})}^{\hat{\mathcal{R}}_t} \longrightarrow \overbrace{(\varepsilon_{\hat{z}}, \delta_{\hat{z}})}^{\hat{\mathcal{Z}}_t},$$

follows $(\varepsilon_{\hat{u}}, \delta_{\hat{u}}) = (\varepsilon_{\hat{x}}, \delta_{\hat{x}}) = (\varepsilon_{\hat{\rho}}, \delta_{\hat{\rho}}) = (\varepsilon_{\hat{z}}, \delta_{\hat{z}})$.

Learning Strategy: example **System (1)**

fsSFL with relative degree vector $\boldsymbol{r} = (4)^\top$

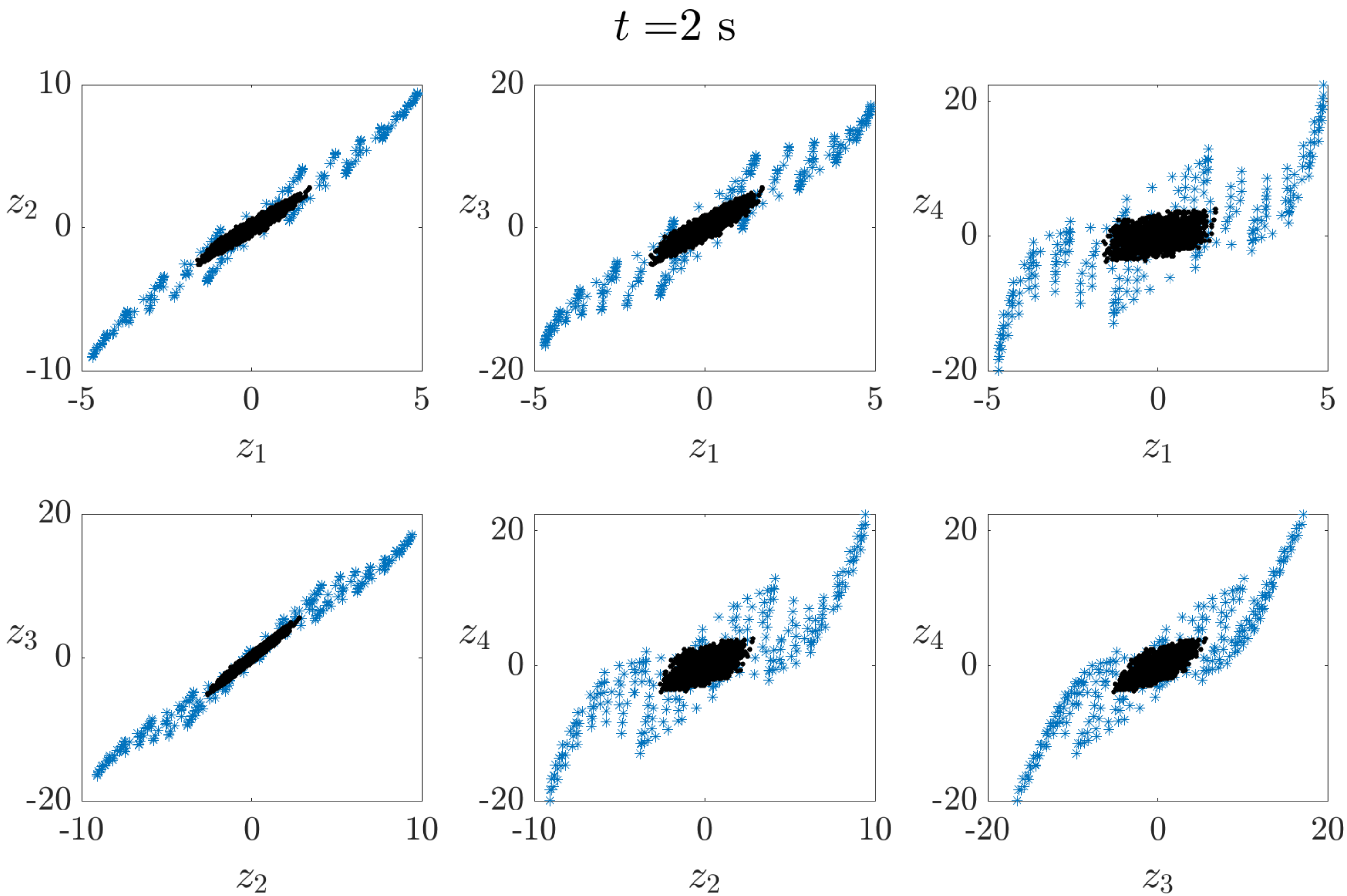


Serial computation time =1.13 s and $N = 1410$.

Learning Strategy: example

System (1)

