Contraction and Reaction in Generalized Schrödinger Bridges

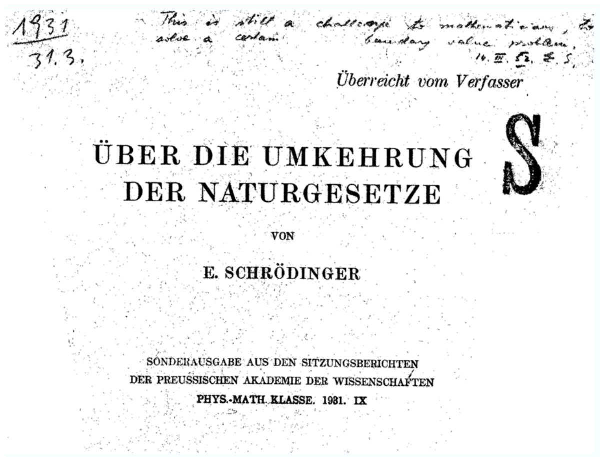
Alexis M.H. Teter

Department of Applied Mathematics
University of California, Santa Cruz
Santa Cruz, CA 95064

Advancement to Ph.D. candidacy

January 30, 2024

What is a Schrödinger Bridge Problem (SBP)



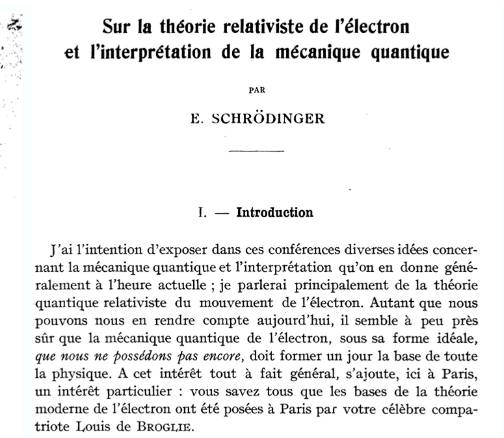
1931
31.3.
This is still a challenge to mathematicians, to solve a certain boundary value problem.
Überreicht vom Verfasser

ÜBER DIE UMKEHRUNG
DER NATURGESETZE

VON

E. SCHRÖDINGER

SONDERAUSGABE AUS DEN SITZUNGSBERICHTEN
DER PREUSSISCHEN AKADEMIE DER WISSENSCHAFTEN
PHYS.-MATH. KLASSE. 1931. IX



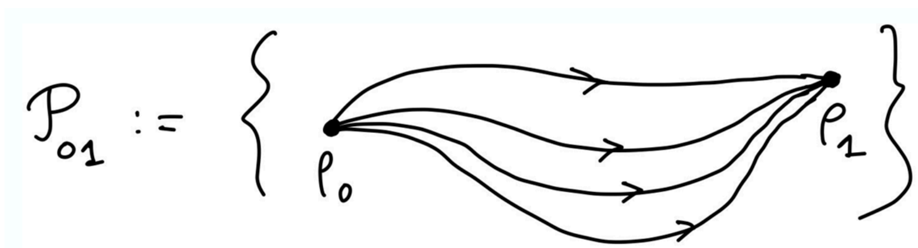
Sur la théorie relativiste de l'électron
et l'interprétation de la mécanique quantique

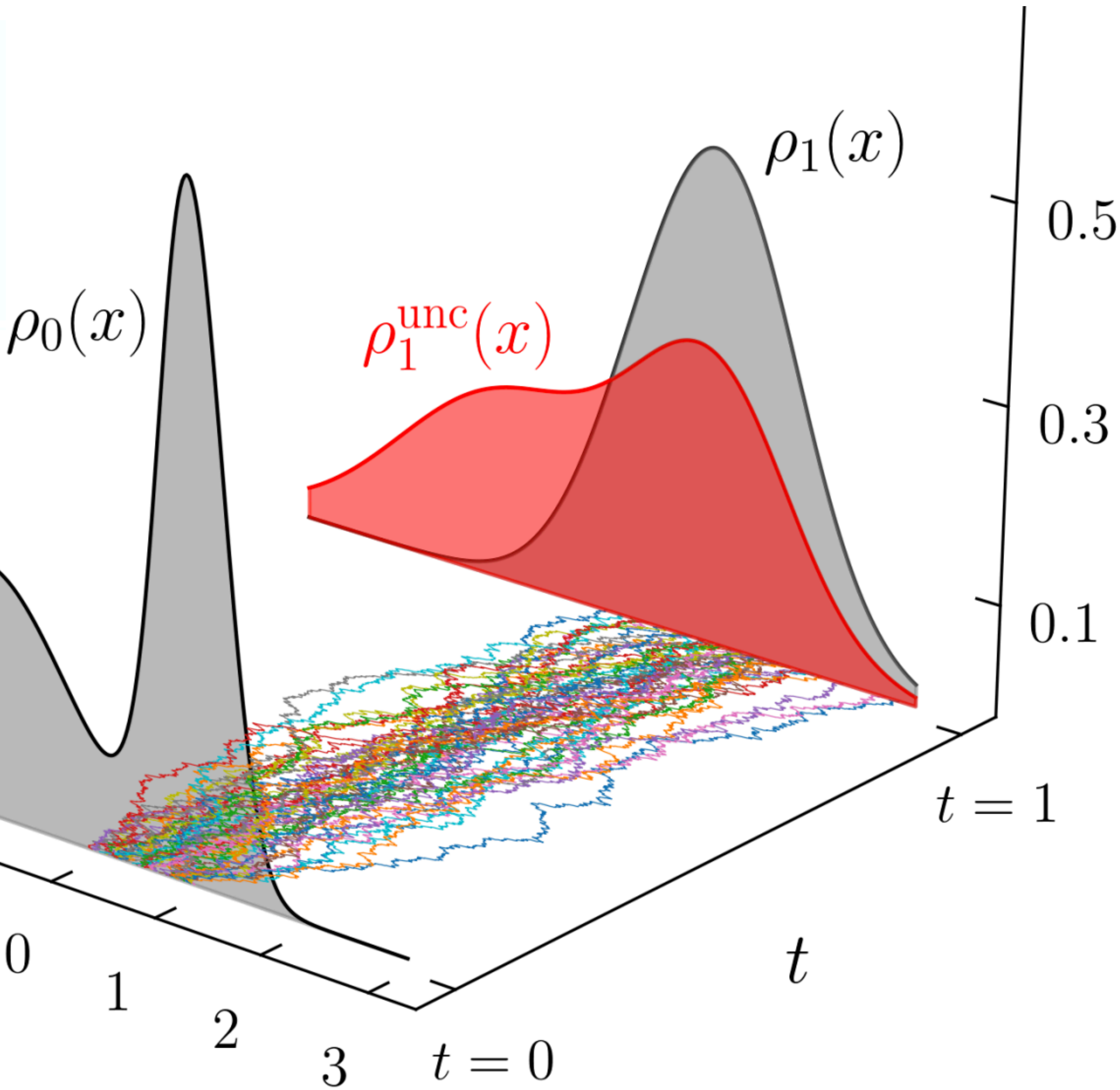
PAR

E. SCHRÖDINGER

I. — Introduction

J'ai l'intention d'exposer dans ces conférences diverses idées concernant la mécanique quantique et l'interprétation qu'on en donne généralement à l'heure actuelle ; je parlerai principalement de la théorie quantique relativiste du mouvement de l'électron. Autant que nous pouvons nous en rendre compte aujourd'hui, il semble à peu près sûr que la mécanique quantique de l'électron, sous sa forme idéale, *que nous ne possédons pas encore*, doit former un jour la base de toute la physique. A cet intérêt tout à fait général, s'ajoute, ici à Paris, un intérêt particulier : vous savez tous que les bases de la théorie moderne de l'électron ont été posées à Paris par votre célèbre compatriote Louis de Broglie.





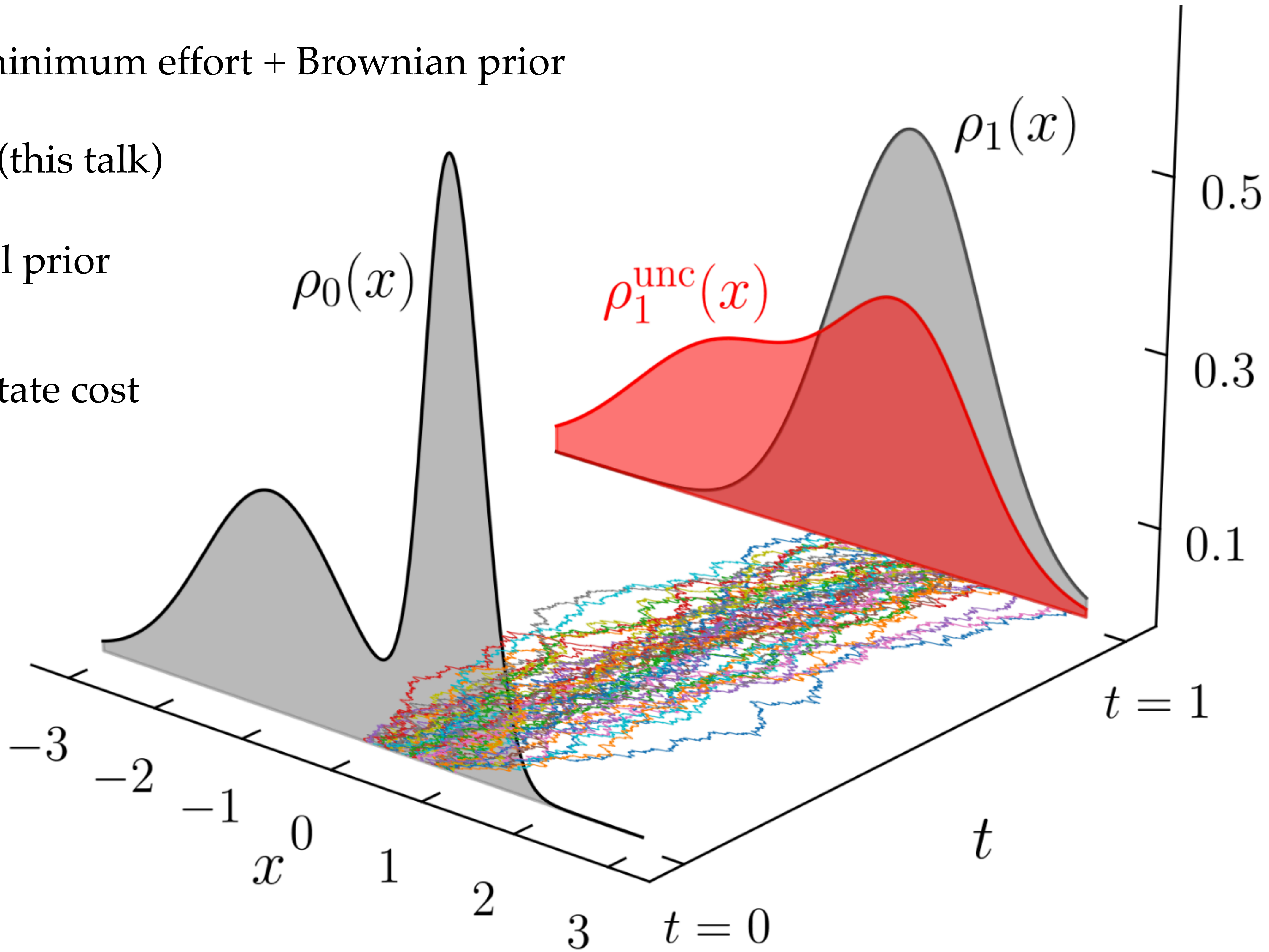
Most likely evolution between 2 distributional snapshots