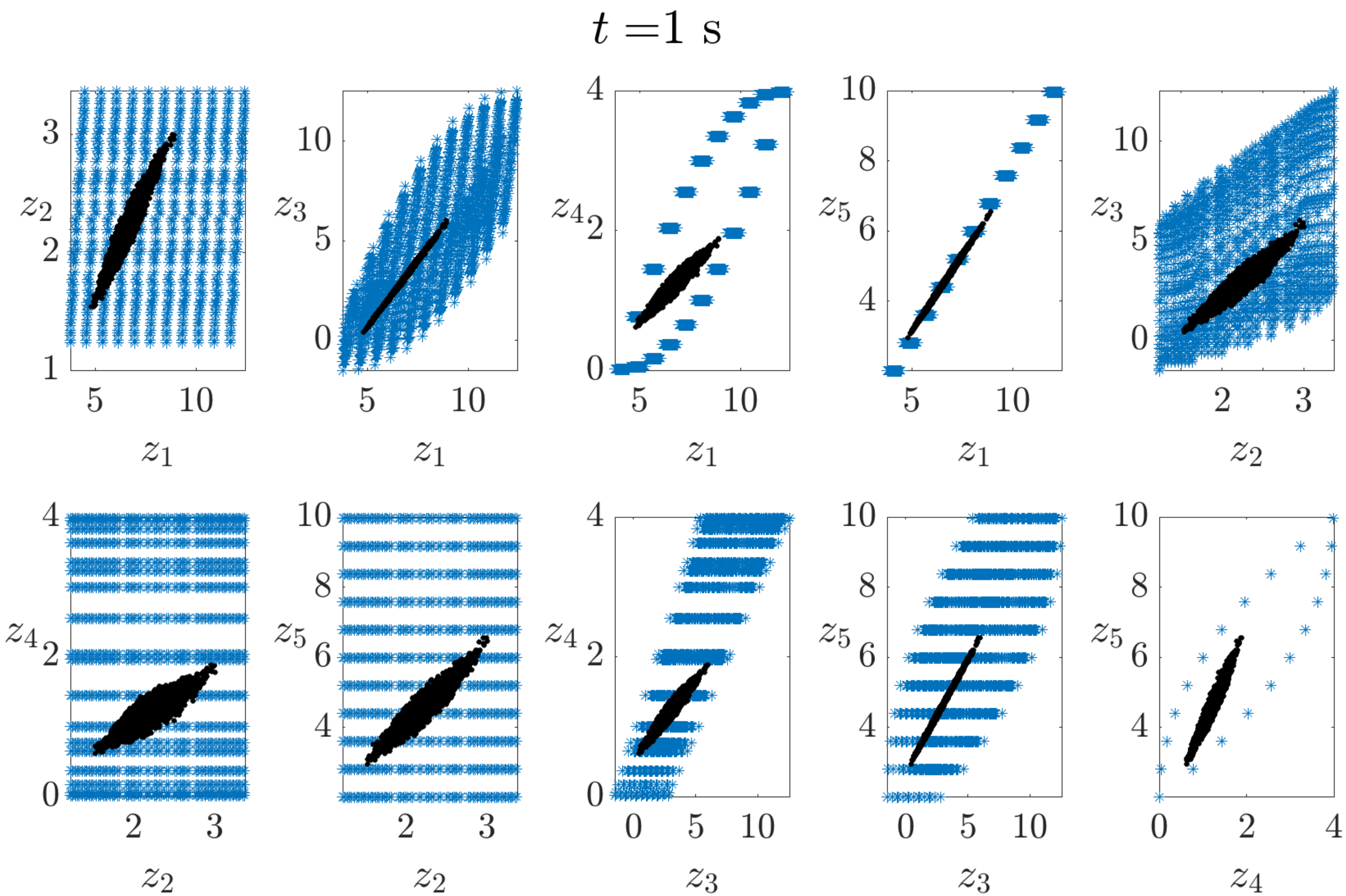
fsSFL with relative degree vector $\boldsymbol{r} = (4)^\top$



Serial computation time =1.20 s and $N = 1410$.

Learning Strategy: example System (2)