This talk: Generalized SBP

Classical SBP = minimum effort + Brownian prior

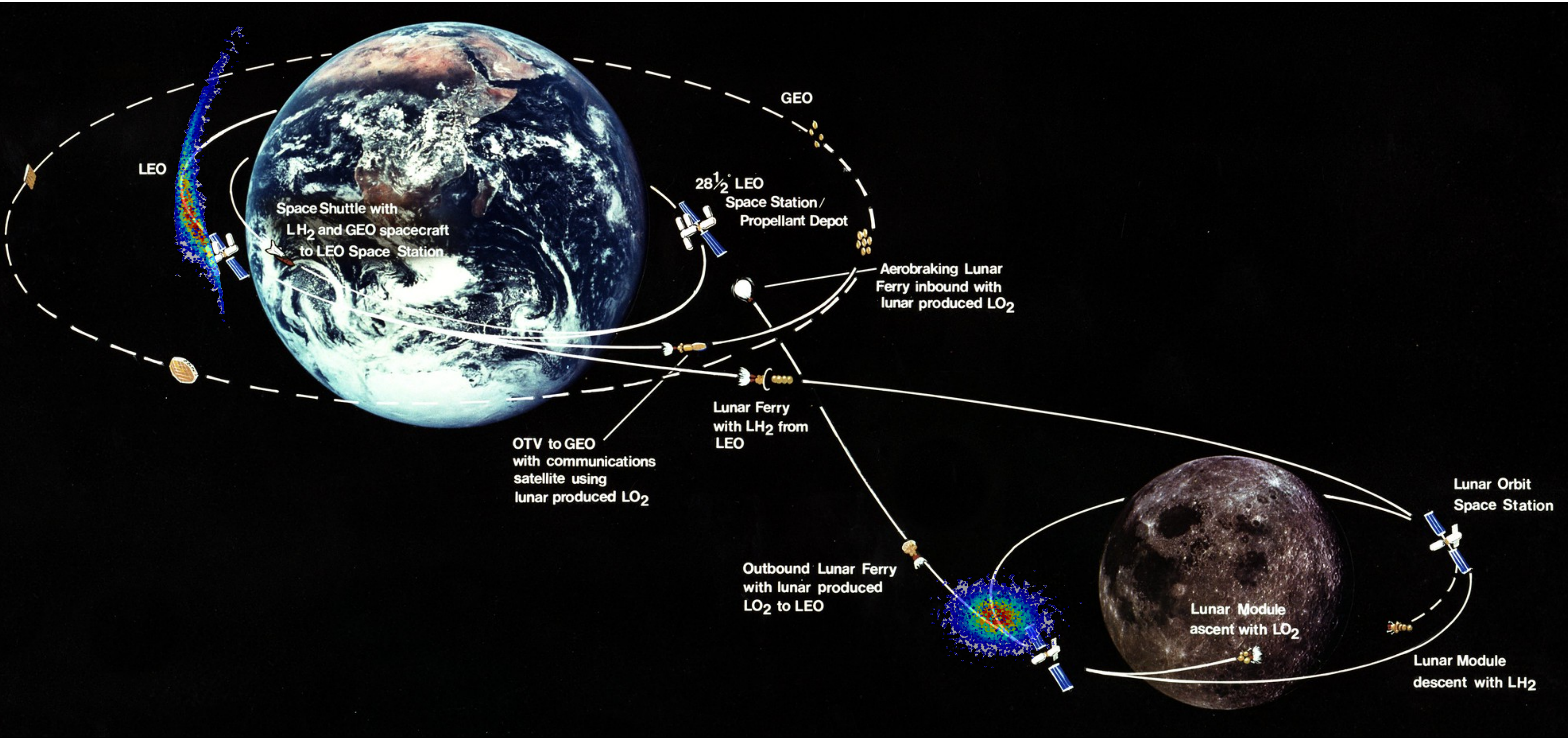
Generalized SBP (this talk)

- More general prior
- Additional state cost



Most likely evolution between 2 distributional snapshots

Motivating Application: Generalized SBP



Stochastic guidance and control of a spacecraft

Background

Optimal Mass Transport (OMT)

Static (Monge) formulation [1781]

$$\underset{\text{measurable } \boldsymbol{\tau}:\mathbb{R}^n \mapsto \mathbb{R}^n}{\arg\inf} \quad \mathbb{E}_{\rho_0} \frac{1}{2} |\boldsymbol{x}_0 - \boldsymbol{\tau}(\boldsymbol{x}_0)|^2$$

$$\text{subject to} \quad \boldsymbol{x}_0 \sim \rho_0, \quad \boldsymbol{\tau}(\boldsymbol{x}_0) \sim \rho_1$$

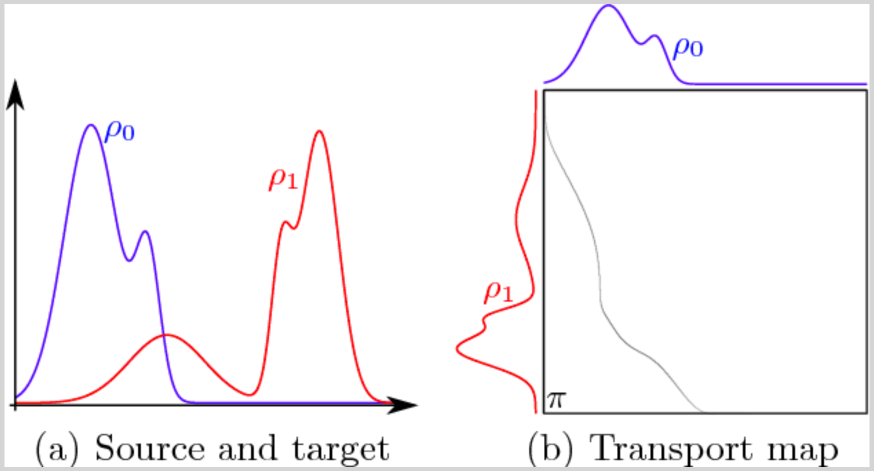


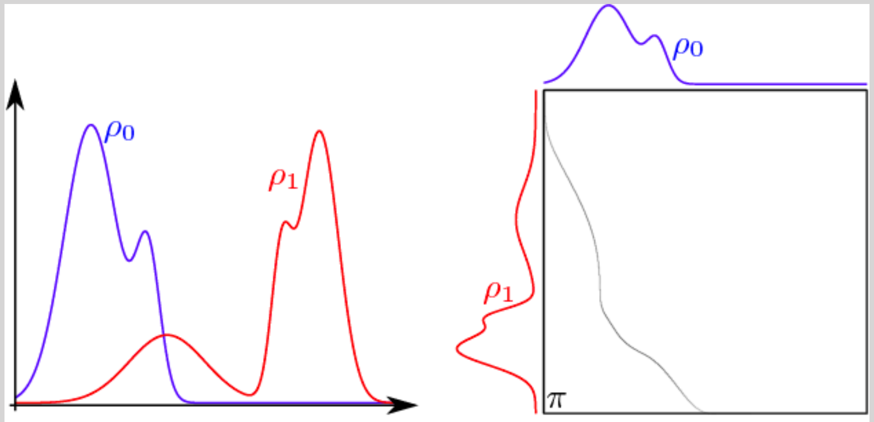
Image credit: Justin Solomon

Optimal Mass Transport (OMT)

Static (Monge) formulation [1781]

$$\underset{\text{measurable } \boldsymbol{\tau}:\mathbb{R}^n\mapsto\mathbb{R}^n}{\arg\inf} \quad \mathbb{E}_{\rho_0}\frac{1}{2}\left|\boldsymbol{x}_0 - \boldsymbol{\tau}(\boldsymbol{x}_0)\right|^2$$

$$\text{subject to} \qquad \boldsymbol{x}_0 \sim \rho_0, \quad \boldsymbol{\tau}(\boldsymbol{x}_0) \sim \rho_1$$



(a) Source and target (b) Transport map

Image credit: Justin Solomon

Static (Kantorovich-Rubinstein) reformulation [1941]

$$\underset{\pi\in\Pi(\rho_0,\rho_1)}{\arg\inf}\ \mathbb{E}_{\pi}\frac{1}{2}\left|\boldsymbol{x}_0 - \boldsymbol{x}_1\right|^2$$

$$\text{subject to} \qquad \boldsymbol{x}_0 \sim \rho_0, \quad \boldsymbol{x}_1 \sim \rho_1$$

Infinite dimensional linear program

Optimal Mass Transport (OMT)

Static (Monge) formulation [1781]

$$\underset{\text{measurable } \boldsymbol{\tau}:\mathbb{R}^n \mapsto \mathbb{R}^n}{\arg\inf} \quad \mathbb{E}_{\rho_0} \frac{1}{2} \left| \boldsymbol{x}_0 - \boldsymbol{\tau}(\boldsymbol{x}_0) \right|^2$$

$$\text{subject to} \quad \boldsymbol{x}_0 \sim \rho_0, \quad \boldsymbol{\tau}(\boldsymbol{x}_0) \sim \rho_1$$

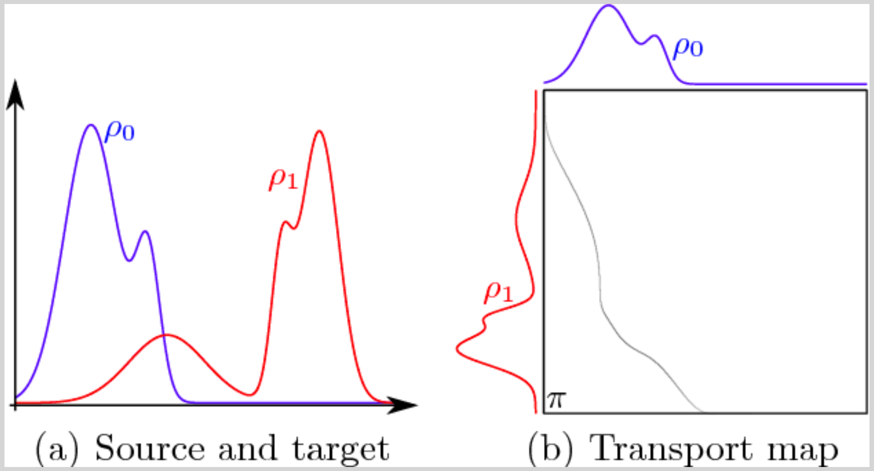


Image credit: Justin Solomon

Static (Kantorovich-Rubinstein) reformulation [1941]

$$\underset{\pi \in \Pi(\rho_0, \rho_1)}{\arg\inf} \; \mathbb{E}_{\pi} \frac{1}{2} \left| \boldsymbol{x}_0 - \boldsymbol{x}_1 \right|^2$$

$$\text{subject to} \quad \boldsymbol{x}_0 \sim \rho_0, \quad \boldsymbol{x}_1 \sim \rho_1$$

Infinite dimensional linear program

Dynamic (Benamou-Brenier) formulation [1999]

$$\underset{(\rho, \boldsymbol{v}) \in \mathcal{P}_{01} \times \mathcal{V}}{\arg\inf} \int_{t_0}^{t_1} \int_{\mathbb{R}^n} \frac{1}{2} |\boldsymbol{v}|^2 \rho(\boldsymbol{x}, t) \mathrm{d}\boldsymbol{x} \mathrm{d}t$$

$$\frac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v}) = 0,$$

$$\rho(\boldsymbol{x}, t = t_0) = \rho_0, \quad \rho(\boldsymbol{x}, t = t_1) = \rho_1$$

Stochastic optimal control problem

Classical SBP as Stochastic Optimal Control

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\frac{1}{2}\left|\boldsymbol{v}\right|^2\rho(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t$$

$$\frac{\partial\rho}{\partial t}+\nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v})=\varepsilon\Delta_{\boldsymbol{x}}\rho,\qquad\varepsilon>0,$$

$$\rho(\boldsymbol{x},t=t_0)=\rho_0,\qquad\rho(\boldsymbol{x},t=t_1)=\rho_1,$$

Fokker-Planck-Kolmogorov PDE

Controlled sample path dynamics

$$\mathrm{d}\boldsymbol{x}=\boldsymbol{v}(\boldsymbol{x},t)\mathrm{d}t+\sqrt{2\varepsilon}\mathrm{d}\boldsymbol{w}(t)$$

Classical OMT vs. Classical SBP

Classical OMT

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\frac{1}{2}|\boldsymbol{v}|^2\rho(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t$$

$$\frac{\partial\rho}{\partial t}+\nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v})=0,$$ **Liouville PDE**

$$\rho(\boldsymbol{x},t=t_0)=\rho_0,\quad \rho(\boldsymbol{x},t=t_1)=\rho_1$$

Classical SBP

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\frac{1}{2}|\boldsymbol{v}|^2\rho(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t$$

Fokker-Planck-Kolmogorov PDE

$$\frac{\partial\rho}{\partial t}+\nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v})=\varepsilon\Delta_{\boldsymbol{x}}\rho,\quad \varepsilon>0,$$

$$\rho(\boldsymbol{x},t=t_0)=\rho_0,\quad \rho(\boldsymbol{x},t=t_1)=\rho_1,$$

Generalized SBP

$$\underset{(\rho, \boldsymbol{v}) \in \mathcal{P}_{01} \times \mathcal{V}}{\arg\inf} \int_{t_0}^{t_1} \int_{\mathbb{R}^n} \left(\frac{1}{2} |\boldsymbol{v}|^2 + q(\boldsymbol{x}) \right) \rho(\boldsymbol{x}, t) \, d\boldsymbol{x} \, dt$$

$$\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(\left(\boldsymbol{g}\boldsymbol{g}^\top \right)_{ij} \rho \right)$$

$$\frac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{f}(\boldsymbol{x}, t, \boldsymbol{v})) = \varepsilon \langle \mathrm{Hess}, \boldsymbol{g}\boldsymbol{g}^\top \rho \rangle$$

$$\rho(\boldsymbol{x}, t = t_0) = \rho_0, \quad \rho(\boldsymbol{x}, t = t_1) = \rho_1$$

Controlled sample path dynamics

$$d\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}, t, \boldsymbol{v}) \, dt + \sqrt{2\varepsilon} \boldsymbol{g}(\boldsymbol{x}, t, \boldsymbol{v}) \, d\boldsymbol{w}(t)$$

Linear SBP: Contraction Coefficient

Related works

Y. Chen, T. Georgiou, and M. Pavon, “Entropic and displacement interpolation: a computational approach using the Hilbert metric,” SIAM Journal on Applied Mathematics, vol. 76, no. 6, pp. 2375–2396, 2016

M. Kuang and E. G. Tabak, “Preconditioning of optimal transport,” SIAM Journal on Scientific Computing, vol. 39, no. 4, pp. A1793–A1810, 2017

Linear SBP

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf} \quad \int_{t_0}^{t_1}\int_{\mathbb{R}^n} \frac{1}{2}|\boldsymbol{v}|^2 \rho(\boldsymbol{x},t)\, d\boldsymbol{x}\, dt$$

$$\frac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho(\boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{v})) = \varepsilon \langle \mathrm{Hess}, \boldsymbol{B}(t)\boldsymbol{B}(t)^\top \rho \rangle$$

resp. compact supports $\mathcal{X}_0, \mathcal{X}_1$

$$\rho(\boldsymbol{x}, t = t_0) = \rho_0, \qquad \rho(\boldsymbol{x}, t = t_1) = \rho_1$$