fsSFL with relative degree vector $\boldsymbol{r} = (3,2)^\top$

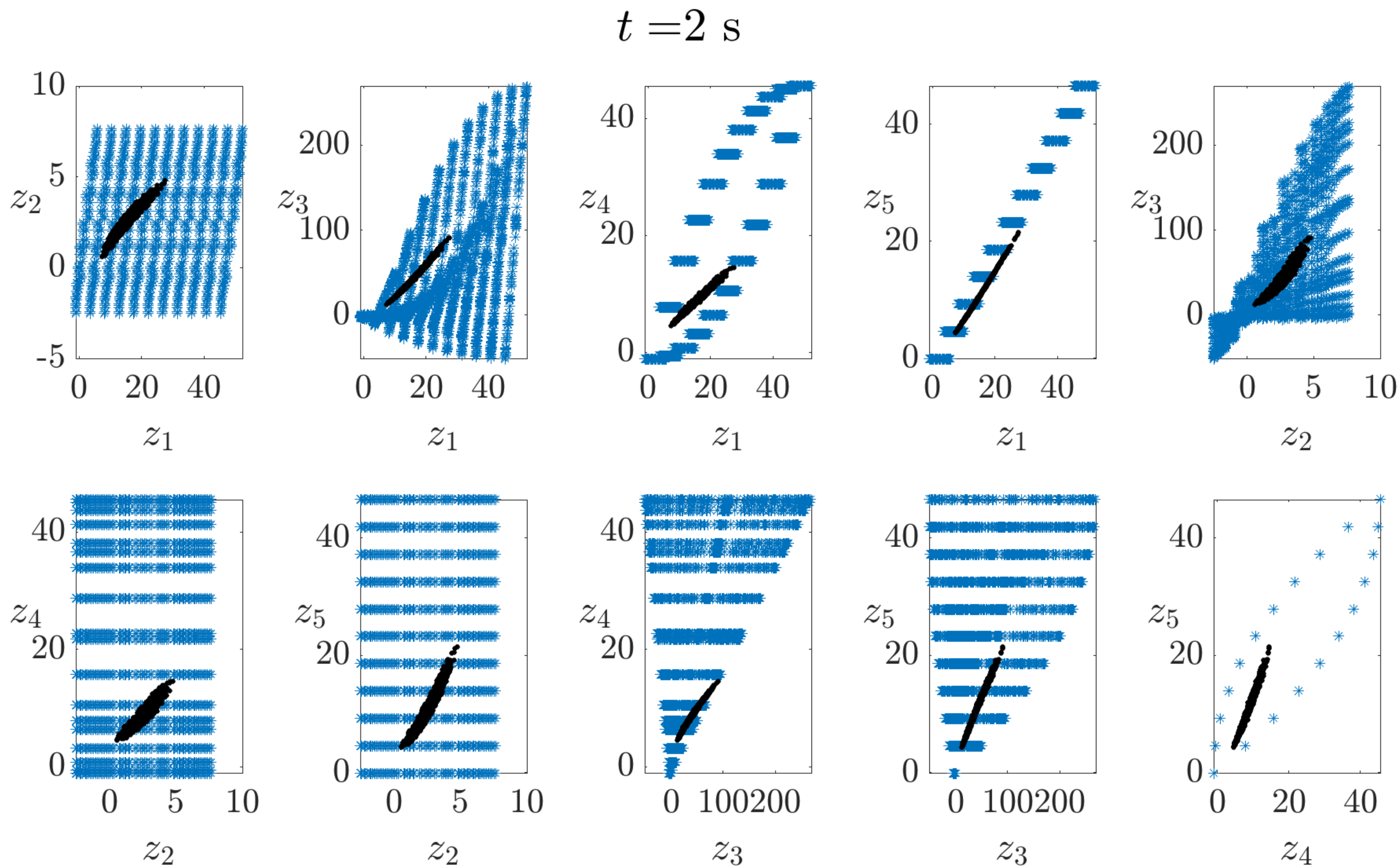


Serial computation time = 0.94 s and N = 15640.

Learning Strategy: example

System (2)

fsSFL with relative degree vector $\boldsymbol{r} = (3,2)^\top$

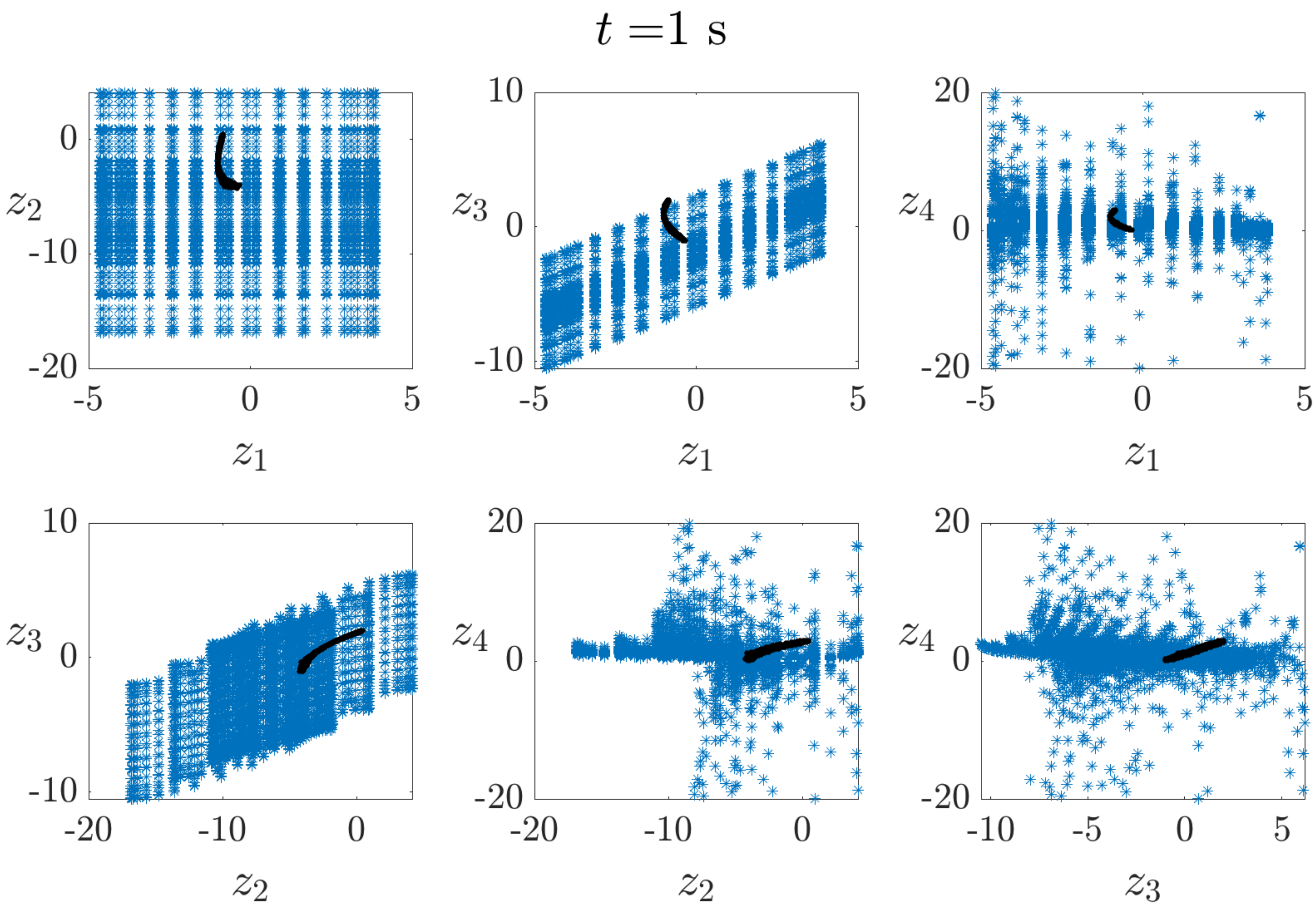


Serial computation time = 1.13 s and N = 15640.

Learning Strategy: example

System (3)

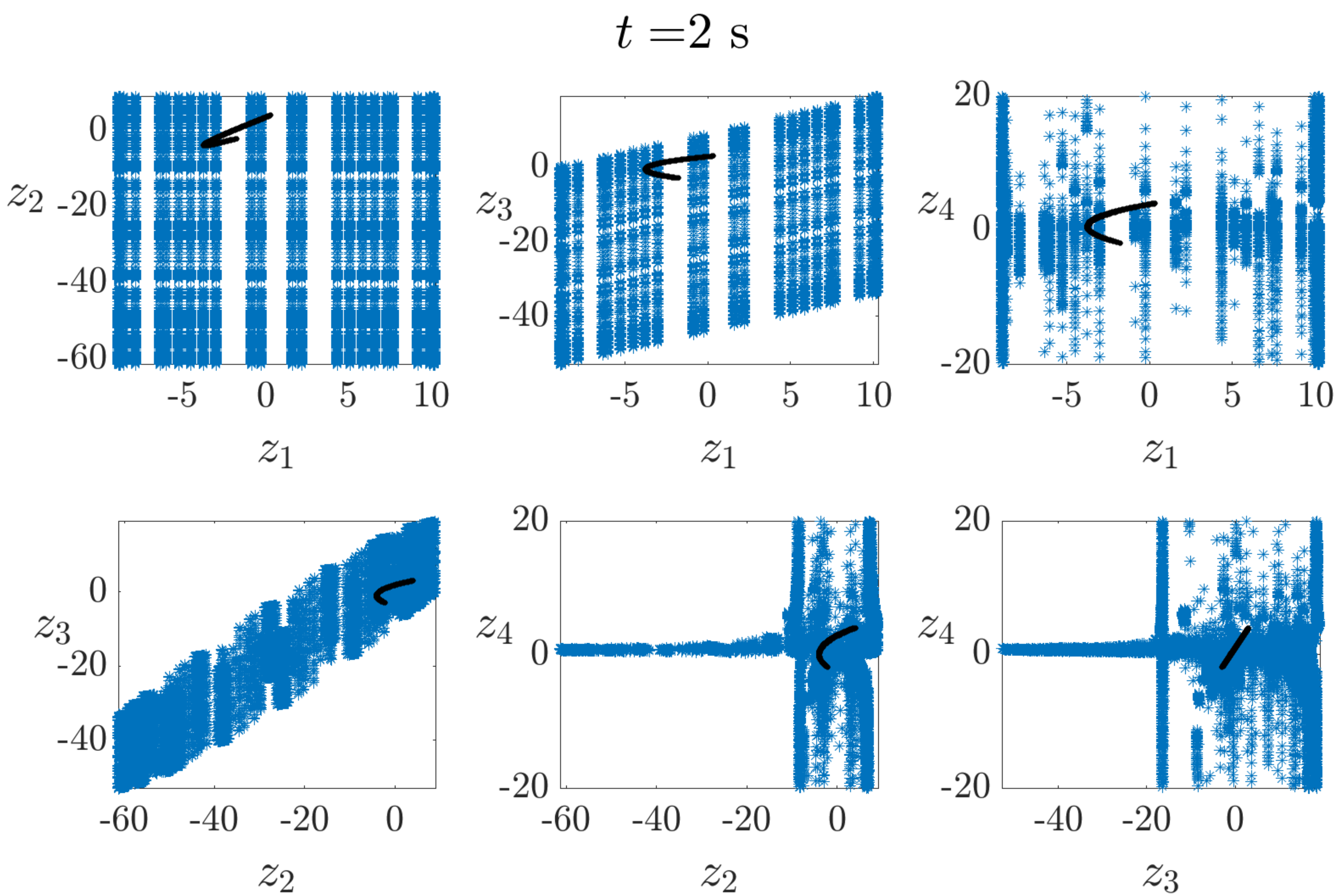
fsDFL with relative degree vector $\boldsymbol{r} = (2,3)^\top$



Serial computational time = 1.45 s and $N = 448686$.

Learning Strategy: example **System (3)**

fsDFL with relative degree vector $r = (2,3)^\top$



Serial computational time = 3.55 s and N = 448686.

Boundary of the Minkowski sum of $\mathcal{X}_t$ with a line segment

In dynamic feedback linearizable systems:

The state diffeomorphism, $\boldsymbol{\tau}(\boldsymbol{z}, w)$ is a function of set valued uncertainty $\mathcal{V}$.

The initial condition in the corresponding integrator coordinates, $\mathcal{X}_0$ is an interval.

Example: system (3)

$$\mathcal{X}_{01} = \begin{bmatrix} z_{01} \\ w \end{bmatrix} = \begin{bmatrix} z_{01} \\ z_{02} - [-1,\ 1] \end{bmatrix}, \qquad \mathcal{X}_{02} = \begin{bmatrix} z_{01} + z_{03} \\ z_{02} \\ z_{04}(z_{02} - w) \end{bmatrix} = \begin{bmatrix} z_{01} + z_{03} \\ z_{02} \\ z_{04}[-1,\ 1] \end{bmatrix}.$$

The matrix vector product $\exp(\boldsymbol{A}_j t)\mathcal{X}_{j0}$, will return a tilted line segment, $\boldsymbol{\ell}_j$ embedded in $\mathbb{R}^{r_j}$, for $j = 1, \cdots, m$.

$$\boldsymbol{\ell}_1 = \begin{bmatrix} z_{01} + tz_{02} \\ z_{01} \end{bmatrix} + \begin{bmatrix} -t \\ -1 \end{bmatrix} [-1,\ 1], \quad \boldsymbol{\ell}_2 = \begin{bmatrix} z_{01} + z_{03} + tz_{02} \\ z_{02} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}t^2 z_{04} \\ tz_{04} \\ z_{04} \end{bmatrix} [-1,\ 1].$$