Controlled sample path dynamics

$$\mathrm{d}\boldsymbol{x}(t) = (\boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{v}(\boldsymbol{x},t))\mathrm{d}t + \sqrt{2\varepsilon}\boldsymbol{B}(t)\mathrm{d}\boldsymbol{w}(t)$$

State transition matrix $\boldsymbol{\Phi}_{t\tau} := \boldsymbol{\Phi}(t,\tau) \quad \forall t_0 \leq \tau \leq t \leq t_1$

Assume controllability: $\boldsymbol{M}_{10} := \int_{t_0}^{t_1} \boldsymbol{\Phi}_{t_1\tau}\boldsymbol{B}(\tau)\boldsymbol{B}^\top(\tau)\boldsymbol{\Phi}_{t_1\tau}^\top \mathrm{d}\tau \succ \boldsymbol{0}$

Classical SBP is special case: $\boldsymbol{A}(t) \equiv \boldsymbol{0}, \boldsymbol{B}(t) \equiv \boldsymbol{I}$

Structure of the Solution for Linear SBP

Optimally controlled joint state PDF: $\rho_{\varepsilon}^{\text{opt}}(\cdot,t) = \widehat{\varphi}_{\varepsilon}(\cdot,t)\varphi_{\varepsilon}(\cdot,t)$

Optimal control: $\boldsymbol{v}_{\varepsilon}^{\text{opt}}(\cdot,t) = 2\varepsilon\nabla_{\boldsymbol{x}}\log\varphi_{\varepsilon}(\cdot,t)$

Schrödinger factors

Define: $\widehat{\varphi}_{\varepsilon,0}(\cdot) := \widehat{\varphi}_{\varepsilon}(\cdot,t=t_0), \quad \varphi_{\varepsilon,1}(\cdot) := \varphi_{\varepsilon}(\cdot,t=t_1)$

Schrödinger system

Markov kernel

$$\rho_0(\boldsymbol{x}) = \widehat{\varphi}_{\varepsilon,0}(\boldsymbol{x})\int_{\mathbb{R}^n} k(t_0,\boldsymbol{x},t_1,\boldsymbol{y})\varphi_{\varepsilon,1}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

$$\rho_1(\boldsymbol{x}) = \varphi_{\varepsilon,1}(\boldsymbol{x})\int_{\mathbb{R}^n} k(t_0,\boldsymbol{y},t_1,\boldsymbol{x})\widehat{\varphi}_{\varepsilon,0}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

Coupled nonlinear integral equations

Here

$$k(t_0,\boldsymbol{x}_0,t_1,\boldsymbol{x}_1) := \frac{\exp\left(-\frac{\left(\boldsymbol{\Phi}_{t_1t_0}\boldsymbol{x}_0-\boldsymbol{x}_1\right)^{\top}\boldsymbol{M}_{10}^{-1}\left(\boldsymbol{\Phi}_{t_1t_0}\boldsymbol{x}_0-\boldsymbol{x}_1\right)}{4\varepsilon}\right)}{\sqrt{(4\pi\varepsilon)^n\det(\boldsymbol{M}_{10})}}$$

Contractive Fixed Point Algorithm

Fixed point recursion over pair $(\varphi_{\varepsilon,1}, \widehat{\varphi}_{\varepsilon,0})$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\widehat{\varphi}_{\varepsilon,0}(\cdot) \arrow[r, "\int", "\text{IVP forward in time}"'] & \widehat{\varphi}_{\varepsilon}(\cdot, t = t_1) \arrow[d, "\rho_1(\cdot)/\widehat{\varphi}_{\varepsilon}(\cdot, t = t_1)"] \\
\varphi_{\varepsilon}(\cdot, t = t_0) \arrow[u, "\rho_0(\cdot)/\varphi_{\varepsilon}(\cdot, t = t_0)"] & \varphi_{\varepsilon,1}(\cdot) \arrow[l, "\text{IVP backward in time}"', "\int"]
\end{tikzcd}
$$

Guaranteed linear convergence with contraction rate $\kappa \in (0,1)$

But exact rate depends on problem data $(\mathcal{X}_0, \mathcal{X}_1, \varepsilon, \boldsymbol{A}(t), \boldsymbol{B}(t))$

Worst case contraction coefficient $\gamma := \underset{\text{Linear SBPs with fixed } (\mathcal{X}_0,\mathcal{X}_1,\varepsilon,\boldsymbol{A}(t),\boldsymbol{B}(t))}{\sup} \kappa$

γ in Classical SBP

Let

$$\alpha_{\mathrm{B}} = \frac{\exp(-\tilde{\alpha}_{\mathrm{B}}/(4\varepsilon))}{\sqrt{(4\pi\varepsilon)^n}}, \quad \beta_{\mathrm{B}} = \frac{\exp\left(-\tilde{\beta}_{\mathrm{B}}/(4\varepsilon)\right)}{\sqrt{(4\pi\varepsilon)^n}}.$$

where

$$\tilde{\beta}_{\mathrm{B}} := \min_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} |\boldsymbol{x}_0 - \boldsymbol{x}_1|^2 \quad \text{and} \quad \tilde{\alpha}_{\mathrm{B}} := \max_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} |\boldsymbol{x}_0 - \boldsymbol{x}_1|^2$$

$$\boxed{\gamma_{\mathrm{B}} := \tanh^2\left(\frac{1}{2}\log\left(\frac{\beta_{\mathrm{B}}}{\alpha_{\mathrm{B}}}\right)\right) \in (0,1)}$$

Chen, Georgiou, Pavon, *SIAM J. Applied Math, 2016*

γ in Linear SBP

Thm. (informal)

State transition matrix **Controllability Gramian**

Let

$$\tilde{\alpha}_{\mathrm{L}} := \max_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)^\top \boldsymbol{M}_{10}^{-1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)$$

$$\tilde{\beta}_{\mathrm{L}} := \min_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)^\top \boldsymbol{M}_{10}^{-1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)$$

Then

$$\gamma_{\mathrm{L}} = \tanh^2 \left(\frac{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}}}{8\varepsilon} \right)$$

γ in Linear SBP

Thm. (informal)

Let

$$\tilde{\alpha}_{\mathrm{L}} := \max_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)^\top \boldsymbol{M}_{10}^{-1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)$$

$$\tilde{\beta}_{\mathrm{L}} := \min_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)^\top \boldsymbol{M}_{10}^{-1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)$$

Then

$$\gamma_{\mathrm{L}} = \tanh^2 \left(\frac{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}}}{8\varepsilon} \right)$$

Note:

$$\begin{aligned} \boldsymbol{A}(t) &\equiv \boldsymbol{0} \\ \boldsymbol{B}(t) &\equiv \boldsymbol{I} \end{aligned} \rightsquigarrow \begin{aligned} \boldsymbol{\Phi}_{t_1 t_0} &= \boldsymbol{I} \\ \boldsymbol{M}_{10} &= \frac{1}{t_1 - t_0} \boldsymbol{I} \end{aligned} \rightsquigarrow \begin{aligned} \tilde{\alpha}_{\mathrm{B}} &:= \max_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} \frac{1}{t_1 - t_0} |\boldsymbol{x}_0 - \boldsymbol{x}_1|^2 \\ \tilde{\beta}_{\mathrm{B}} &:= \min_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} \frac{1}{t_1 - t_0} |\boldsymbol{x}_0 - \boldsymbol{x}_1|^2 \end{aligned}$$

Control-theoretic Interpretation for γ_{L}

$$\tilde{\alpha}_{\mathrm{L}} := \max_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} (\boldsymbol{\Phi}_{t_1 t_0}\boldsymbol{x}_0 - \boldsymbol{x}_1)^\top \boldsymbol{M}_{10}^{-1} (\boldsymbol{\Phi}_{t_1 t_0}\boldsymbol{x}_0 - \boldsymbol{x}_1)$$

$$\tilde{\boldsymbol{\beta}}_{\mathrm{L}} := \min_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} (\boldsymbol{\Phi}_{t_1 t_0}\boldsymbol{x}_0 - \boldsymbol{x}_1)^\top \boldsymbol{M}_{10}^{-1} (\boldsymbol{\Phi}_{t_1 t_0}\boldsymbol{x}_0 - \boldsymbol{x}_1)$$

$$\begin{aligned} &\underset{\boldsymbol{v}}{\text{minimum}} \int_{t_0}^{t_1} \frac{1}{2} |\boldsymbol{v}|^2 dt \\ &\text{subject to} \quad \dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{v} \\ &\qquad\qquad\quad \boldsymbol{x}(t = t_0) = \boldsymbol{x}_0, \boldsymbol{x}(t = t_1) = \boldsymbol{x}_1 \end{aligned}$$

Minimum cost for deterministic OCP

Control-theoretic Interpretation for γ_{L}

Range of optimal state transfer cost

$$\gamma_{\mathrm{L}} = \tanh^2\left(\frac{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}}}{8\varepsilon}\right)$$

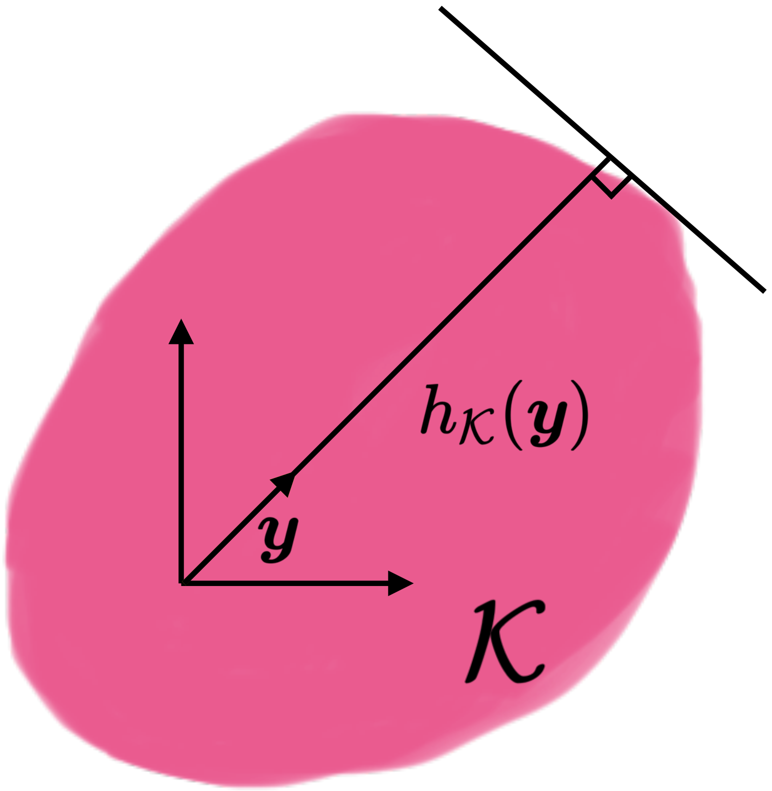
Process noise

Conforms with intuition:

$$\boxed{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}} \uparrow \quad \Rightarrow \quad \gamma_{\mathrm{L}} \uparrow}$$

$$\boxed{\varepsilon \uparrow \quad \Rightarrow \quad \gamma_{\mathrm{L}} \downarrow}$$

Support Functions



The support function $h_{\mathcal{K}}(\cdot)$ for closed convex set $\mathcal{K}$ is

$$h_{\mathcal{K}}(\boldsymbol{y}) := \sup_{\boldsymbol{x} \in \mathcal{K}} \langle \boldsymbol{y}, \boldsymbol{x} \rangle, \quad \boldsymbol{y} \in \mathbb{R}^n$$

e.g., distance from the origin to a supporting hyperplane of $\mathcal{K}$ with normal in direction of $\boldsymbol{y}$

γ in Linear SBP

$$\gamma_{\mathrm{L}} = \tanh^2\left(\frac{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}}}{8\varepsilon}\right)$$

Thm. (informal)

With support functions of $\mathcal{X}_0$ and $\mathcal{X}_1$, and Euclidean unit sphere $\mathcal{S}^{n-1}$

$$\tilde{\alpha}_{\mathrm{L}} = \{\max_{\boldsymbol{y}\in\mathcal{S}^{n-1}} (h_{\mathcal{X}_0}(\boldsymbol{\Phi}_{t_1 t_0}^{\top}\boldsymbol{M}_{10}^{-1/2}\boldsymbol{y}) + h_{\mathcal{X}_1}(-\boldsymbol{M}_{10}^{-1/2}\boldsymbol{y}))\}^2$$

$$\tilde{\beta}_{\mathrm{L}} = \{\min_{\boldsymbol{y}\in\mathcal{S}^{n-1}} (h_{\mathcal{X}_0}(\boldsymbol{\Phi}_{t_1 t_0}^{\top}\boldsymbol{M}_{10}^{-1/2}\boldsymbol{y}) + h_{\mathcal{X}_1}(-\boldsymbol{M}_{10}^{-1/2}\boldsymbol{y}))\}^2$$

γ in Linear SBP

$$\gamma_{\mathrm{L}} = \tanh^2\left(\frac{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}}}{8\varepsilon}\right)$$

Thm. (informal)