We need to obtain

$$\partial \mathcal{X}_j\left(\{\mathcal{X}_{0j}\}, t\right) = \partial\left(\boldsymbol{\ell}_j \dotplus \mathcal{X}_j(\{\mathbf{0}\}, t)\right), \qquad \partial \mathcal{X}_t = \partial \mathcal{X}_1 \dotplus \cdots \dotplus \partial \mathcal{X}_m.$$

Boundary of the Minkowski sum of $\mathcal{X}_t$ with a line segment

Parametric line segment:

$$\boldsymbol{\ell}(c) = \boldsymbol{\ell}_0 + (\boldsymbol{\ell}_1 - \boldsymbol{\ell}_0)\, c, \quad 0 \le c \le 1, \quad \boldsymbol{\ell}_0, \boldsymbol{\ell}_1 \in \mathbb{R}^{n_x}.$$

Define:

$$\widehat{\boldsymbol{\ell}} := (\boldsymbol{\ell}_1 - \boldsymbol{\ell}_0) \,/ \|\boldsymbol{\ell}_1 - \boldsymbol{\ell}_0\| = (\ell_1, \ell_2, \cdots, \ell_{n_x}).$$

There exist a parametric surface $\sigma(\mathbf{s})$ such that

$$\sigma(\boldsymbol{s}) := \langle \widehat{\boldsymbol{n}}(\boldsymbol{s}), \widehat{\boldsymbol{\ell}} \rangle = 0, \quad \text{for } \boldsymbol{s} \in \mathcal{S} \setminus \left\{ \boldsymbol{s} \in \mathcal{S} \;\middle|\; s_i = s_j, \text{ where } i \neq j, \;\; i, j = 1, \cdots, n_x - 1 \right\},$$

where $\widehat{\boldsymbol{n}}$ is the outward normal vector on $\partial\mathcal{X}$.

The parametric surface $\sigma(\mathbf{s})$ divides the parameter space into two parts:

$$\mathcal{S}_0 \cup \mathcal{S}_1 \cup \sigma = \mathcal{S},$$

such that $\mathcal{S}_0$ contains

$$\left\{ s \in \mathcal{S} \;\middle|\; s_1 = s_2 = \cdots = s_{n_x - 1} \right\}.$$

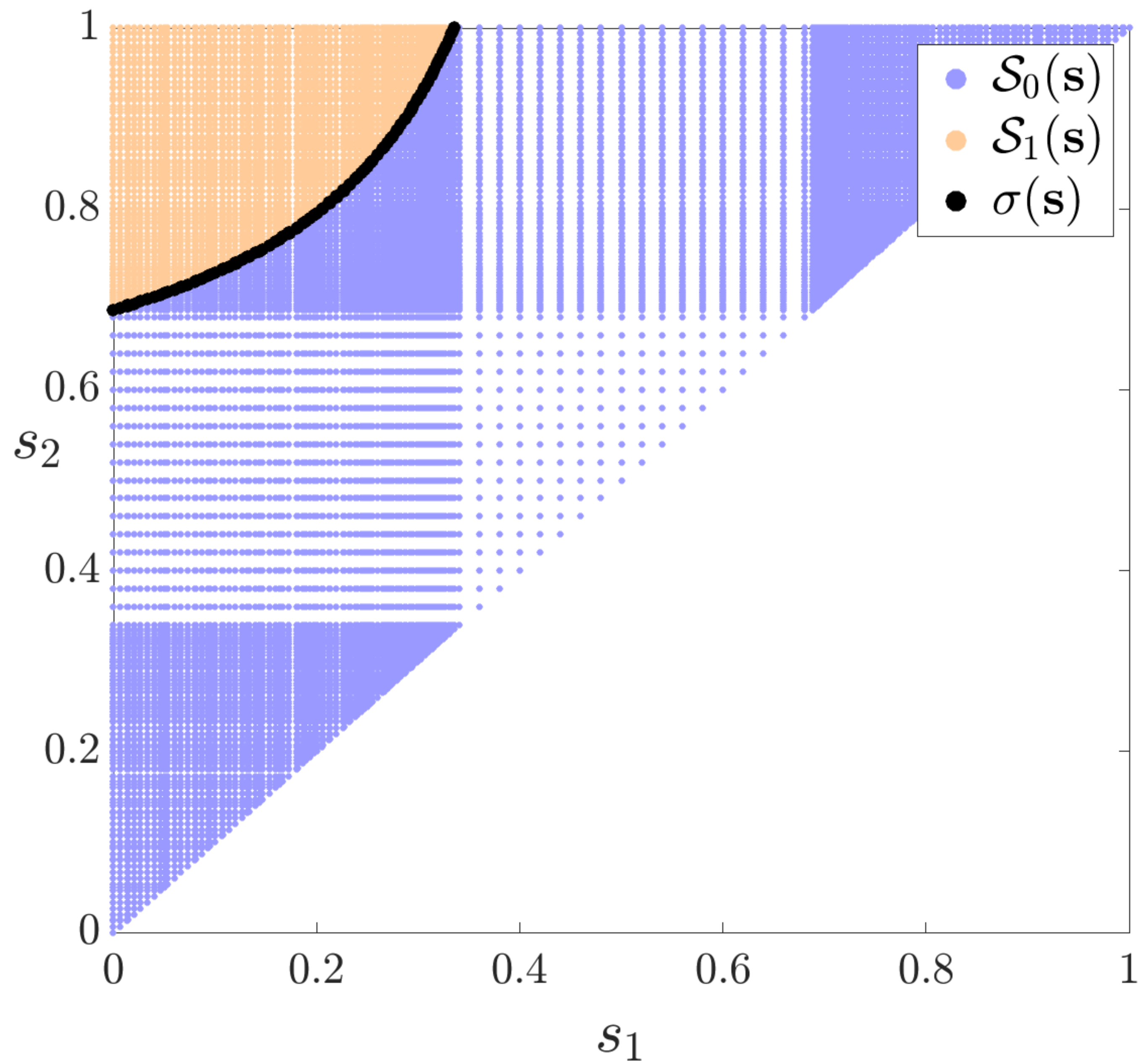
Boundary of the Minkowski sum of $\mathcal{X}_t$ with a line segment

Ex: 3-D integrator reach set with $\mathbf{x}_0 = [0\ 0\ 0]^\top$, and $u(t) \in [-0.5, 1.5]$, for all $s \in [0,1]$ s, and

$$\sigma(\boldsymbol{s}) := \left\{ \boldsymbol{s} \in \mathbb{R}^2 \;\middle|\; 0 \leq s_1 < s_2 \leq t, \quad (s_1 - s_2)\,\ell_1 - \left(\frac{s_1^2}{2} - \frac{s_2^2}{2}\right)\ell_2 + \left(\frac{s_1^2 s_2}{2} - \frac{s_2^2 s_1}{2}\right)\ell_3 = 0 \right\}.$$

$$\boldsymbol{\ell}_0 = \begin{bmatrix} -0.5 & -0.54 & -0.80 \end{bmatrix},$$

$$\ell_1 = \begin{bmatrix} -0.24 & 0.22 & 0.67 \end{bmatrix}.$$



The parameter space for $\mathcal{X}_t \in \mathbb{R}^3$ and $\boldsymbol{\ell} \in \mathbb{R}^2$.

Boundary of the Minkowski sum of $\mathcal{X}_t$ with a line segment

Theorem. The parametric surface $\boldsymbol{\sigma}(\mathbf{s})$, where $\mathbf{s} = s_1, \cdots, s_{n_x-1}$, is given by

$$\{\boldsymbol{s} \in \mathbb{R}^{n_x-1} \;\mid\; 0 \leq s_1 < s_2 < \ldots < s_{n_x-1} \leq t, \;\; \sum^{n_x} (n_x - i)!\,(-1)^{n_x-i} \ell_i e_{i-1} = 0\},$$

where e_r denotes the rth elementary symmetric polynomial.

Define:

$$\partial\mathcal{X}_{\text{cut}} := \left\{\boldsymbol{x}^{\text{bdy}}(\boldsymbol{s}) \in \partial\mathcal{X}^{\text{upper}}(\boldsymbol{s}) \;\middle|\; \boldsymbol{s} \in \partial\mathcal{S}_1\right\} \,\Delta\, \left\{\boldsymbol{x}^{\text{bdy}}(\boldsymbol{s}) \in \partial\mathcal{X}^{\text{lower}}(\boldsymbol{s}) \;\middle|\; \boldsymbol{s} \in \partial\mathcal{S}_0\right\},$$

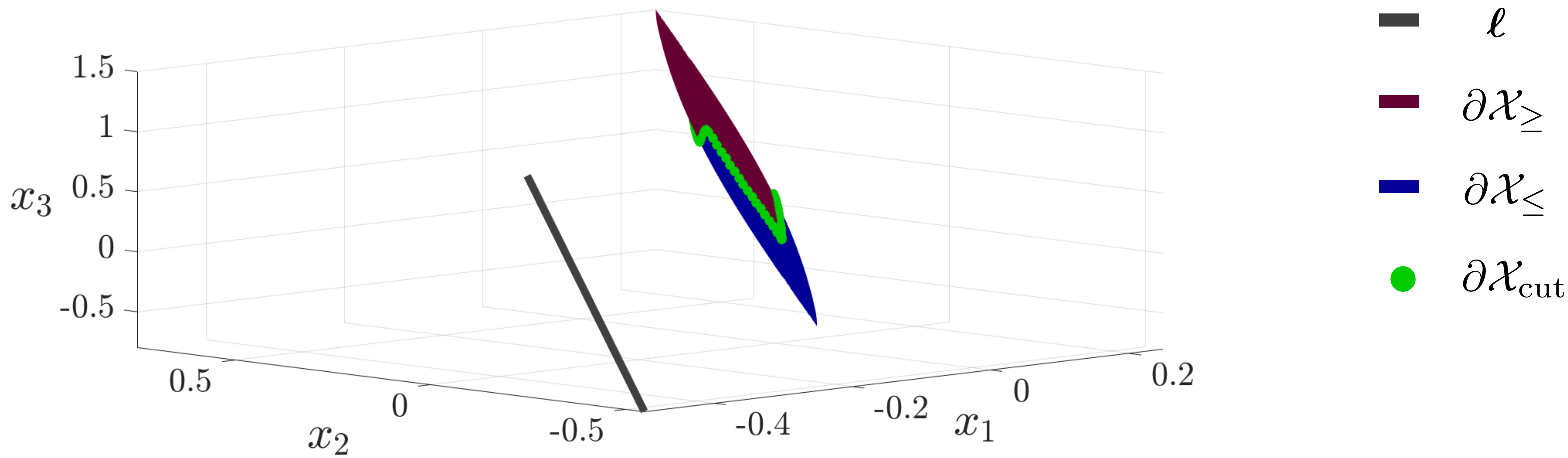
which divides $\partial\mathcal{X}$ into two sets: $\partial\mathcal{X}_{\geq}$, $\partial\mathcal{X}_{\leq}$, **such that** $\partial\mathcal{X}_{\geq} \cup \partial\mathcal{X}_{\leq} \cup \partial\mathcal{X}_{\text{cut}} = \partial\mathcal{X}_t$, **where**