With support functions of $\mathcal{X}_0$ and $\mathcal{X}_1$, and Euclidean unit sphere $\mathcal{S}^{n-1}$

$$\tilde{\alpha}_{\mathrm{L}} = \{\max_{\boldsymbol{y}\in\mathcal{S}^{n-1}} (h_{\mathcal{X}_0}(\boldsymbol{\Phi}_{t_1t_0}^{\top}\boldsymbol{M}_{10}^{-1/2}\boldsymbol{y}) + h_{\mathcal{X}_1}(-\boldsymbol{M}_{10}^{-1/2}\boldsymbol{y}))\}^2$$
$$\tilde{\beta}_{\mathrm{L}} = \{\min_{\boldsymbol{y}\in\mathcal{S}^{n-1}} (h_{\mathcal{X}_0}(\boldsymbol{\Phi}_{t_1t_0}^{\top}\boldsymbol{M}_{10}^{-1/2}\boldsymbol{y}) + h_{\mathcal{X}_1}(-\boldsymbol{M}_{10}^{-1/2}\boldsymbol{y}))\}^2$$

Note:

$$\boldsymbol{\Phi}_{t_1t_0} = \boldsymbol{I}$$
$$\boldsymbol{M}_{10} = \frac{1}{t_1 - t_0}\boldsymbol{I}$$

$\rightsquigarrow$

$$\tilde{\alpha}_{\mathrm{B}} = \frac{1}{t_1 - t_0}\{\max_{\boldsymbol{y}\in\mathcal{S}^{n-1}} (h_{\mathcal{X}_0}(\boldsymbol{y}) + h_{\mathcal{X}_1}(-\boldsymbol{y})\}^2$$
$$\tilde{\alpha}_{\mathrm{B}} = \frac{1}{t_1 - t_0}\{\min_{\boldsymbol{y}\in\mathcal{S}^{n-1}} (h_{\mathcal{X}_0}(\boldsymbol{y}) + h_{\mathcal{X}_1}(-\boldsymbol{y})\}^2$$

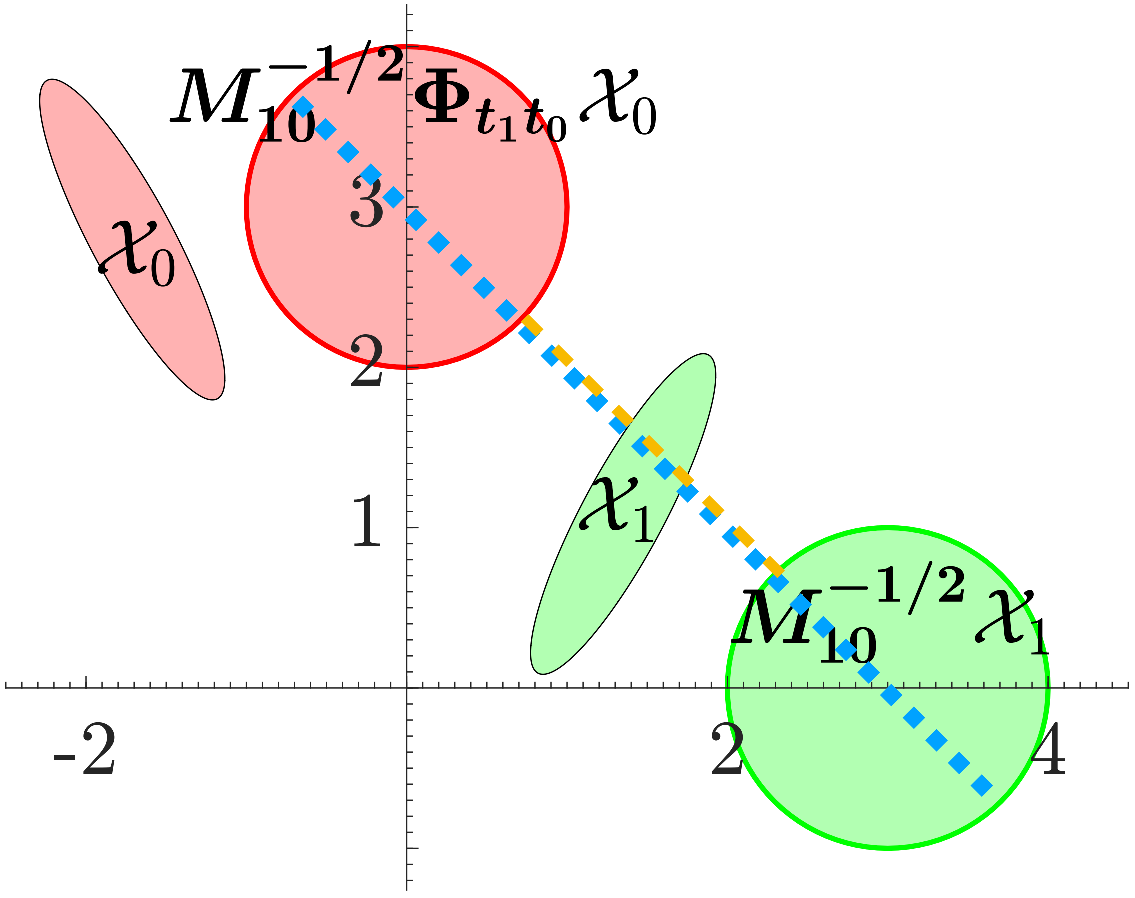
Geometric Interpretation for γ_{L}

$$\gamma_{\mathrm{L}} = \tanh^2\left(\frac{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}}}{8\varepsilon}\right)$$

Geometric interpretation:

$\tilde{\alpha}_{\mathrm{L}}$ and $\tilde{\beta}_{\mathrm{L}}$ can be considered the maximum and minimal separation of

$M_{10}^{-1/2}\mathbf{\Phi}_{t_1 t_0}\mathcal{X}_0$ and $M_{10}^{-1/2}\mathcal{X}_1$



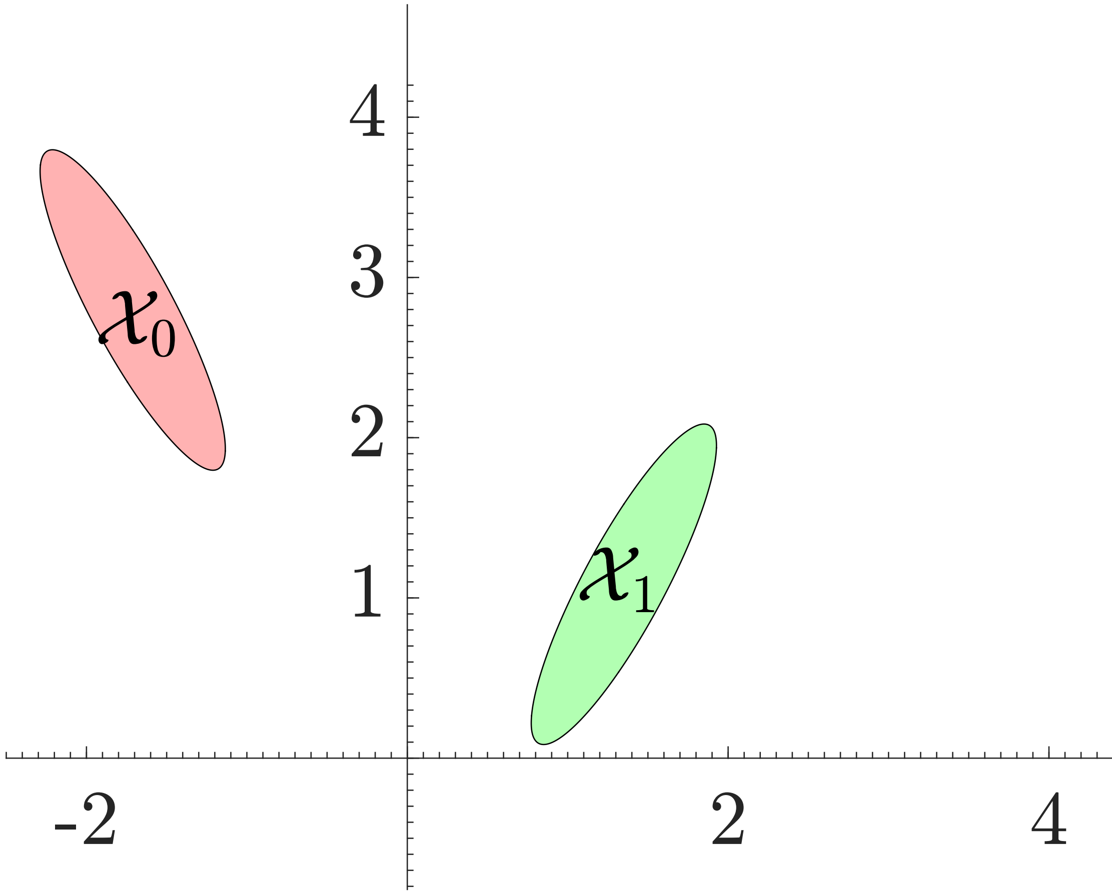
Applications to Preconditioning:

Preconditioning to improve optimal transport algorithms
~ Kuang and Tabak, *SIAM J. Scientific Computing*, 2017

Example: Linear SBP: $\varepsilon = 0.5$

$$d\boldsymbol{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}(t)dt + \sqrt{2\varepsilon}\begin{bmatrix} 0 \\ 1 \end{bmatrix} d\boldsymbol{w}(t)$$

$$\boldsymbol{\Phi}_{t_1 t_0} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{M}_{10}^{-1} = \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}.$$



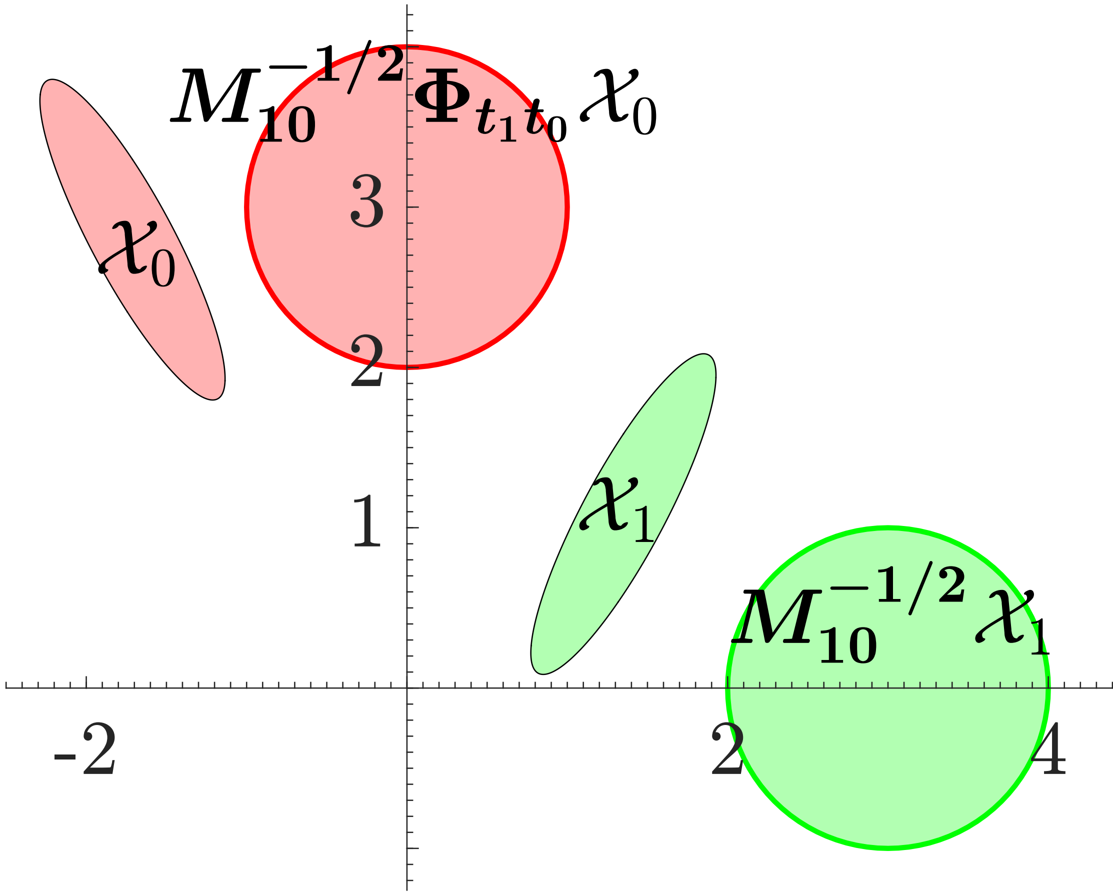
Applications to Preconditioning:

Preconditioning to improve optimal transport algorithms
~ Kuang and Tabak, *SIAM J. Scientific Computing*, 2017

Example: Linear SBP: $\varepsilon = 0.5$

$$d\boldsymbol{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}(t)dt + \sqrt{2\varepsilon} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\boldsymbol{w}(t)$$

$$\boldsymbol{\Phi}_{t_1 t_0} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{M}_{10}^{-1} = \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}.$$



No Preconditioning:

$$\tilde{\alpha}_{\mathrm{L}} = 2 + 2\sqrt{3}$$
$$\tilde{\beta}_{\mathrm{L}} = -2 + 2\sqrt{3}$$
$$\longrightarrow \quad \gamma_{\mathrm{L}} = \tanh^2(1) \approx 0.580$$