$$\partial\mathcal{X}_{\geq} := \{\boldsymbol{x}^{\text{bdy}}(\boldsymbol{s}) \in \partial\mathcal{X}_t \mid \langle \widehat{\boldsymbol{n}}(\boldsymbol{s}), \widehat{\boldsymbol{\ell}} \rangle \geq 0\},$$
$$\partial\mathcal{X}_{\leq} := \{\boldsymbol{x}^{\text{bdy}}(\boldsymbol{s}) \in \partial\mathcal{X}_t \mid \langle \widehat{\boldsymbol{n}}(\boldsymbol{s}), \widehat{\boldsymbol{\ell}} \rangle \leq 0\}.$$

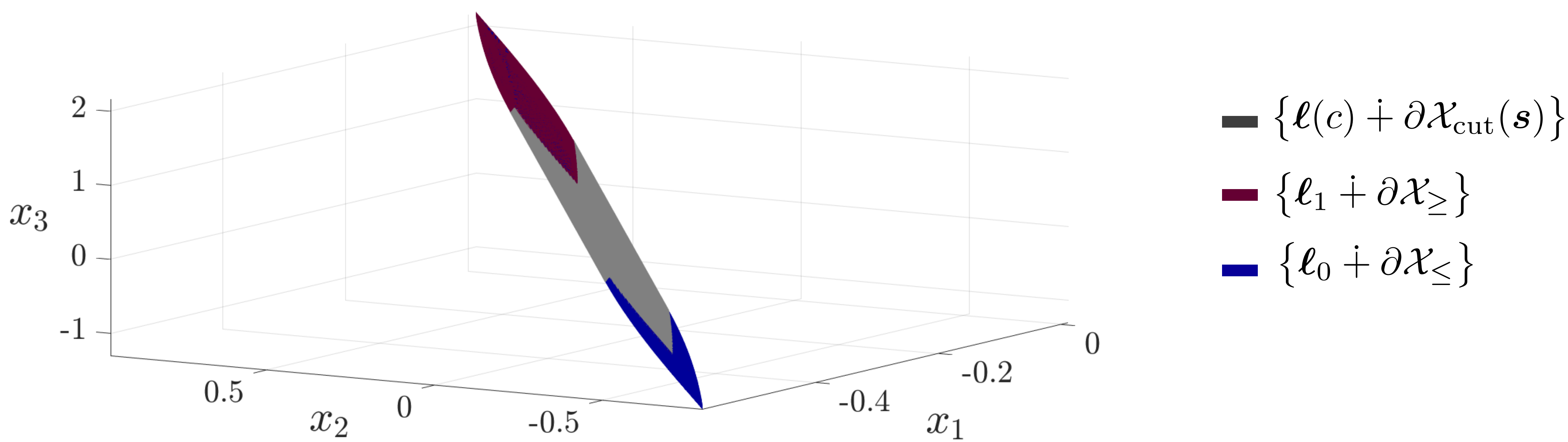
Theorem.

$$\partial\left(\boldsymbol{\ell} \dot{+} \mathcal{X}_t\right)(c, \boldsymbol{s}) = \left\{\boldsymbol{\ell}(c) \dot{+} \partial\mathcal{X}_{\text{cut}}(\boldsymbol{s})\right\} \cup \left\{\boldsymbol{\ell}_1 \dot{+} \partial\mathcal{X}_{\geq}\right\} \cup \left\{\boldsymbol{\ell}_0 \dot{+} \partial\mathcal{X}_{\leq}\right\}$$