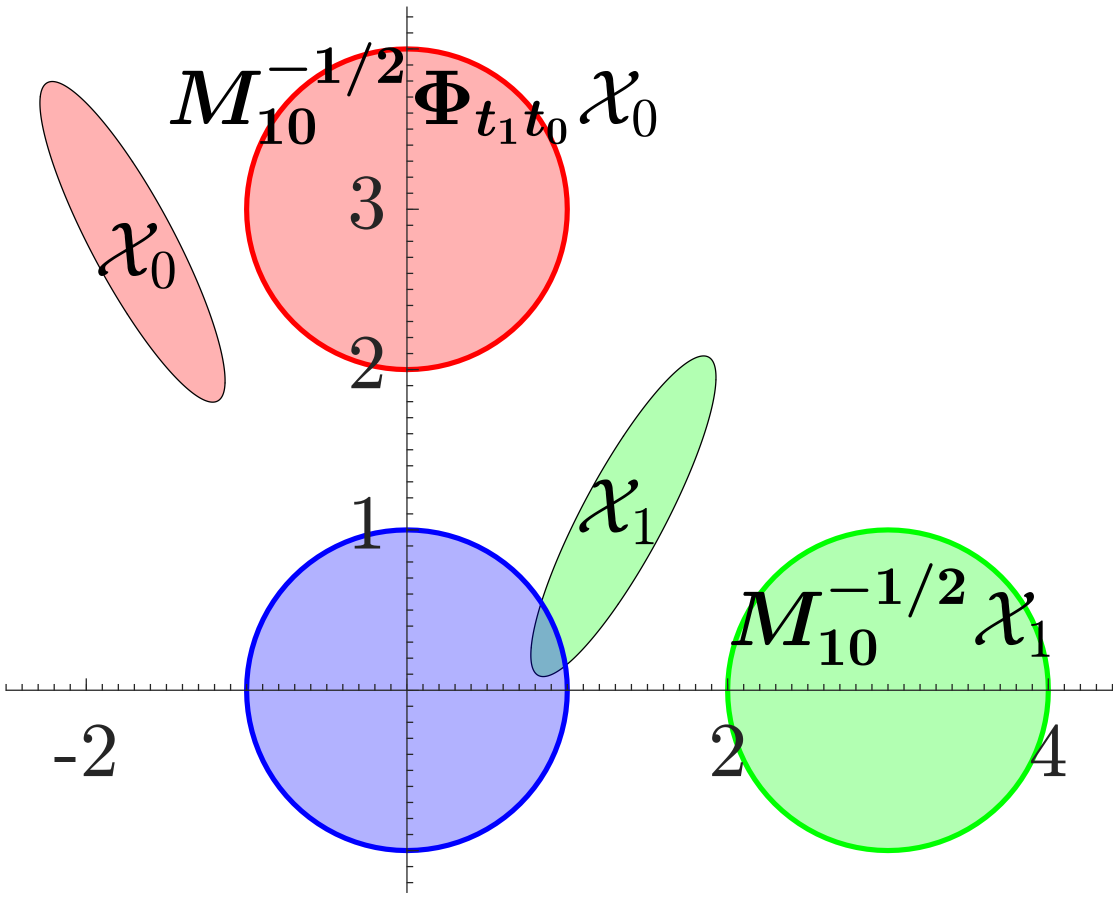
Applications to Preconditioning:

Preconditioning to improve optimal transport algorithms
~ Kuang and Tabak, *SIAM J. Scientific Computing*, 2017

Example: Linear SBP: $\varepsilon = 0.5$

$$d\boldsymbol{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}(t)dt + \sqrt{2\varepsilon}\begin{bmatrix} 0 \\ 1 \end{bmatrix} d\boldsymbol{w}(t)$$

$$\boldsymbol{\Phi}_{t_1 t_0} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{M}_{10}^{-1} = \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}.$$



With Preconditioning:

$$\tilde{\alpha}_{\mathrm{L}}^{\mathrm{precond}} = 2, \; \tilde{\beta}_{\mathrm{L}}^{\mathrm{precond}} = 0 \longrightarrow \gamma_{\mathrm{L}}^{\mathrm{precond}} = \tanh^2(0.5) = 0.214$$

SBP with State Cost

Related works

Dawson, D., Gorostiza, L., and Wakolbinger, A., "Schrödinger processes and large deviations," Journal of mathematical physics, Vol. 31, No. 10, 1990, pp. 2385–2388. https://doi.org/10.1063/1.528840

Aebi, R., and Nagasawa, M., "Large deviations and the propagation of chaos for Schrödinger processes," Probability Theory and Related Fields, Vol. 94, No. 1, 1992, pp. 53–68. https://doi.org/10.1007/BF01222509

SBP with State Cost

$$\arg\inf_{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}} \quad \int_{t_0}^{t_1}\int_{\mathbb{R}^n}\left(\frac{1}{2}|\boldsymbol{v}|^2 + q(\boldsymbol{x})\right)\rho(\boldsymbol{x},t)\, d\boldsymbol{x}\, dt$$

$$\frac{\partial\rho}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v}) = \varepsilon\Delta_{\boldsymbol{x}}\rho$$

$$\boldsymbol{x}(t=t_0)\sim\rho_0\ (\text{given}), \quad \boldsymbol{x}(t=t_1)\sim\rho_1\ (\text{given})$$

Controlled sample path dynamics

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x},t)\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}\boldsymbol{w}(t)$$

Solution for the SBP with State Cost

Thm. (informal)

SBP with state cost admits a unique solution

Proof idea:

Reformulate as Kullback-Leibler minimization over path space:

$$\underbrace{\arg\inf_{\mathbb{P}\in\Pi_{01}} \quad D_{\mathrm{KL}}\left(\mathbb{P} \,\Big\|\, \frac{\exp\left(-\frac{1}{2\varepsilon}\int_{t_0}^{t_1} q(\boldsymbol{x})dt\right)\mathbb{W}}{Z}\right)}_{\text{large deviation principle}}$$

Conditions for Optimality

Necessary conditions of optimality for the SBP with state cost

The pair $\left(\rho_{\varepsilon}^{\text{opt}}, \boldsymbol{v}_{\varepsilon}^{\text{opt}}\right)$ solves the coupled nonlinear PDEs

$$\frac{\partial \psi_{\varepsilon}}{\partial t} + \frac{1}{2}|\nabla_{\boldsymbol{x}}\psi_{\varepsilon}|^{2} + \varepsilon\Delta_{\boldsymbol{x}}\psi_{\varepsilon} = q(\boldsymbol{x})$$

$$\frac{\partial \rho_{\varepsilon}^{\text{opt}}}{\partial t} + \nabla_{\boldsymbol{x}}\cdot\left(\rho_{\varepsilon}^{\text{opt}}\nabla_{\boldsymbol{x}}\psi_{\varepsilon}\right) = \varepsilon\Delta_{\boldsymbol{x}}\rho_{\varepsilon}^{\text{opt}}$$

with boundary conditions

$$\rho_{\varepsilon}^{\text{opt}}\left(\boldsymbol{x}, t = t_0\right) = \rho_0(\boldsymbol{x})$$

$$\rho_{\varepsilon}^{\text{opt}}(\boldsymbol{x}, t = t_1) = \rho_1(\boldsymbol{x})$$

Structure of the solution for SBP with State Cost

Boundary-coupled system of linear PDEs for the Schrödinger factors

Reaction-diffusion PDEs

$$\frac{\partial \widehat{\varphi}_\varepsilon}{\partial t} = \left(\varepsilon \Delta_{\boldsymbol{x}} - \frac{1}{2\varepsilon} q(\boldsymbol{x})\right) \widehat{\varphi}_\varepsilon \leftarrow \mathcal{L}_{\text{forward}} \widehat{\varphi}$$

$$\frac{\partial \varphi_\varepsilon}{\partial t} = \left(-\varepsilon \Delta_{\boldsymbol{x}} + \frac{1}{2\varepsilon} q(\boldsymbol{x})\right) \varphi_\varepsilon \leftarrow \mathcal{L}_{\text{backward}} \varphi$$

$$\widehat{\varphi}_\varepsilon(\cdot, t = t_0) \varphi_\varepsilon(\cdot, t = t_0) = \rho_0$$

$$\widehat{\varphi}_\varepsilon(\cdot, t = t_1) \varphi_\varepsilon(\cdot, t = t_1) = \rho_1.$$

Optimally controlled joint state PDF

$$\rho_\varepsilon^{opt}(\cdot, t) = \hat{\varphi}_\varepsilon(\cdot, t) \varphi_\varepsilon(\cdot, t)$$

Optimal control

$$\boldsymbol{v}_\varepsilon^{\text{opt}}(\cdot, t) = 2\varepsilon \nabla_{\boldsymbol{x}} \log \varphi_\varepsilon(\cdot, t)$$

Algorithm

Fixed point recursion over pair $\left(\varphi_{\varepsilon,1}, \widehat{\varphi}_{\varepsilon,0}\right)$

$$\begin{tikzcd}
\widehat{\varphi}_{\varepsilon,0}(\cdot) \arrow[r, "\int \mathcal{L}_{\text{forward}} \widehat{\varphi}"] & \widehat{\varphi}_{\varepsilon}(\cdot, t = t_1) \arrow[d, "\rho_1(\cdot)/\widehat{\varphi}_{\varepsilon}(\cdot, t = t_1)"] \\
\varphi_{\varepsilon}(\cdot, t = t_0) \arrow[u, "\rho_0(\cdot)/\varphi_{\varepsilon}(\cdot, t = t_0)"] & \varphi_{\varepsilon,1}(\cdot) \arrow[l, "\int \mathcal{L}_{\text{backward}} \varphi"]
\end{tikzcd}$$

Schrödinger system:

$$\rho_0(\boldsymbol{x}) = \widehat{\varphi}_{\varepsilon,0}(\boldsymbol{x}) \int_{\mathbb{R}^n} k(t_0, \boldsymbol{x}, t_1, \boldsymbol{y}) \varphi_{\varepsilon,1}(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}$$

$$\rho_1(\boldsymbol{x}) = \varphi_{\varepsilon,1}(\boldsymbol{x}) \int_{\mathbb{R}^n} k(t_0, \boldsymbol{y}, t_1, \boldsymbol{x}) \widehat{\varphi}_{\varepsilon,0}(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}$$

Fredholm Integral Equation of 2nd Kind

Thm. (informal)

Solution of linear reaction-diffusion PDE IVP with state-dependent reaction rate:

$$\frac{\partial u}{\partial t} = a\Delta_{\boldsymbol{x}} u + q(\boldsymbol{x})u, \quad \boldsymbol{x} \in \mathbb{R}^n, \quad u(\boldsymbol{x}, t = t_0) = u_0(\boldsymbol{x}) \text{ given}$$

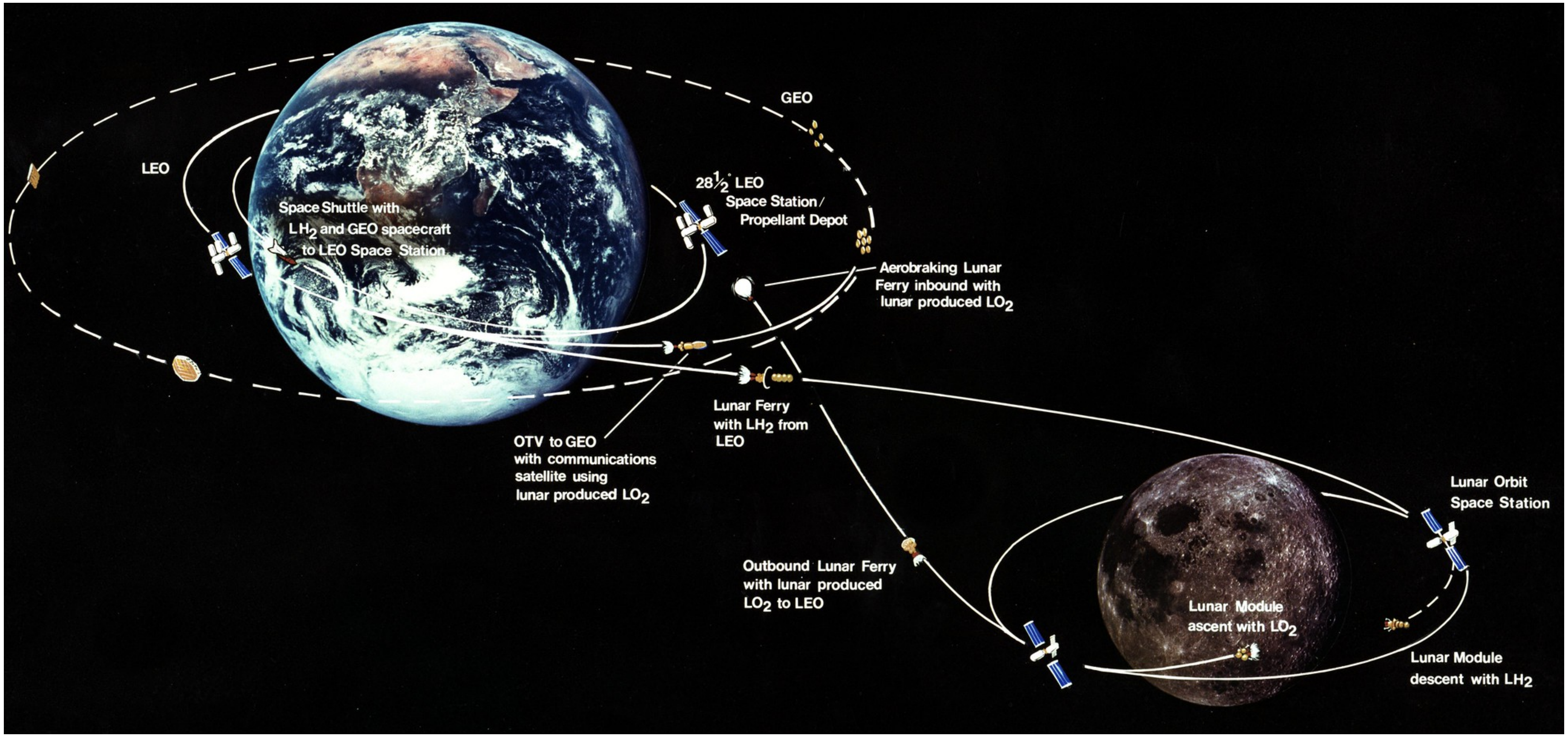
admits space-time Fredholm integral representation

$$u(\boldsymbol{x}, t) = \underbrace{\frac{1}{\sqrt{(4\pi a t)^n}} \int_{\mathbb{R}^n} \exp\left(-\frac{|\boldsymbol{x} - \boldsymbol{y}|^2}{4at}\right) u_0(\boldsymbol{y}) d\boldsymbol{y}}_{\text{term 1}}$$

$$+ \underbrace{\int_{t_0}^{t} \frac{1}{\sqrt{(4\pi a(t-\tau))^n}} \int_{\mathbb{R}^n} \exp\left(-\frac{|\boldsymbol{x} - \boldsymbol{y}|^2}{4a(t-\tau)}\right) q(\boldsymbol{y}) u(\boldsymbol{y}, \tau)\, d\boldsymbol{y}\, d\tau}_{\text{term 2}}$$

Case Study
Probabilistic Lambert's Problem

Lambert's Problem

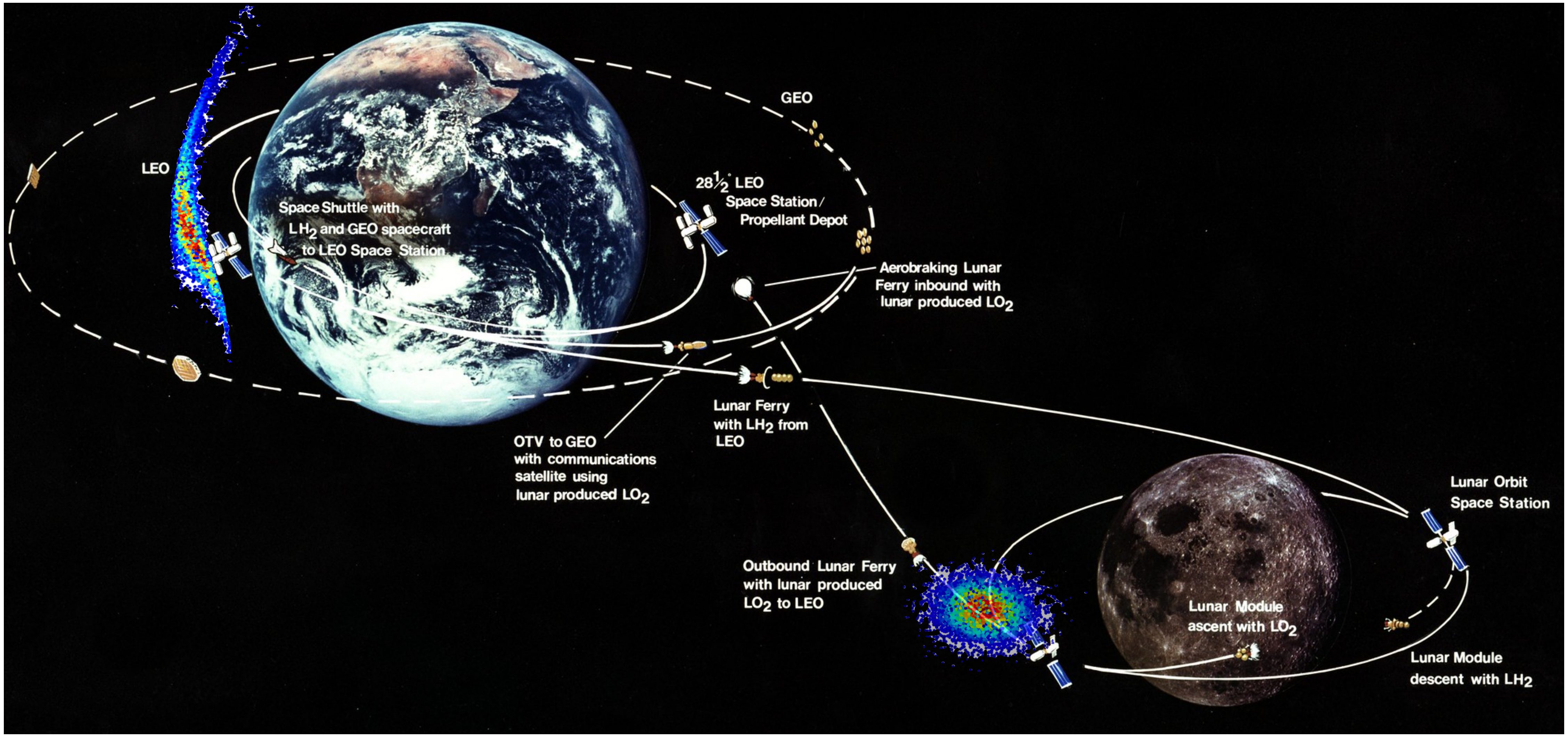


3D position coordinate $\boldsymbol{x} := \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

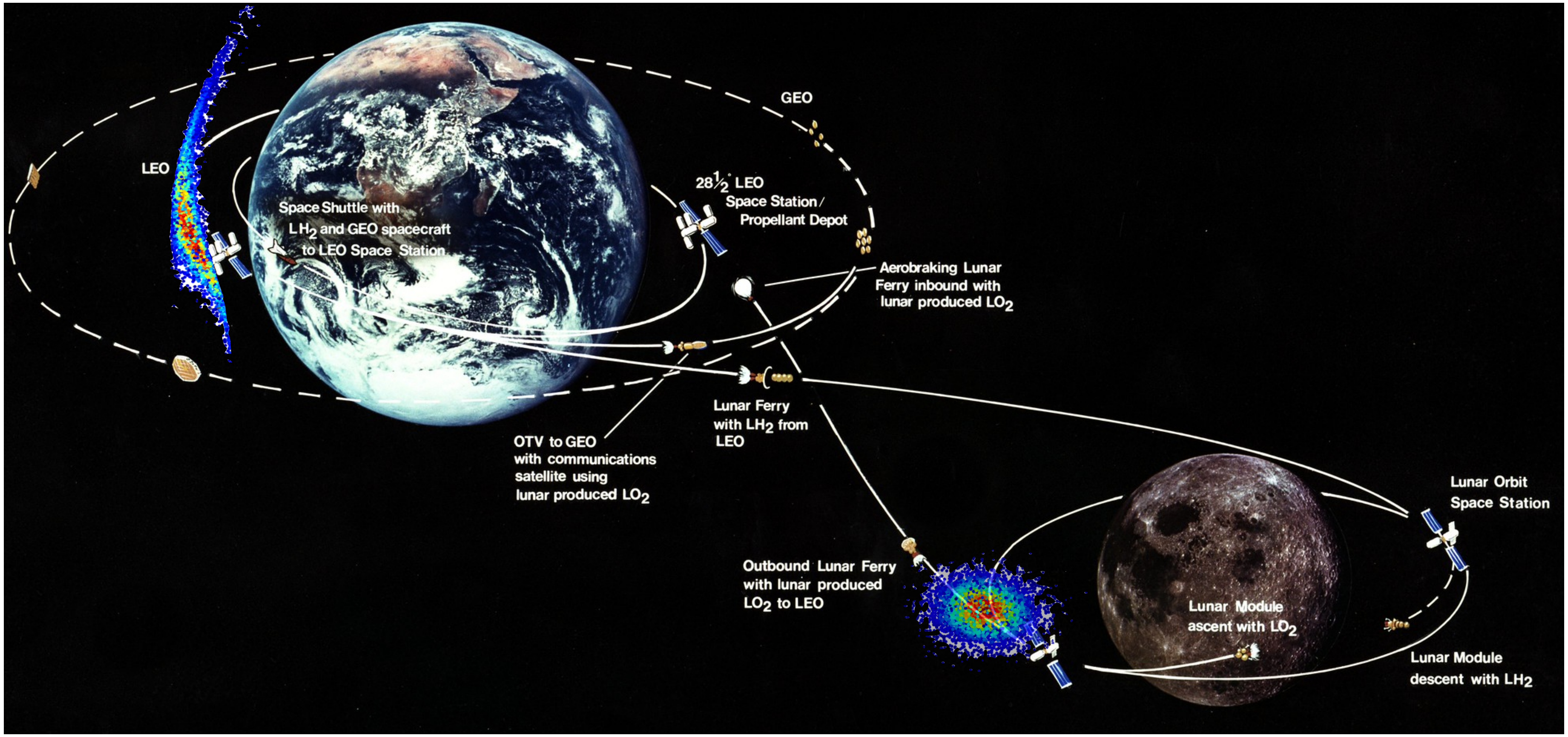
Find velocity control policy $\dot{\boldsymbol{x}} := \boldsymbol{v}(\boldsymbol{r}, t)$ such that

$$\ddot{\boldsymbol{x}} = -\nabla_{\boldsymbol{x}} V(\boldsymbol{x}), \quad \boxed{\boldsymbol{x}(t = t_0) = \boldsymbol{x}_0 \text{ (given)}, \quad \boldsymbol{x}(t = t_1) = \boldsymbol{x}_1 \text{ (given)}}$$

Probabilistic Lambert's Problem



Probabilistic Lambert's Problem



3D position coordinate $\boldsymbol{x} := \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

Find velocity control policy $\dot{\boldsymbol{x}} := \boldsymbol{v}(\boldsymbol{r}, t)$ such that

$$\ddot{\boldsymbol{x}} = -\nabla_{\boldsymbol{x}} V(\boldsymbol{x}), \quad \boxed{\boldsymbol{x}(t = t_0) \sim \rho_0 \,(\text{given}), \quad \boldsymbol{x}(t = t_1) \sim \rho_1 \,(\text{given})}$$

Connection with OMT

Lambert Problem $\Leftrightarrow$ Deterministic OCP

Reformulate Lambert's problem as deterministic OCP
[Bando and Yamakawa, *JGCD*, 2010]

$$\ddot{\boldsymbol{x}} = -\nabla_{\boldsymbol{x}} V(\boldsymbol{x}), \quad \boldsymbol{x}(t = t_0) = \boldsymbol{x}_0 \text{ (given)}, \quad \boldsymbol{x}(t = t_1) = \boldsymbol{x}_1 \text{ (given)}$$

$$\Updownarrow$$

$$\arg\inf_{\boldsymbol{v}} \int_{t_0}^{t_1} \left(\frac{1}{2} |\boldsymbol{v}|^2 - V(\boldsymbol{x}) \right) dt$$

$$\dot{\boldsymbol{x}} = \boldsymbol{v}$$

Potential as state cost

$$\boldsymbol{x}(t = t_0) = \boldsymbol{x}_0 \text{ (given)}, \quad \boldsymbol{x}(t = t_1) = \boldsymbol{x}_1 \text{ (given)}$$

Lambertian OMT (L-OMT)

Probabilistic Lambert's Problem $\Leftrightarrow$ Generalized OMT

$$\ddot{\boldsymbol{x}} = -\nabla_{\boldsymbol{x}} V(\boldsymbol{x}), \quad \boldsymbol{x}(t = t_0) \sim \rho_0 \text{ (given)}, \quad \boldsymbol{x}(t = t_1) \sim \rho_1 \text{ (given)}$$

$$\Updownarrow$$

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf} \int_{t_0}^{t_1} \int_{\mathbb{R}^n} \left(\frac{1}{2} |\boldsymbol{v}|^2 \underbrace{- V(\boldsymbol{x})}_{V = 0 \text{ is OMT}} \right) \rho(\boldsymbol{x}, t) \, d\boldsymbol{x} dt$$

$$\dot{\boldsymbol{x}} = \boldsymbol{v}$$

$$\boldsymbol{x}(t = t_0) = \boldsymbol{x}_0 \text{ (given)}, \quad \boldsymbol{x}(t = t_1) = \boldsymbol{x}_1 \text{ (given)}$$

Existence and Uniqueness of Solution for L-OMT

Thm. (informal)

L-OMT with negative potential admits a unique solution

Proof Idea:

Consider Lagrangian for L-OMT problem

Show that the Lagrangian is strictly convex and superlinear in $\boldsymbol{v}$

Use Figalli's theory for Tonelli Lagrangians induced by action integrals

Connection to SBP with state cost

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\left(\frac{1}{2}|\boldsymbol{v}|^2 - V(\boldsymbol{x})\right)\rho(\boldsymbol{x},t)\,d\boldsymbol{x}dt$$

$$\frac{\partial\rho}{\partial t}+\nabla_{\boldsymbol{r}}\cdot(\rho\boldsymbol{v})=0,$$

$$\rho(t=t_0,\cdot)=\rho_0,\quad \rho(t=t_1,\cdot)=\rho_1$$

Lambertian SBP (L-SBP)

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\left(\frac{1}{2}|\boldsymbol{v}|^2 - V(\boldsymbol{x})\right)\rho(\boldsymbol{x},t)\,d\boldsymbol{x}dt$$