Boundary of the Minkowski sum of $\mathcal{X}_t$ with a line segment

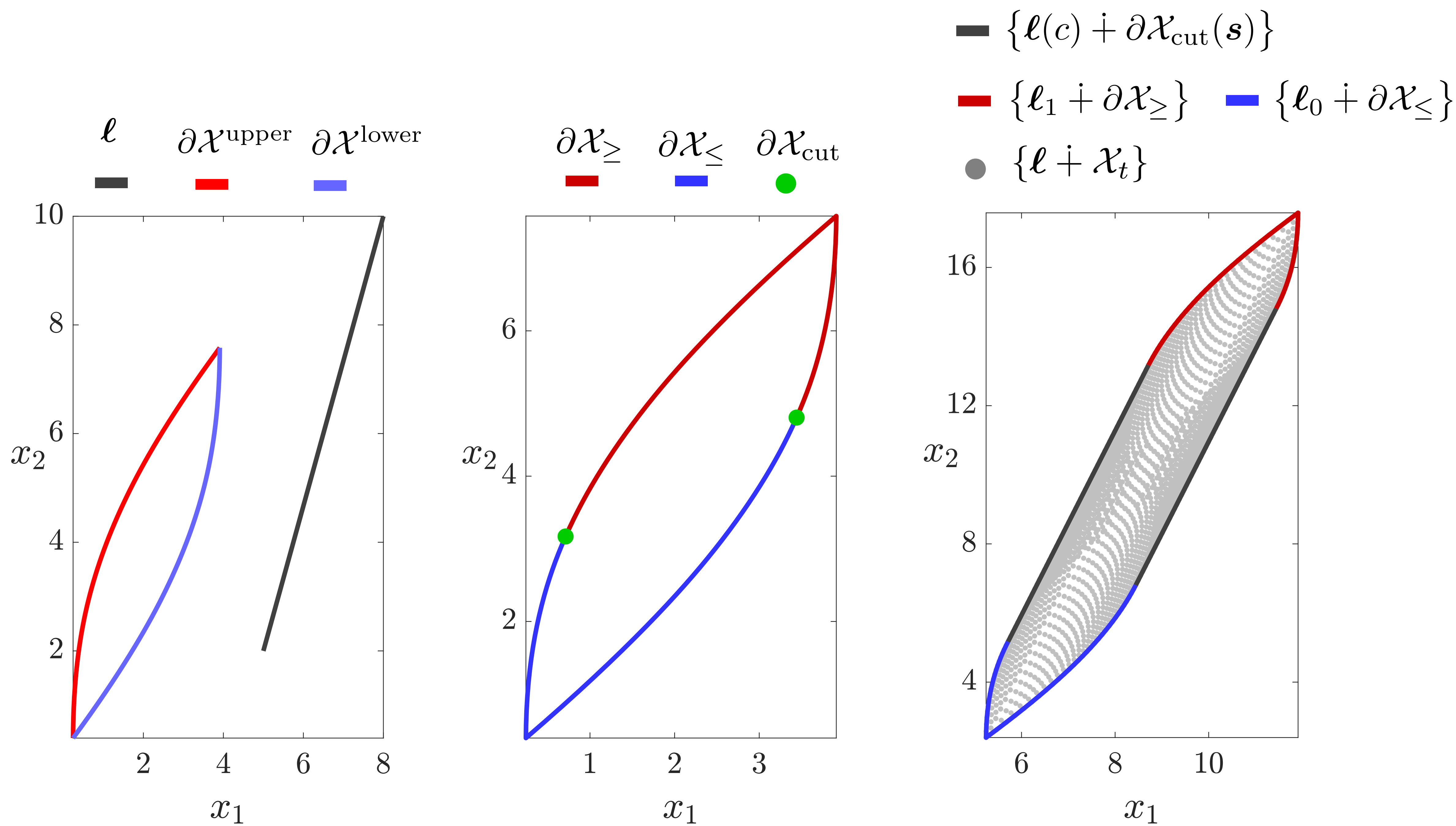


The integrator reach set $\mathcal{X}_t \in \mathbb{R}^3$ and the line segment $\boldsymbol{\ell} \in \mathbb{R}^2$



The boundary of the Minkowski sum of integrator reach set $\mathcal{X}_t \in \mathbb{R}^3$ with the line segment $\boldsymbol{\ell} \in \mathbb{R}^2$.

Boundary of the Minkowski sum of $\mathcal{X}_t$ with a line segment



The boundary of the Minkowski sum of integrator reach set $\mathcal{X}_t \in \mathbb{R}^2$ with the line segment $\ell \in \mathbb{R}$.

Parallelization

Parallelization

The proposed learning algorithm allows multiple layers of parallel computation

Computing $\partial\mathcal{X}_j$, parallelization across components: $n^{\text{FLOPS}}_{\boldsymbol{x}^{\text{bdy}}_{kj}} = \mathcal{O}\big(\big(4 + \mathbf{1}_{k<r_j-1}2(r_j - k - 1)\big)N_s\big)$

Number of discretization points of each component of the parameter space

Computing $\partial\mathcal{X}_j$, parallelization across blocks: $n^{\text{FLOPS}}_{\boldsymbol{x}^{\text{bdy}}_{j}} = \mathcal{O}\left(\sum_{k=1}^{r_j}\big[4 + \mathbf{1}_{k<r_j-1}2(r_j - k - 1)\big]N_s\right)$

Minkowski sum of integrator blocks: $\partial\mathcal{X}_t = \partial\mathcal{X}_1 \dotplus \partial\mathcal{X}_2 \dotplus \ldots \dotplus \partial\mathcal{X}_m$

Transforming $\partial\mathcal{X}_t$ back to original coordinates $\partial\mathcal{Z}_t$

Publications

1. **S.H.** and Abhishek Halder, The convex geometry of integrator reach sets, *Proceedings of the American Control Conference*, 2020.

2. **S.H.** and Abhishek Halder, Anytime Ellipsoidal Over-approximation of Forward Reach Sets of Uncertain Linear Systems, *Proceedings of the Workshop on Computation-aware Algorithmic Design for Cyber-physical Systems*, 2021.

3. **S.H.**, K.F. Caluya, A. Halder and B. Singh, Prediction and optimal feedback steering of probability density functions for safe automated driving, *IEEE Control Systems Letters*, 2021.

4. **S.H.,** A. Halder and B. Singh, Density-based stochastic reachability computation for occupancy prediction in automated driving, *IEEE Transactions on Control Systems Technology*, 2021.

5. **S.H.** and Abhishek Halder, The boundary and taxonomy of integrator reach sets, *Proceedings of the American Control Conference*, 2022. [Under Review]

6. **S.H.** and Abhishek Halder, The curious case of integrator reach sets, Part I: Basic theory, *IEEE Transactions on Automatic Control*, 2022. [In Revision]

7. **S.H.** and Abhishek Halder, The curious case of integrator reach sets, Part II: Applications to feedback linearizable systems. [To Be Submitted]

Future Timeline

[i] **Accounting for singularity (Winter 2022 - Spring 2022)**

[ii] **Generic dynamic state feedback linearizable systems (Summer 2022 - Winter 2023)**

[iii] **Collision avoidance applications (Winter 2022 - Spring 2023)**

[iv] **Partial state feedback linearizable systems (Winter 2022 - Summer 2023)**

[v] **Completing Dissertation and Graduation (Summer 2023 - Fall 2023)**

Thank You