Regularization > 0

$$\frac{\partial\rho}{\partial t}+\nabla_{\boldsymbol{r}}\cdot(\rho\boldsymbol{v})=\varepsilon\Delta_{\boldsymbol{r}}\rho,$$

$$\rho(t=t_0,\cdot)=\rho_0,\quad \rho(t=t_1,\cdot)=\rho_1$$

L-SBP Solution

Thm. (informal) Existence and uniqueness of L-SBP is guaranteed

$$V(\boldsymbol{x}) = -\frac{\mu}{|\boldsymbol{x}|}\left(1 + \frac{J_2 R_{\text{Earth}}^2}{2|\boldsymbol{x}|^2}\left(1 - \frac{3z^2}{|\boldsymbol{x}|^2}\right)\right) \longrightarrow$$

Bounded and negative for $|\boldsymbol{x}|^2 \geq \mathrm{R}_{\text{Earth}}^2$

Thm. (Necessary conditions of optimality for L-SBP)

$$\frac{\partial \psi_\varepsilon}{\partial t} + \frac{1}{2}|\nabla_{\boldsymbol{x}}\psi_\varepsilon|^2 + \varepsilon\Delta_{\boldsymbol{x}}\psi_\varepsilon = -V(\boldsymbol{x})$$

$$\frac{\partial \rho_\varepsilon^{\text{opt}}}{\partial t} + \nabla_{\boldsymbol{x}} \cdot \left(\rho_\varepsilon^{\text{opt}}\nabla_{\boldsymbol{x}}\psi_\varepsilon\right) = \varepsilon\Delta_{\boldsymbol{x}}\rho_\varepsilon^{\text{opt}}$$

$$\rho_\varepsilon^{\text{opt}}(t = t_0, \cdot) = \rho_0, \quad \rho_\varepsilon^{\text{opt}}(t = t_1, \cdot) = \rho_1$$

L-SBP Computation via Schrödinger Factors

Recursion over pair $\left(\varphi_1, \hat{\varphi}_0\right)$

$$
\begin{tikzcd}
\widehat{\varphi}_{\varepsilon,0}(\cdot) \arrow[r, "\int"] & \widehat{\varphi}_{\varepsilon}(\cdot, t = t_1) \arrow[d, "\rho_1(\cdot)/\widehat{\varphi}_{\varepsilon}(\cdot, t = t_1)"] \\
\varphi_{\varepsilon}(\cdot, t = t_0) \arrow[u, "\rho_0(\cdot)/\varphi_{\varepsilon}(\cdot, t = t_0)"] & \varphi_{\varepsilon,1}(\cdot) \arrow[l, "\int"]
\end{tikzcd}
$$

$$\frac{\partial \widehat{\varphi}_{\varepsilon}}{\partial t} = \left(\varepsilon \Delta_{\boldsymbol{x}} + \frac{1}{2\varepsilon} V(\boldsymbol{x})\right) \widehat{\varphi}_{\varepsilon}$$

$$\frac{\partial \varphi_{\varepsilon}}{\partial t} = -\left(\varepsilon \Delta_{\boldsymbol{x}} + \frac{1}{2\varepsilon} V(\boldsymbol{x})\right) \varphi_{\varepsilon}$$

$$\rho_{\varepsilon}^{\text{opt}}(t = t_0, \cdot) = \rho_0, \quad \rho_{\varepsilon}^{\text{opt}}(t = t_1, \cdot) = \rho_1$$

L-SBP Computation via Schrödinger Factors

Recursion over pair $\left(\varphi_1, \hat{\varphi}_0\right)$

$$\begin{tikzcd}
\widehat{\varphi}_{\varepsilon,0}(\cdot) \arrow[r, "\int"] & \widehat{\varphi}_{\varepsilon}(\cdot, t = t_1) \arrow[d, "\rho_1(\cdot)/\widehat{\varphi}_{\varepsilon}(\cdot, t=t_1)"] \\
\varphi_{\varepsilon}(\cdot, t = t_0) \arrow[u, "\rho_0(\cdot)/\varphi_{\varepsilon}(\cdot, t=t_0)"] & \varphi_{\varepsilon,1}(\cdot) \arrow[l, "\int"']
\end{tikzcd}$$

Thm. (Fredholm Integral Representation)

$$\widehat{\varphi}_{\varepsilon}(\boldsymbol{x}, t) = \frac{1}{\sqrt{(4\pi\varepsilon t)^3}} \int_{\mathbb{R}^3} \exp\left(-\frac{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|^2}{4\varepsilon t}\right) \widehat{\varphi}_{\varepsilon,0}(\tilde{\boldsymbol{x}})\, d\tilde{\boldsymbol{x}}$$
$$- \int_{t_0}^{t} \frac{1}{2\varepsilon\sqrt{(4\pi\varepsilon(t-\tau))^3}} \int_{\mathbb{R}^3} \exp\left(-\frac{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|^2}{4\varepsilon(t-\tau)}\right) V(\tilde{\boldsymbol{x}}) \widehat{\varphi}_{\varepsilon}(\tilde{\boldsymbol{x}}, \tau)\, d\tilde{\boldsymbol{x}}\, d\tau$$

Numerical Case Study

Prescribed time horizon $[t_0, t_1] \equiv [0,1]$ hours

Endpoint joint PDFs

$$\boldsymbol{x}_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$$
$$\boldsymbol{x}_1 \sim \mathcal{N}(\mu_1, \Sigma_1)$$

where

$$\mu_0 = \begin{pmatrix} 5000 \\ 10000 \\ 2100 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} -14600 \\ 2500 \\ 7000 \end{pmatrix}$$

$$\Sigma_0 = \frac{1}{100}\mathrm{diag}(\mu_0^2), \quad \Sigma_1 = \frac{1}{100}\mathrm{diag}(\mu_1^2),$$

Solution: Computation

IDEA: Fixed point recursion over pair $\left(\varphi_1, \hat{\varphi}_0\right)$

$$\begin{tikzcd}
\widehat{\varphi}_{\varepsilon,0}(\cdot) \arrow[r, "\int"] & \widehat{\varphi}_{\varepsilon}(\cdot, t=t_1) \arrow[d, "\rho_1(\cdot)/\widehat{\varphi}_{\varepsilon}(\cdot,t=t_1)"] \\
\varphi_{\varepsilon}(\cdot, t=t_0) \arrow[u, "\rho_0(\cdot)/\varphi_{\varepsilon}(\cdot,t=t_0)"] & \varphi_{\varepsilon,1}(\cdot) \arrow[l, "\int"]
\end{tikzcd}$$

Idea:
Left Riemann
Approximation
of Second Term

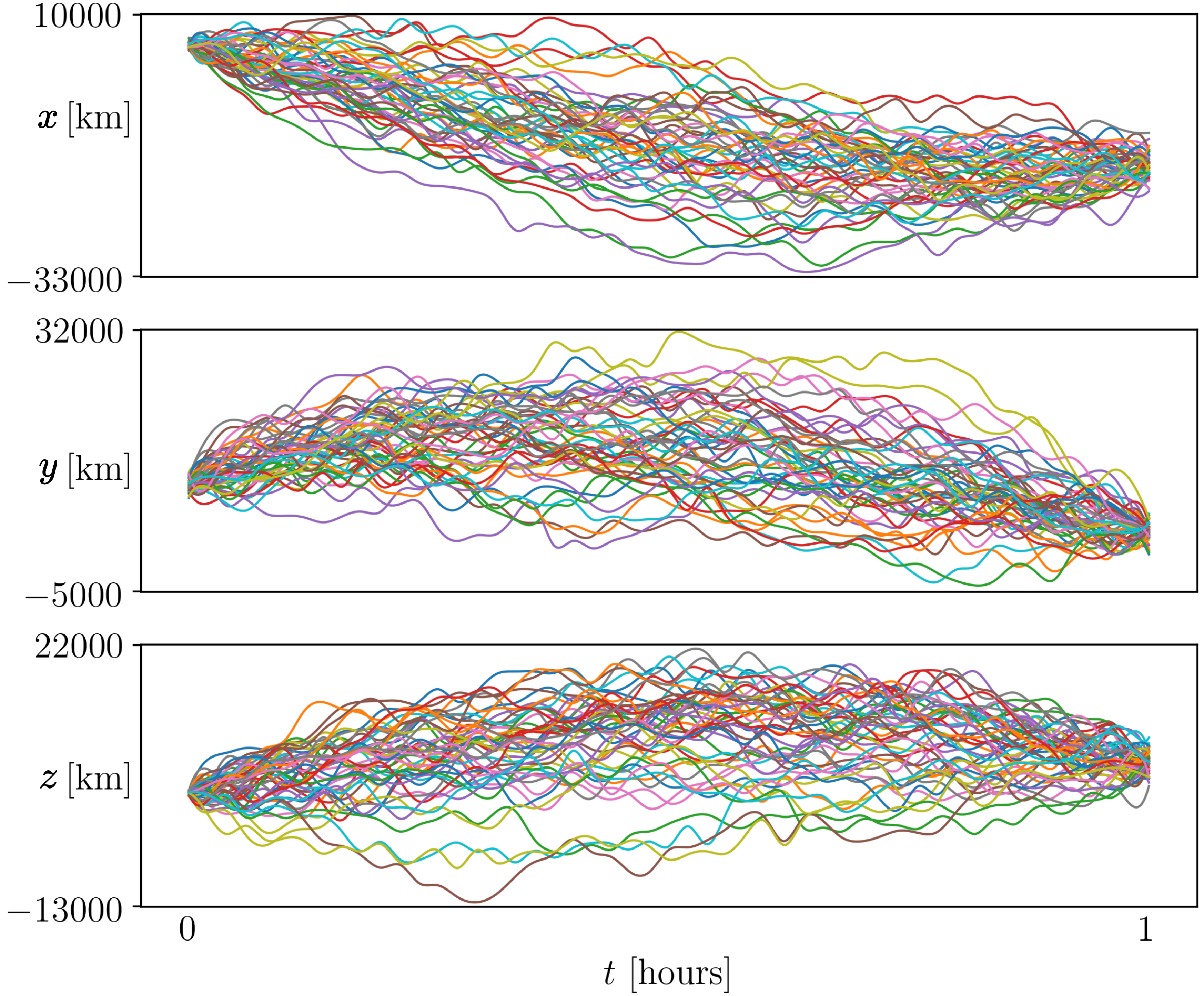
$$\int_{t_0}^{t_1}\int_{\mathbb{R}^n} f(\widetilde{\boldsymbol{x}}, \boldsymbol{x}, \tau, t)\, d\widetilde{\boldsymbol{x}}\, d\tau$$

$$\approx \sum_{q=0}^{k-1}\sum_{m=0}^{N_x}\sum_{n=0}^{N_y}\sum_{j=0}^{N_z} f(\widetilde{\boldsymbol{x}}_{(m,n,j)}, \boldsymbol{x}, t_0 + k\Delta t, t)\Delta z \Delta y \Delta x \Delta t$$

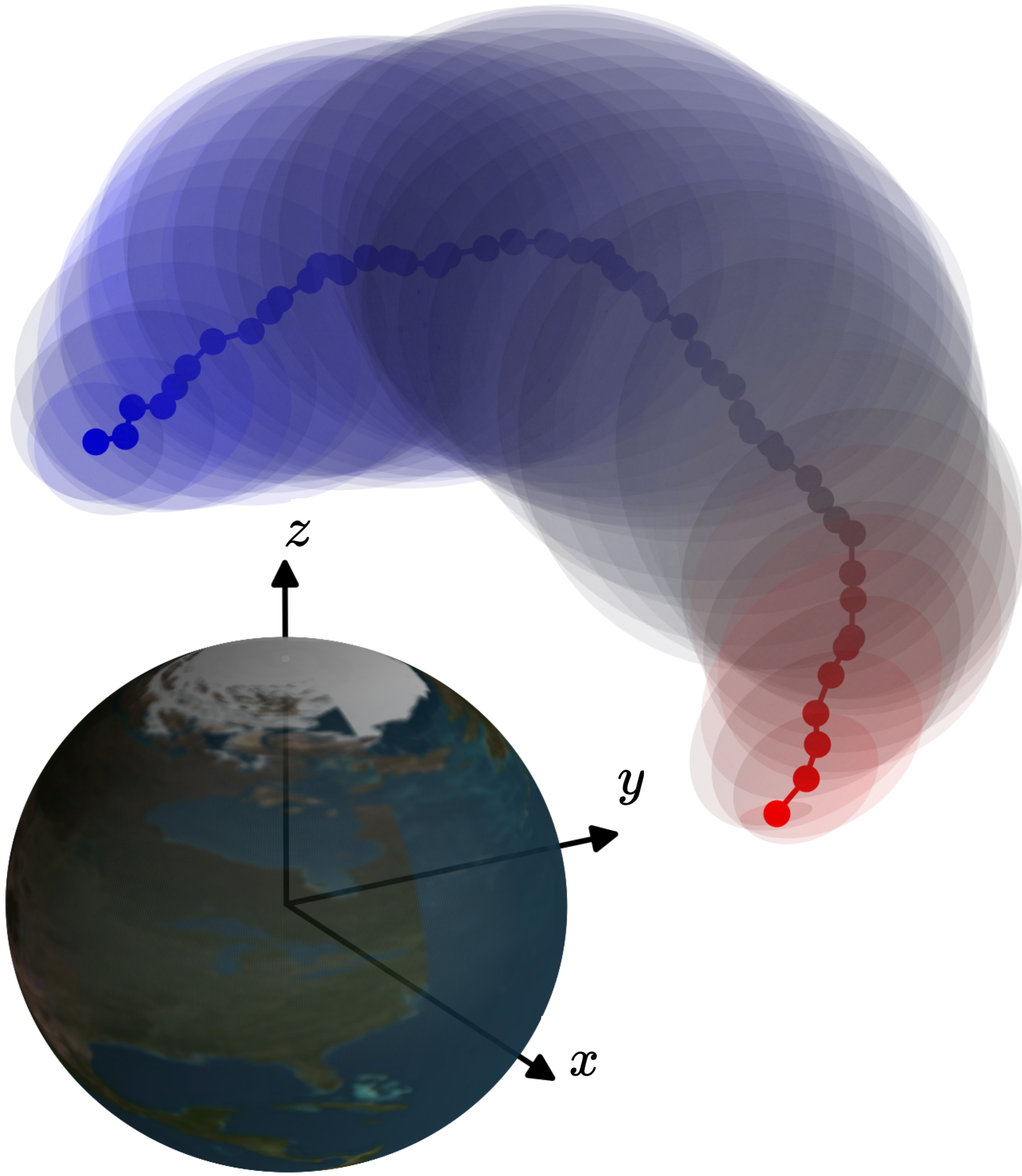
where $\widetilde{\boldsymbol{x}}_{(m,n,j)} = (x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$

Numerical Case Study (cont.)

Optimally controlled closed loop state sample paths

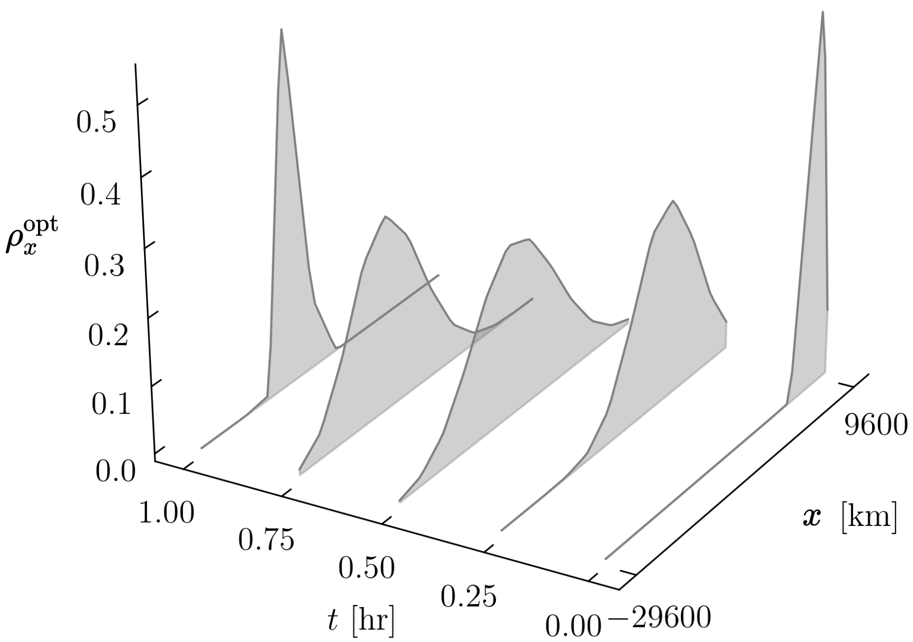


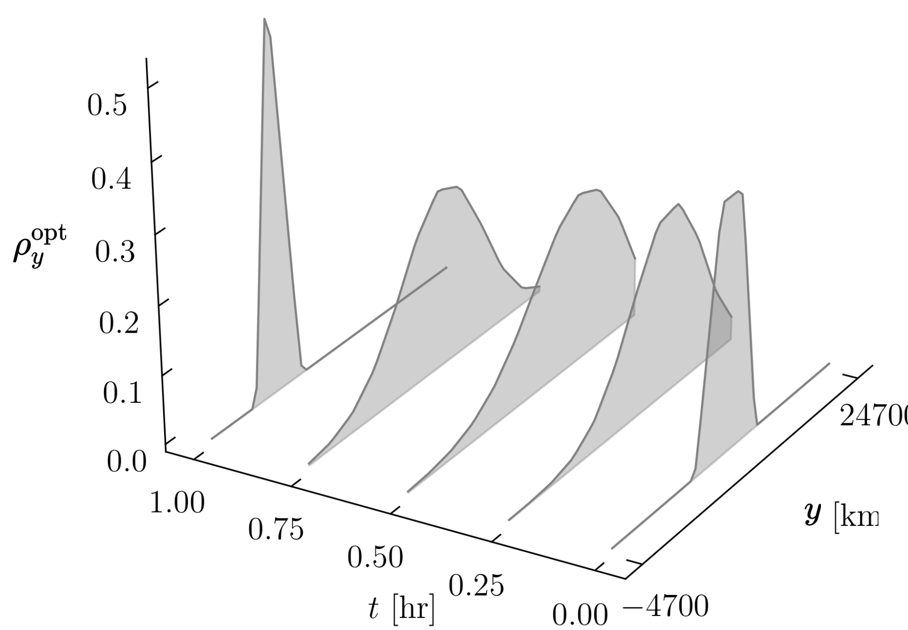
Numerical Case Study (cont.)

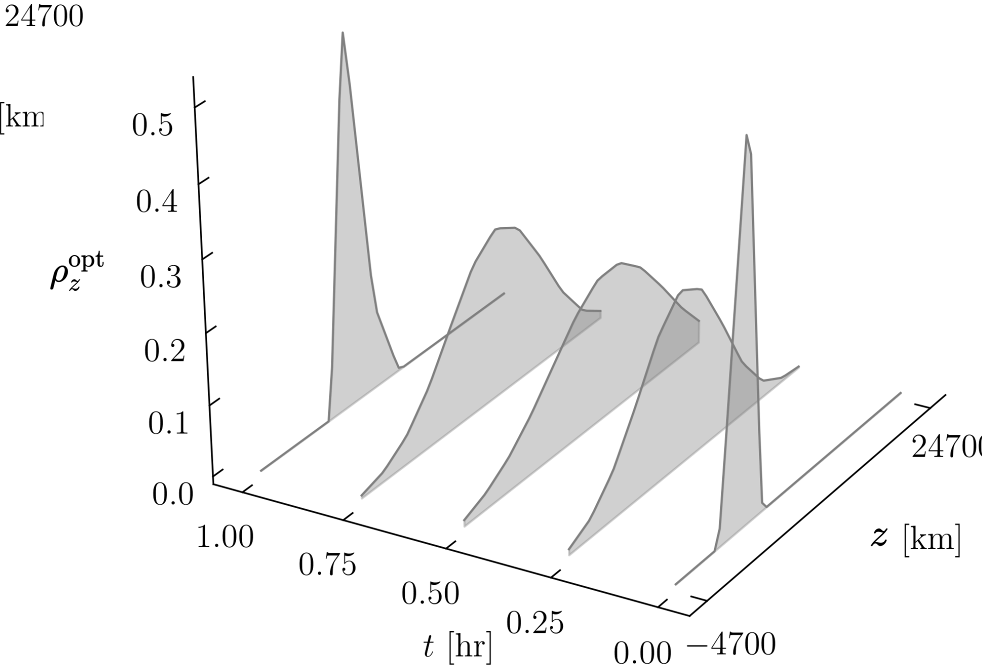


Numerical Case Study (cont.)

Univariate marginals for optimally controlled joint PDFs







Tentative Timeline for Research

Winter-Spring 2024: Further investigation of convergence guarantees for reaction-diffusion PDEs associated with SBPs with additive state costs.

Summer-Fall 2024: Deriving conditions for optimality of generalized SBPs.

Winter-Spring 2025: Publishing results, writing my dissertation.

Summer 2025: Ph.D. defense.

Publications

Alexis M.H. Teter, Yongxin Chen, Abhishek Halder
"On the contraction coefficient of the Schrödinger bridge for stochastic linear systems"
IEEE Control Systems Letters, Vol. 7, pp. 3325–3330, 2023
doi: 10.1109/LCSYS.2023.3326836
(also accepted for presentation at the *2024 American Control Conference*)

Alexis M.H. Teter, Iman Nodozi, Abhishek Halder
"Probabilistic Lambert Problem: Connections with Optimal Transport, Schrödinger Bridge and Reaction-Diffusion PDEs"
under review
arXiv:2401.07961, 2024

Alexis M.H. Teter, Iman Nodozi, Abhishek Halder
"Proximal mean field learning in shallow neural networks"
Transactions on Machine Learning Research, 2024
URL: https://openreview.net/forum?id=vyRBsqj5iG

Thank You

Backup Slides

γ in Linear SBP

Thm. (informal)

$$\tilde{\alpha}_{\mathrm{L}} = \{\max_{\boldsymbol{y}\in\mathcal{S}^{n-1}} (h_{\mathcal{X}_0}(\boldsymbol{\Phi}_{t_1t_0}^{\top}\boldsymbol{M}_{10}^{-1/2}\boldsymbol{y}) + h_{\mathcal{X}_1}(-\boldsymbol{M}_{10}^{-1/2}\boldsymbol{y}))\}^2$$

$$\tilde{\beta}_{\mathrm{L}} = \{\min_{\boldsymbol{y}\in\mathcal{S}^{n-1}} (h_{\mathcal{X}_0}(\boldsymbol{\Phi}_{t_1t_0}^{\top}\boldsymbol{M}_{10}^{-1/2}\boldsymbol{y}) + h_{\mathcal{X}_1}(-\boldsymbol{M}_{10}^{-1/2}\boldsymbol{y}))\}^2$$

Proof idea:

$$\tilde{\alpha}_{\mathrm{L}} = \max_{\boldsymbol{x}_0\in\mathcal{X}_0,\boldsymbol{x}_1\in\mathcal{X}_1} (\boldsymbol{\Phi}_{t_1t_0}\boldsymbol{x}_0 - \boldsymbol{x}_1)^{\top}\boldsymbol{M}_{10}^{-1}(\boldsymbol{\Phi}_{t_1t_0}\boldsymbol{x}_0 - \boldsymbol{x}_1)$$

$$\tilde{\alpha}_{\mathrm{L}} = \max_{\boldsymbol{x}\in\boldsymbol{M}_{10}^{-1/2}\boldsymbol{\Phi}_{10}\mathcal{X}_0-\boldsymbol{M}_{10}^{-1/2}\mathcal{X}_1} |\boldsymbol{x}|^2 = \{\max_{\boldsymbol{x}\in\boldsymbol{M}_{10}^{-1/2}\boldsymbol{\Phi}_{10}\mathcal{X}_0-\boldsymbol{M}_{10}^{-1/2}\mathcal{X}_1} \left\langle \frac{\boldsymbol{x}}{|\boldsymbol{x}|}, \boldsymbol{x} \right\rangle\}^2$$

$$\tilde{\alpha}_{\mathrm{L}} = \{\max_{\boldsymbol{y}\in\mathbb{S}^{n-1}} h_{\boldsymbol{M}_{10}^{-1/2}\boldsymbol{\Phi}_{10}\mathcal{X}_0-\boldsymbol{M}_{10}^{-1/2}\mathcal{X}_1}(\boldsymbol{y})\}^2$$

Solution to the Classical SBP

Thm. (Necessary conditions of optimality for the classical SBP):

The pair $\left(\rho_\varepsilon^{\text{opt}}, \boldsymbol{v}_\varepsilon^{\text{opt}}\right)$ solves the coupled PDEs

Value function

$$\frac{\partial \psi_\varepsilon}{\partial t} + \frac{1}{2}\left|\nabla_x \psi_\varepsilon\right|^2 + \varepsilon \Delta_x \psi_\varepsilon = 0,$$

$$\frac{\partial \rho_\varepsilon^{\text{opt}}}{\partial t} + \nabla_x \cdot \left(\rho_\varepsilon^{\text{opt}} \nabla_x \psi_\varepsilon\right) = \varepsilon \Delta_x \rho_\varepsilon^{\text{opt}}$$

with boundary conditions

$$\rho_\varepsilon^{\text{opt}}\left(\boldsymbol{x}, t = t_0\right) = \rho_0(\boldsymbol{x})$$

$$\rho_\varepsilon^{\text{opt}}\left(\boldsymbol{x}, t = t_1\right) = \rho_1(\boldsymbol{x})$$

Solution to the Classical SBP

Hopf-Cole transform

$$\underbrace{\varphi_\varepsilon := \exp\left(\frac{\psi_\varepsilon}{2\varepsilon}\right), \quad \widehat{\varphi}_\varepsilon}_{\text{Schrödinger factors}} := \rho_\varepsilon^{\text{opt}} \exp\left(-\frac{\psi_\varepsilon}{2\varepsilon}\right)$$

results in

$$\boxed{\begin{gathered} \frac{\partial \hat{\varphi}_\varepsilon}{\partial t} = \varepsilon \boldsymbol{\Delta}_x \hat{\varphi}_\varepsilon \\ \frac{\partial \varphi_\varepsilon}{\partial t} = -\varepsilon \boldsymbol{\Delta}_x \varphi_\varepsilon \\ \hat{\varphi}_\varepsilon(\boldsymbol{x}, t = t_0)\varphi_\varepsilon(\boldsymbol{x}, t = t_0) = \rho_0(\boldsymbol{x}), \\ \hat{\varphi}_\varepsilon(\boldsymbol{x}, t = t_1)\varphi_\varepsilon(\boldsymbol{x}, t = t_1) = \rho_1(\boldsymbol{x}) \end{gathered}}$$

Contraction Coefficient for Linear SBP

Thm. (informal)

$$\gamma_{\mathrm{L}} = \tanh^2\left(\frac{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}}}{8\varepsilon}\right)$$

Proof Idea: