Contractions and Reactions in Schrödinger Bridges

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Joint work with


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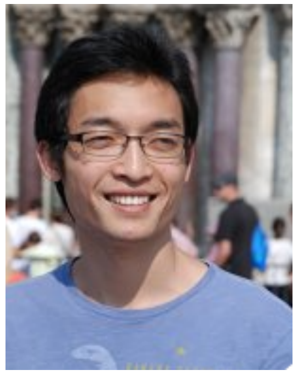
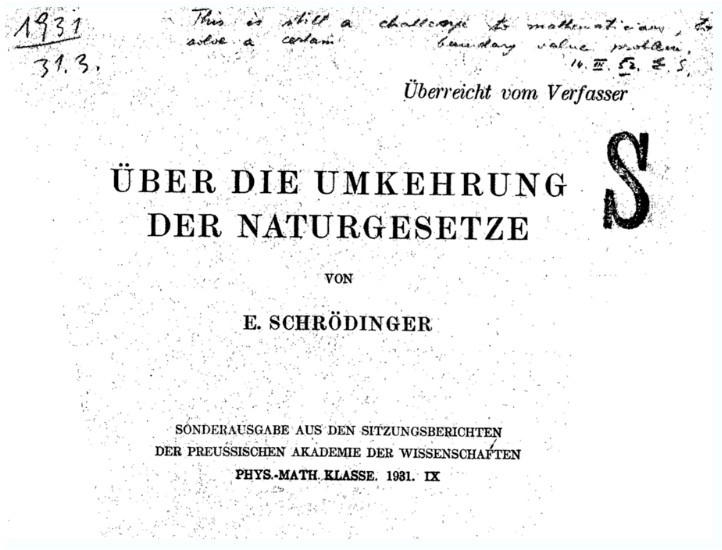

Yongxin Chen

April 14, 2025

Schrödinger Bridge Problem



1931
31.3.

This is still a challenge to mathematicians, to solve a certain boundary value problem. 16. III. 52. E. S.

Überreicht vom Verfasser

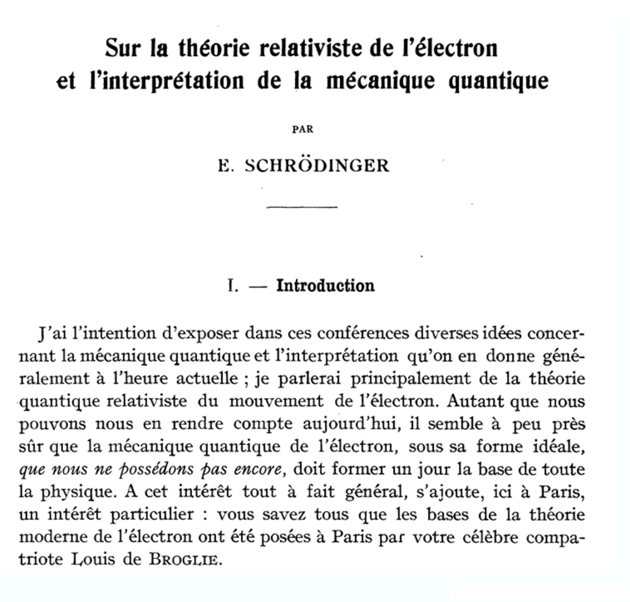
ÜBER DIE UMKEHRUNG
DER NATURGESETZE

VON

E. SCHRÖDINGER

S

SONDERAUSGABE AUS DEN SITZUNGSBERICHTEN
DER PREUSSISCHEN AKADEMIE DER WISSENSCHAFTEN
PHYS.-MATH. KLASSE. 1931. IX



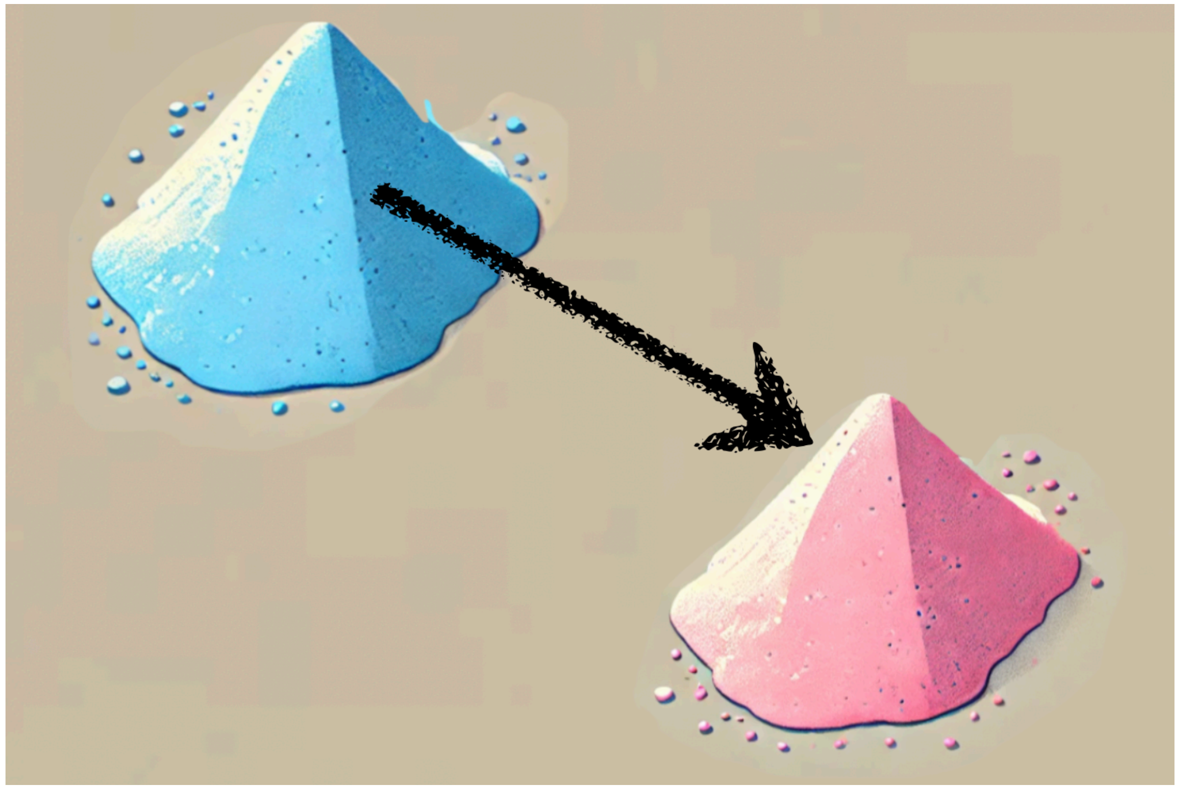
Sur la théorie relativiste de l'électron
et l'interprétation de la mécanique quantique

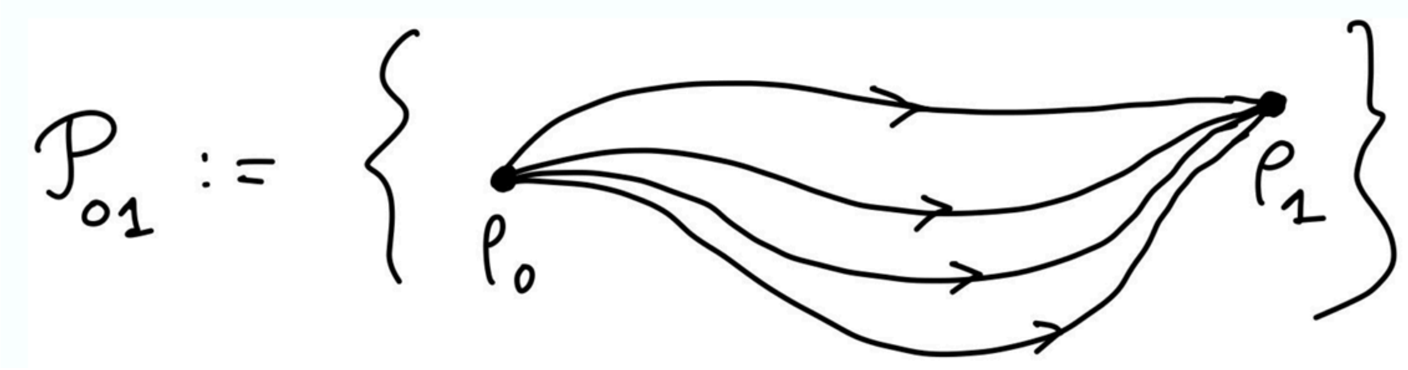
PAR

E. SCHRÖDINGER

I. — **Introduction**

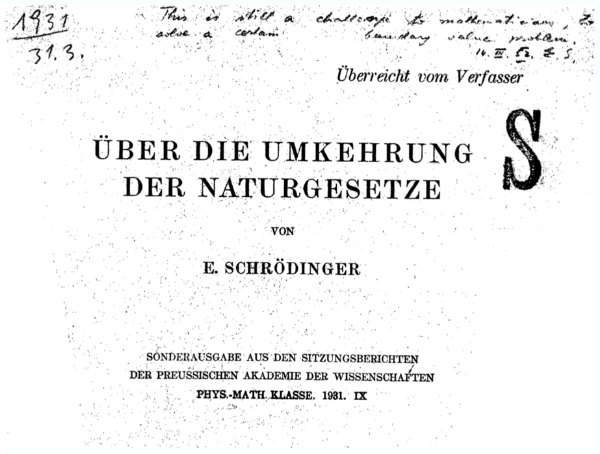
J'ai l'intention d'exposer dans ces conférences diverses idées concernant la mécanique quantique et l'interprétation qu'on en donne généralement à l'heure actuelle ; je parlerai principalement de la théorie quantique relativiste du mouvement de l'électron. Autant que nous pouvons nous en rendre compte aujourd'hui, il semble à peu près sûr que la mécanique quantique de l'électron, sous sa forme idéale, *que nous ne possédons pas encore*, doit former un jour la base de toute la physique. A cet intérêt tout à fait général, s'ajoute, ici à Paris, un intérêt particulier : vous savez tous que les bases de la théorie moderne de l'électron ont été posées à Paris par votre célèbre compatriote Louis de Broglie.





Most likely evolution between 2 distributional snapshots

Schrödinger Bridge Problem



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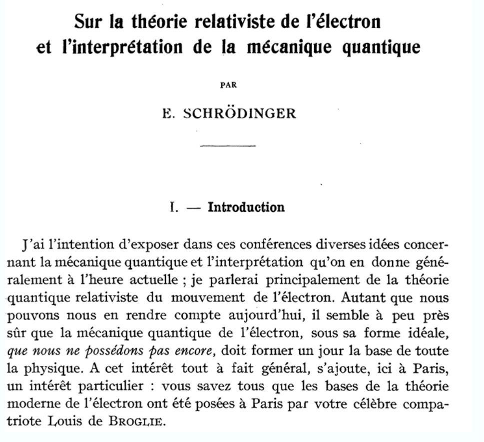

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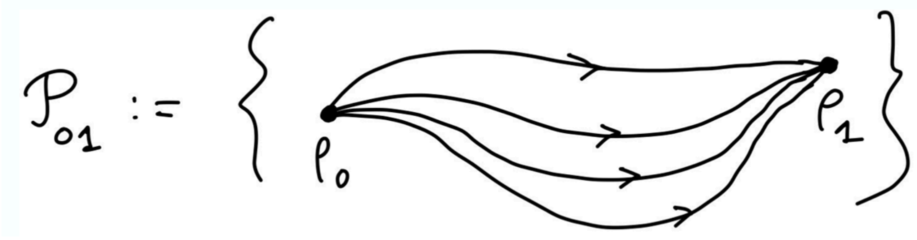

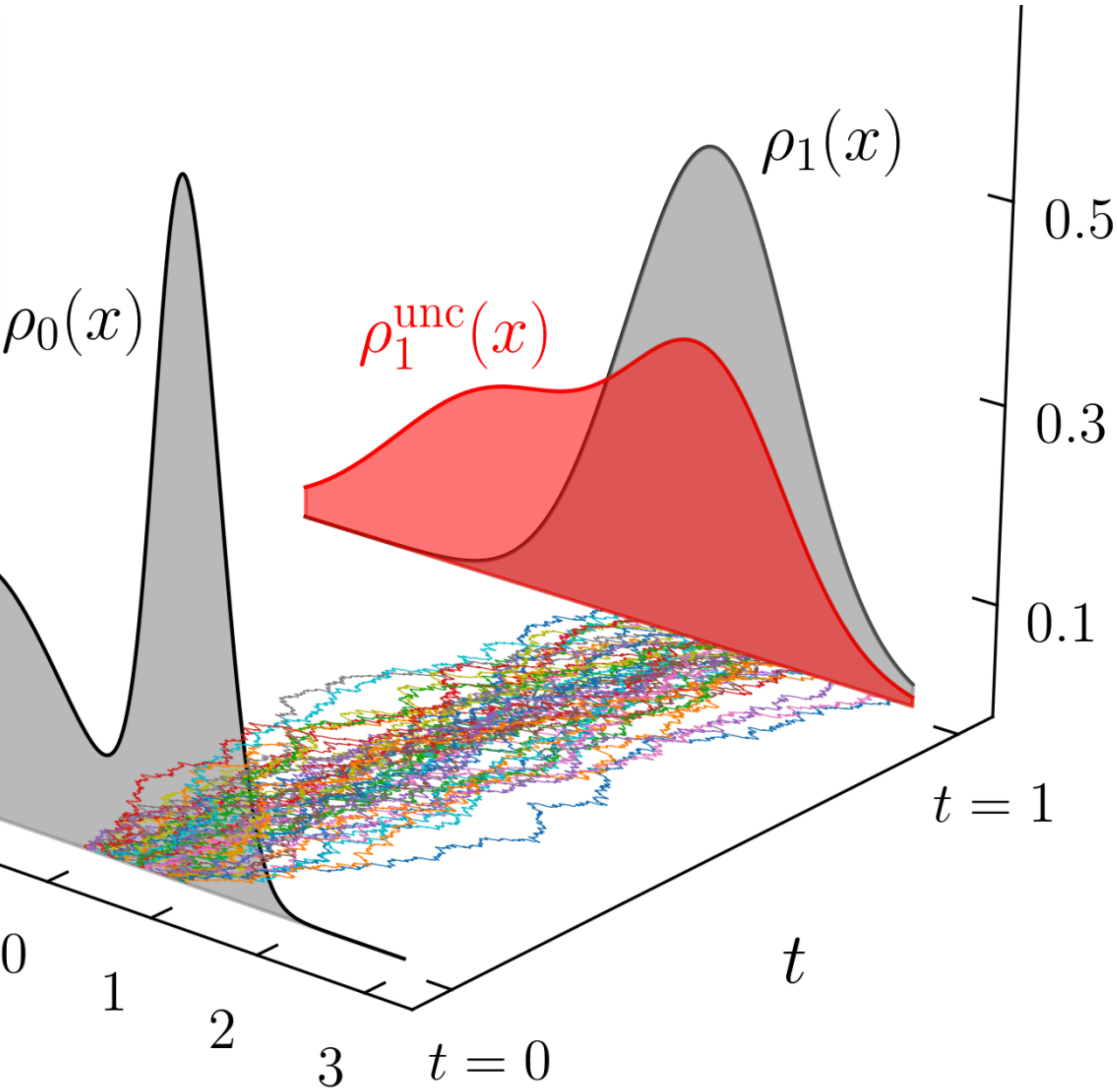

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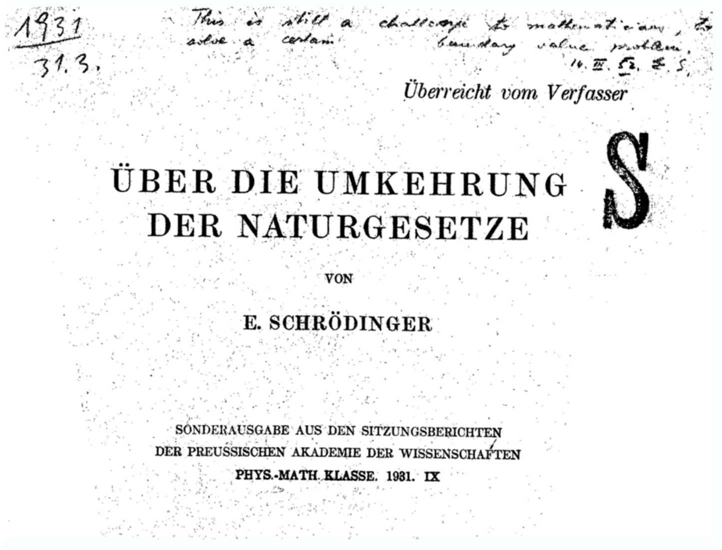

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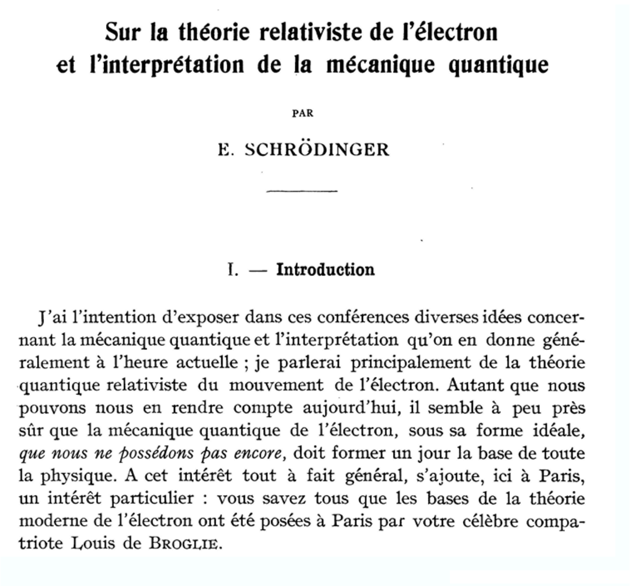

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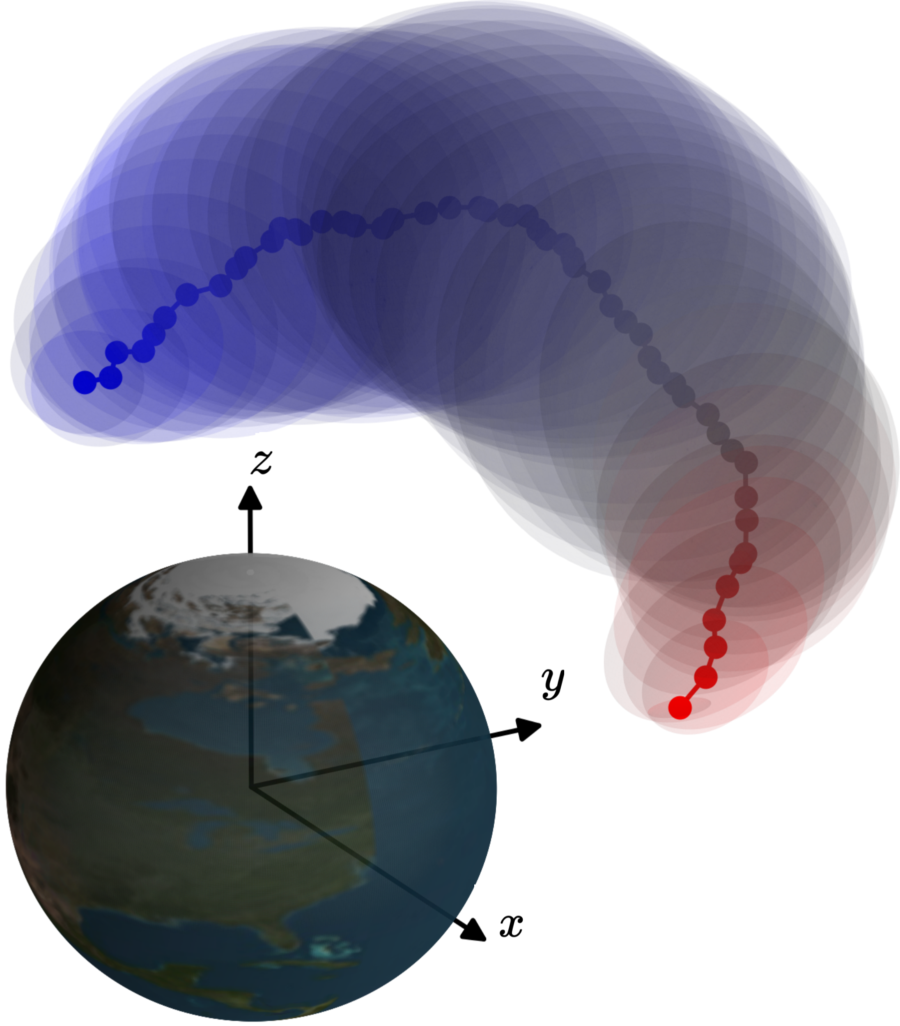
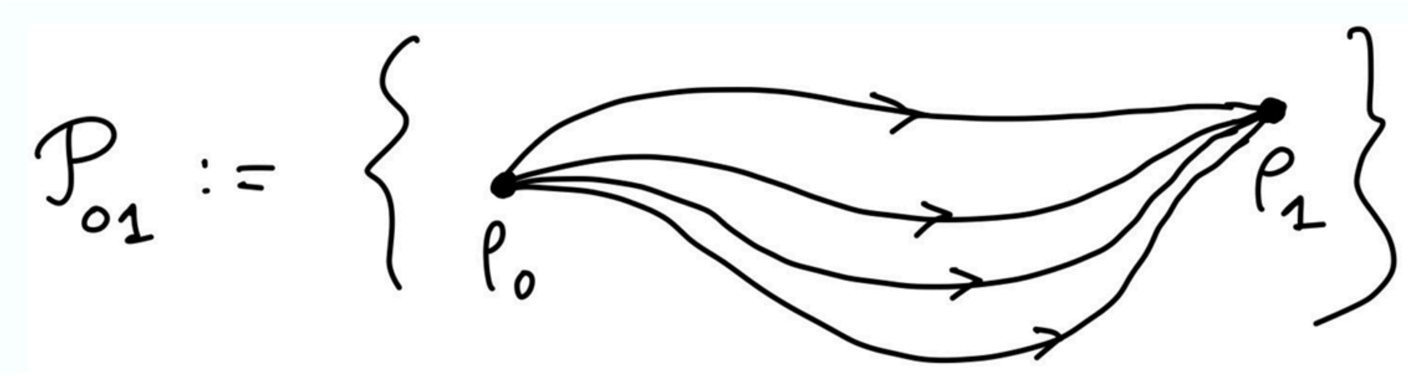


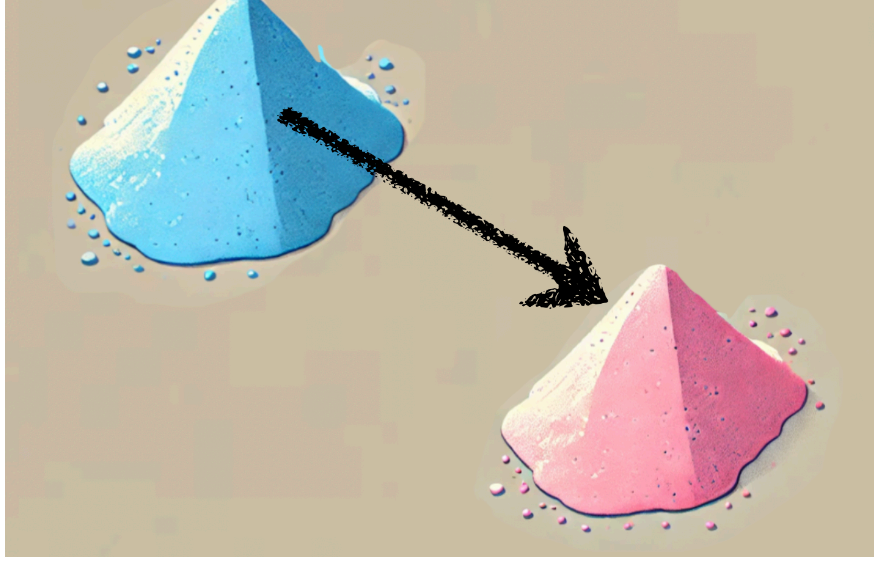


Most likely evolution between 2 distributional snapshots

Classical SBP

Find the best policy to minimize…

…the effort needed to steer…



$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf} \int_{t_0}^{t_1}\int_{\mathbb{R}^n} \frac{1}{2}|\boldsymbol{v}|^2\, \rho(\boldsymbol{x},t)\, d\boldsymbol{x}\;\; dt$$

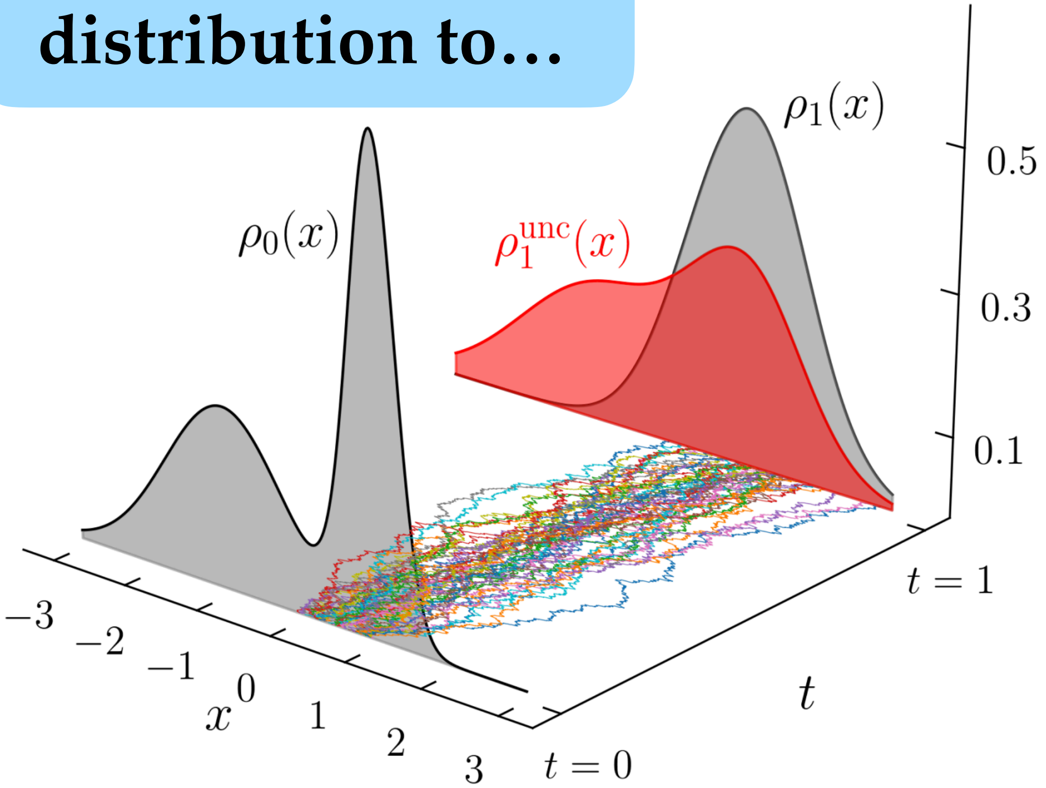
$$d\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x},t)\, dt + \sqrt{2\varepsilon}\, d\boldsymbol{w}(t)$$

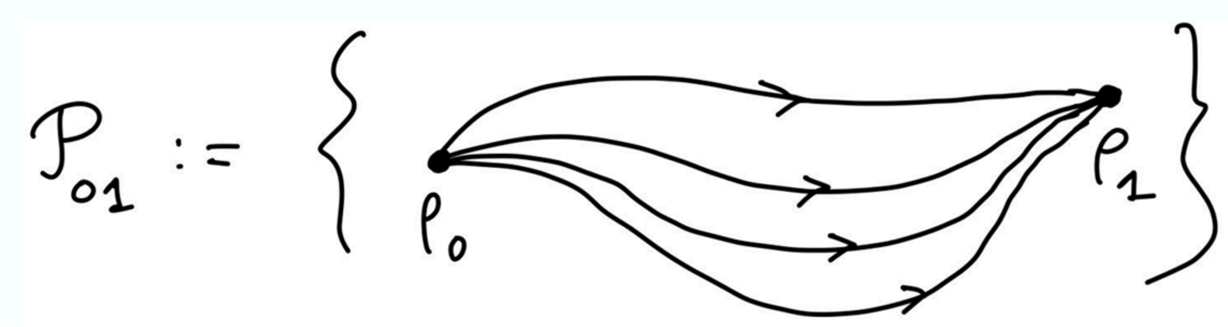
$$\rho(\boldsymbol{x},t=t_0) = \rho_0, \qquad \rho(\boldsymbol{x},t=t_1) = \rho_1,$$

…a given initial distribution to…

…a given final distribution…

…subject to certain sample path dynamics.





Solution to the Classical SBP

Necessary conditions of optimality:

The pair $\left(\rho_\varepsilon^{\text{opt}}, \boldsymbol{v}_\varepsilon^{\text{opt}}\right)$ solves the coupled PDEs

Value function

$$\frac{\partial \psi_\varepsilon}{\partial t} + \frac{1}{2}|\nabla_x \psi_\varepsilon|^2 + \varepsilon \Delta_x \psi_\varepsilon = 0,$$

$$\frac{\partial \rho_\varepsilon^{\text{opt}}}{\partial t} + \nabla_x \cdot \left(\rho_\varepsilon^{\text{opt}} \nabla_x \psi_\varepsilon\right) = \varepsilon \Delta_x \rho_\varepsilon^{\text{opt}}$$

Hopf-Cole transform

$$\varphi_\varepsilon := \exp\left(\frac{\psi_\varepsilon}{2\varepsilon}\right), \quad \widehat{\varphi}_\varepsilon := \rho_\varepsilon^{\text{opt}} \exp\left(-\frac{\psi_\varepsilon}{2\varepsilon}\right)$$

Schrödinger factors

Optimally controlled joint state PDF: $\rho_\varepsilon^{\text{opt}}(\cdot, t) = \widehat{\varphi}_\varepsilon(\cdot, t)\varphi_\varepsilon(\cdot, t)$

Optimal control: $\boldsymbol{v}_\varepsilon^{\text{opt}}(\cdot, t) = 2\varepsilon \nabla_{\boldsymbol{x}} \log \varphi_\varepsilon(\cdot, t)$

Schrödinger factors

Solution to the Classical SBP

Hopf-Cole transform is used to decouple PDEs

The pair $\left(\widehat{\varphi}_{\varepsilon}, \varphi_{\varepsilon}\right)$ solves the linear, uncoupled PDEs

$$\frac{\partial \widehat{\varphi}_{\varepsilon}}{\partial t} = \varepsilon \Delta_{\boldsymbol{x}} \widehat{\varphi}_{\varepsilon}$$

$$\frac{\partial \varphi_{\varepsilon}}{\partial t} = -\varepsilon \Delta_{\boldsymbol{x}} \varphi_{\varepsilon}$$

with coupled boundary conditions

$$\begin{aligned} \widehat{\varphi}_{\varepsilon}(\boldsymbol{x}, t = t_0)\, \varphi_{\varepsilon}(\boldsymbol{x}, t = t_0) &= \rho_0(\boldsymbol{x}), \\ \widehat{\varphi}_{\varepsilon}(\boldsymbol{x}, t = t_1)\, \varphi_{\varepsilon}(\boldsymbol{x}, t = t_1) &= \rho_1(\boldsymbol{x}). \end{aligned}$$

Algorithm

Schrödinger system

Markov kernel

$$\rho_0(\boldsymbol{x}) = \widehat{\varphi}_{\varepsilon,0}(\boldsymbol{x}) \int_{\mathbb{R}^n} k(t_0, \boldsymbol{x}, t_1, \boldsymbol{y}) \varphi_{\varepsilon,1}(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}$$

$$\rho_1(\boldsymbol{x}) = \varphi_{\varepsilon,1}(\boldsymbol{x}) \int_{\mathbb{R}^n} k(t_0, \boldsymbol{y}, t_1, \boldsymbol{x}) \widehat{\varphi}_{\varepsilon,0}(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}$$

Coupled nonlinear integral equations

Fixed point recursion over pair $\left(\varphi_{\varepsilon,1}, \widehat{\varphi}_{\varepsilon,0}\right)$

1.) Make an initial guess.

$$\begin{tikzcd}
\hat{\varphi}_0(\boldsymbol{x}) \arrow[r, "\int \kappa_{\text{forward}} \widehat{\varphi}"] & \hat{\varphi}_1(\boldsymbol{x}) \arrow[d, "\rho_1(\boldsymbol{x})/\hat{\varphi}_1(\boldsymbol{x})"] \\
\varphi_0(\boldsymbol{x}) \arrow[u, "\rho_0(\boldsymbol{x})/\varphi_0(\boldsymbol{x})"] & \varphi_1(\boldsymbol{x}) \arrow[l, "\int \kappa_{\text{backward}} \varphi"]
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2.) Integrate forward in time.

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\end{tikzcd}$$

We can *eventually* solve the Schrödinger Bridge problem, if we have a handle on uncontrolled kernel.

Part I: Background on SBP

Part II: Contraction Coefficient κ:

→ Guarantees on κ for known kernels (classical and linear SBP)

→ Uses of κ

Part III: SBP with state cost:

→ Sources of state cost

→ Approaches to deriving a handle for the kernel

Contraction Coefficient

Algorithm: Convergence

$$\begin{tikzcd}
\hat{\varphi}_0(\boldsymbol{x}) \arrow[r, "\int \kappa_{\text{forward}} \widehat{\varphi}"] & \hat{\varphi}_1(\boldsymbol{x}) \arrow[d, "\rho_1(\boldsymbol{x})/\hat{\varphi}_1(\boldsymbol{x})"] \\
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\end{tikzcd}$$

Contraction rate $\kappa(f)$ of mapping f is maximum α such that

$$d_{\mathrm{H}}(f(\boldsymbol{x}), f(\boldsymbol{y})) \leq \alpha \, d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{y})$$

Hilbert metric

Worst-case contraction coefficient γ

Algorithm has linear convergence with contraction coefficient $\kappa \leq \gamma$

γ in Classical SBP

Let

$$\alpha_{\mathrm{B}} = \frac{\exp(-\tilde{\alpha}_{\mathrm{B}}/(4\varepsilon))}{\sqrt{(4\pi\varepsilon)^{n}}}, \quad \beta_{\mathrm{B}} = \frac{\exp\left(-\tilde{\beta}_{\mathrm{B}}/(4\varepsilon)\right)}{\sqrt{(4\pi\varepsilon)^{n}}}.$$

where

$$\tilde{\beta}_{\mathrm{B}} := \min_{\boldsymbol{x}_0\in\mathcal{X}_0,\boldsymbol{x}_1\in\mathcal{X}_1} \left|\boldsymbol{x}_0 - \boldsymbol{x}_1\right|^2 \quad \text{and} \quad \tilde{\alpha}_{\mathrm{B}} := \max_{\boldsymbol{x}_0\in\mathcal{X}_0,\boldsymbol{x}_1\in\mathcal{X}_1} \left|\boldsymbol{x}_0 - \boldsymbol{x}_1\right|^2$$

$$\boldsymbol{\kappa} \leq \gamma_{\mathrm{B}} := \tanh^2\left(\frac{1}{2}\log\left(\frac{\beta_{\mathrm{B}}}{\alpha_{\mathrm{B}}}\right)\right) \in (0,1)$$

Chen, Georgiou, Pavon, *SIAM J. Applied Math, 2016*

$$\downarrow$$

$$\boldsymbol{\kappa} \leq \gamma_{\mathrm{B}} := \tanh^2\left(\frac{\tilde{\alpha}_{\mathrm{B}} - \tilde{\beta}_{\mathrm{B}}}{8\varepsilon}\right) \in (0,1)$$

Linear SBP

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf} \quad \int_{t_0}^{t_1}\int_{\mathbb{R}^n}\frac{1}{2}|\boldsymbol{v}|^2\rho(\boldsymbol{x},t)\,d\boldsymbol{x}\,dt$$

$$\frac{\partial\rho}{\partial t}+\nabla_{\boldsymbol{x}}\cdot(\rho(\boldsymbol{A}(t)\boldsymbol{x}+\boldsymbol{B}(t)\boldsymbol{v}))=\varepsilon\langle\text{Hess},\boldsymbol{B}(t)\boldsymbol{B}(t)^\top\rho\rangle$$

resp. compact supports $\mathcal{X}_0, \mathcal{X}_1$

$$\rho(\boldsymbol{x},t=t_0)=\rho_0, \quad \rho(\boldsymbol{x},t=t_1)=\rho_1$$

Controlled sample path dynamics

$$\mathrm{d}\boldsymbol{x}(t)=(\boldsymbol{A}(t)\boldsymbol{x}(t)+\boldsymbol{B}(t)\boldsymbol{v}(\boldsymbol{x},t))\mathrm{d}t+\sqrt{2\varepsilon}\boldsymbol{B}(t)\mathrm{d}\boldsymbol{w}(t)$$

State transition matrix $\boldsymbol{\Phi}_{t\tau}:=\boldsymbol{\Phi}(t,\tau) \quad \forall t_0\leq\tau\leq t\leq t_1$

Assume controllability: $\boldsymbol{M}_{10}:=\displaystyle\int_{t_0}^{t_1}\boldsymbol{\Phi}_{t_1\tau}\boldsymbol{B}(\tau)\boldsymbol{B}^\top(\tau)\boldsymbol{\Phi}_{t_1\tau}^\top\mathrm{d}\tau\succ\boldsymbol{0}$

* **Classical SBP is special case:** $\boldsymbol{A}(t)\equiv\boldsymbol{0}, \boldsymbol{B}(t)\equiv\boldsymbol{I}$

Structure of the Solution for Linear SBP

Kernel for Linear SBP:

$$k(t_0, \boldsymbol{x}_0, t_1, \boldsymbol{x}_1) := \frac{\exp\left(-\frac{\left(\boldsymbol{\Phi}_{t_1 t_0}\boldsymbol{x}_0 - \boldsymbol{x}_1\right)^\top \boldsymbol{M}_{10}^{-1}\left(\boldsymbol{\Phi}_{t_1 t_0}\boldsymbol{x}_0 - \boldsymbol{x}_1\right)}{4\varepsilon}\right)}{\sqrt{(4\pi\varepsilon)^n \det(\boldsymbol{M}_{10})}}$$

Guaranteed linear convergence with contraction rate $\kappa \in (0, 1)$

Exact rate depends on problem data $(\mathcal{X}_0, \mathcal{X}_1, \varepsilon, \boldsymbol{A}(t), \boldsymbol{B}(t))$

Worst case contraction coefficient $\gamma := \sup\limits_{\text{Linear SBPs with fixed } (\mathcal{X}_0, \mathcal{X}_1, \varepsilon, \boldsymbol{A}(t), \boldsymbol{B}(t))} \kappa$

γ in Linear SBP

Thm. (informal)

State transition matrix **Controllability Gramian**

Let

$$\tilde{\alpha}_{\mathrm{L}} := \max_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)^\top \boldsymbol{M}_{10}^{-1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)$$

$$\tilde{\beta}_{\mathrm{L}} := \min_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)^\top \boldsymbol{M}_{10}^{-1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)$$

Then

$$\gamma_{\mathrm{L}} = \tanh^2 \left(\frac{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}}}{8\varepsilon} \right)$$

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Then

$$\gamma_{\mathrm{L}} = \tanh^2 \left(\frac{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}}}{8\varepsilon} \right)$$

Note:

$$\begin{aligned} \boldsymbol{A}(t) &\equiv \boldsymbol{0} \\ \boldsymbol{B}(t) &\equiv \boldsymbol{I} \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} \boldsymbol{\Phi}_{t_1 t_0} &= \boldsymbol{I} \\ \boldsymbol{M}_{10} &= \frac{1}{t_1 - t_0} \boldsymbol{I} \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} \tilde{\alpha}_{\mathrm{B}} &:= \max_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} \frac{1}{t_1 - t_0} |\boldsymbol{x}_0 - \boldsymbol{x}_1|^2 \\ \tilde{\beta}_{\mathrm{B}} &:= \min_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} \frac{1}{t_1 - t_0} |\boldsymbol{x}_0 - \boldsymbol{x}_1|^2 \end{aligned}$$

Control-theoretic Interpretation for γ_{L}

$$\tilde{\alpha}_{\mathrm{L}} := \max_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)^\top \boldsymbol{M}_{10}^{-1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)$$

$$\tilde{\beta}_{\mathrm{L}} := \min_{\boldsymbol{x}_0 \in \mathcal{X}_0, \boldsymbol{x}_1 \in \mathcal{X}_1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)^\top \boldsymbol{M}_{10}^{-1} (\boldsymbol{\Phi}_{t_1 t_0} \boldsymbol{x}_0 - \boldsymbol{x}_1)$$

$$\begin{aligned} &\underset{\boldsymbol{v}}{\text{minimum}} \int_{t_0}^{t_1} \frac{1}{2} |\boldsymbol{v}|^2 dt \\ &\text{subject to} \quad \dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{v} \\ &\qquad\qquad\quad\; \boldsymbol{x}(t = t_0) = \boldsymbol{x}_0, \boldsymbol{x}(t = t_1) = \boldsymbol{x}_1 \end{aligned}$$

Minimum cost for deterministic OCP

Control-theoretic Interpretation for γ_{L}

Range of optimal state transfer cost

$$\gamma_{\mathrm{L}} = \tanh^2\left(\frac{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}}}{8\varepsilon}\right)$$

Process noise

Conforms with intuition:

$$\boxed{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}} \uparrow \quad \Rightarrow \quad \gamma_{\mathrm{L}} \uparrow}$$

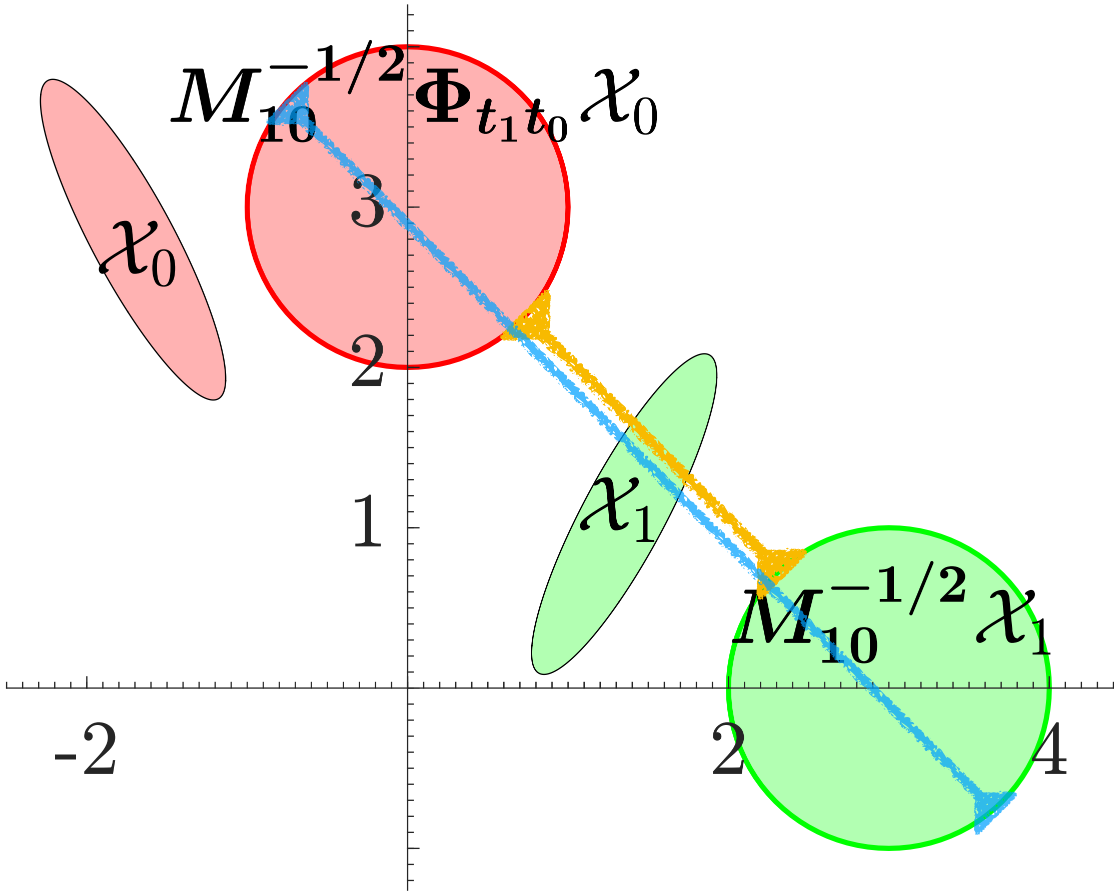
$$\boxed{\varepsilon \uparrow \quad \Rightarrow \quad \gamma_{\mathrm{L}} \downarrow}$$

Geometric Interpretation for γ_{L}

$$\gamma_{\mathrm{L}} = \tanh^2\left(\frac{\tilde{\alpha}_{\mathrm{L}} - \tilde{\beta}_{\mathrm{L}}}{8\varepsilon}\right)$$

$$\tilde{\alpha}_{\mathrm{L}} := \max_{\boldsymbol{x}_0 \in \boldsymbol{M}_{10}^{-1/2}\boldsymbol{\Phi}_{10}\mathcal{X}_0,\ \boldsymbol{x}_1 \in \boldsymbol{M}_{10}^{-1/2}\mathcal{X}_1} |\boldsymbol{x}_0 - \boldsymbol{x}_1|^2$$

$$\tilde{\beta}_{\mathrm{L}} := \min_{\boldsymbol{x}_0 \in \boldsymbol{M}_{10}^{-1/2}\boldsymbol{\Phi}_{10}\mathcal{X}_0,\ \boldsymbol{x}_1 \in \boldsymbol{M}_{10}^{-1/2}\mathcal{X}_1} |\boldsymbol{x}_0 - \boldsymbol{x}_1|^2$$



Geometric interpretation:

$\tilde{\alpha}_{\mathrm{L}}$ and $\tilde{\beta}_{\mathrm{L}}$ are the maximum and minimal separation of $M_{10}^{-1/2}\boldsymbol{\Phi}_{t_1t_0}\mathcal{X}_0$ and $M_{10}^{-1/2}\mathcal{X}_1$

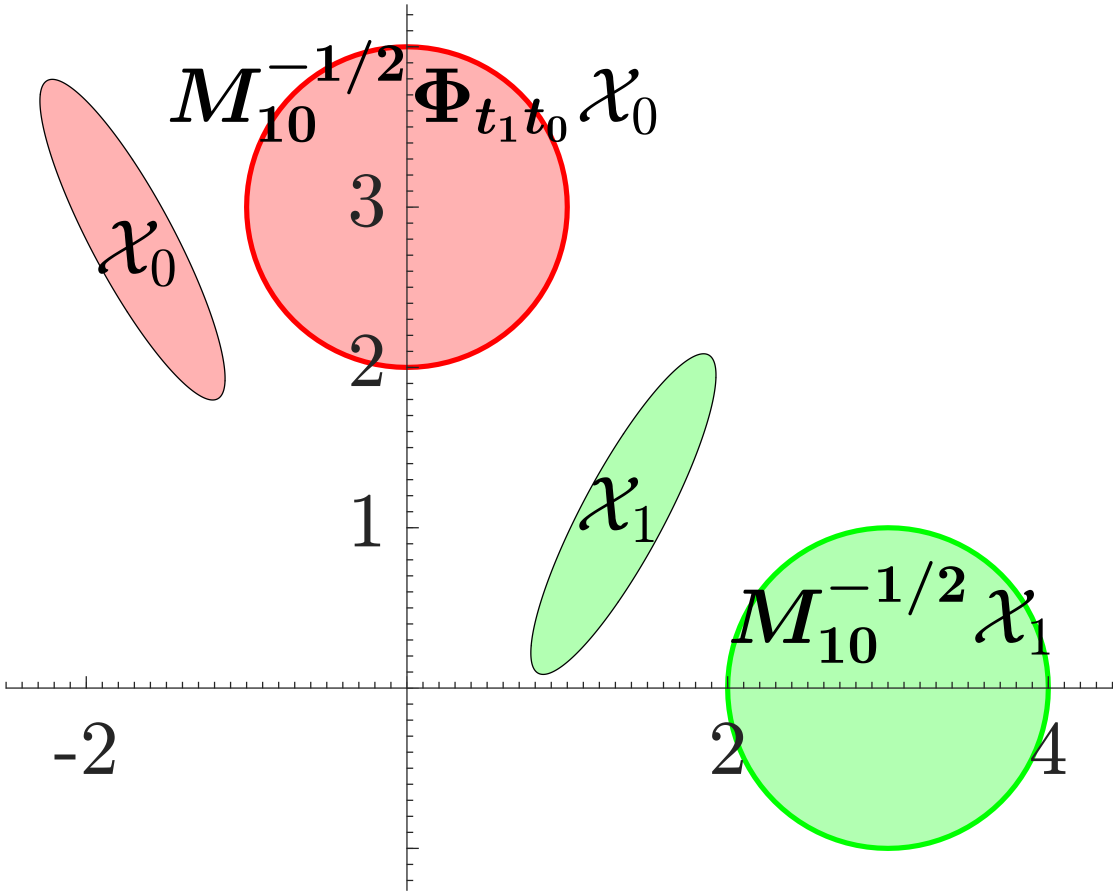
Applications to Preconditioning:

Preconditioning to improve optimal transport algorithms
~ Kuang and Tabak, *SIAM J. Scientific Computing*, 2017

Example: Linear SBP: $\varepsilon = 0.5$

$$d\boldsymbol{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}(t)dt + \sqrt{2\varepsilon}\begin{bmatrix} 0 \\ 1 \end{bmatrix} d\boldsymbol{w}(t)$$

$$\boldsymbol{\Phi}_{t_1 t_0} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{M}_{10}^{-1} = \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}.$$



No Preconditioning:

$$\tilde{\alpha}_{\mathrm{L}} = 2 + 2\sqrt{3}$$
$$\tilde{\beta}_{\mathrm{L}} = -2 + 2\sqrt{3}$$

$$\longrightarrow \quad \gamma_{\mathrm{L}} = \tanh^2(1) \approx 0.580$$

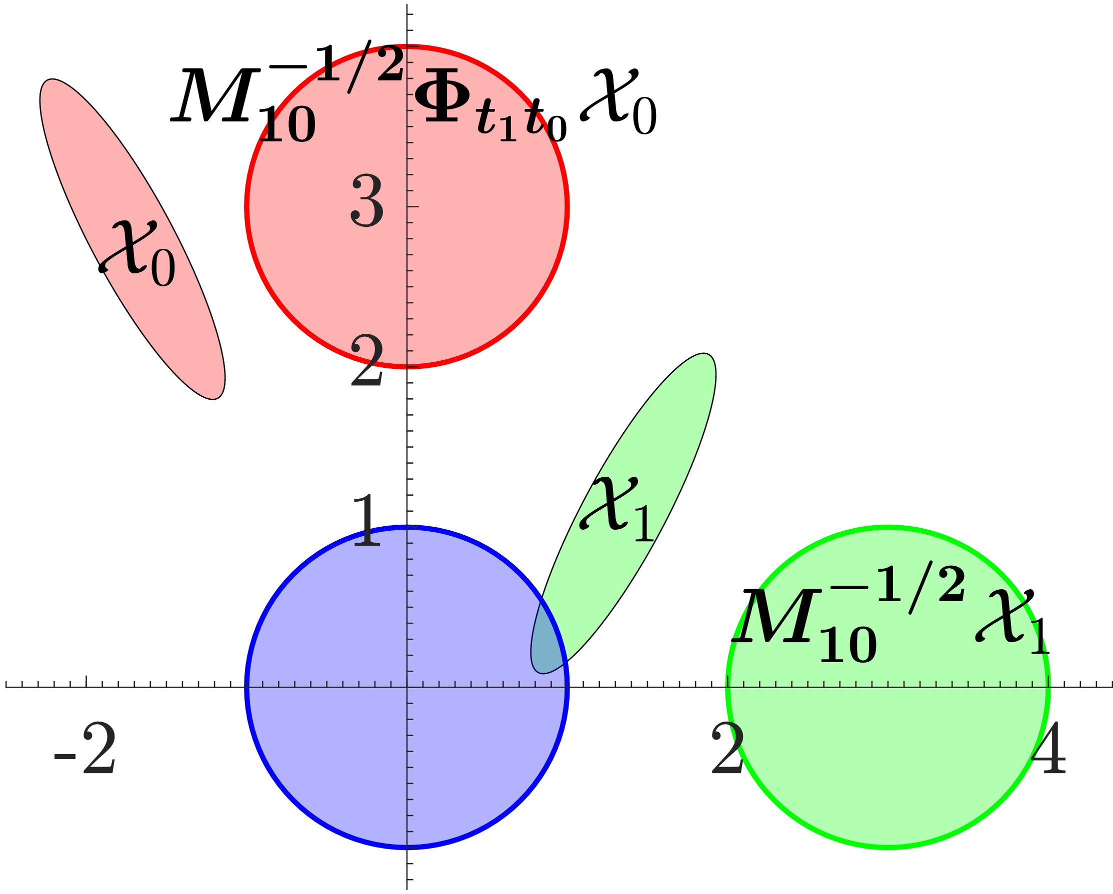
Applications to Preconditioning:

Preconditioning to improve optimal transport algorithms
~ Kuang and Tabak, *SIAM J. Scientific Computing*, 2017

Example: Linear SBP: $\varepsilon = 0.5$

$$d\boldsymbol{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}(t)dt + \sqrt{2\varepsilon} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\boldsymbol{w}(t)$$

$$\boldsymbol{\Phi}_{t_1 t_0} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{M}_{10}^{-1} = \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}.$$



With Preconditioning:

$$\tilde{\alpha}_{\mathrm{L}}^{\mathrm{precond}} = 2, \; \tilde{\beta}_{\mathrm{L}}^{\mathrm{precond}} = 0 \quad \longrightarrow \quad \gamma_{\mathrm{L}}^{\mathrm{precond}} = \tanh^2(0.5) = 0.214$$

SBP with State Cost

SBP with State Cost

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf} \quad \int_{t_0}^{t_1}\int_{\mathbb{R}^n}\left(\frac{1}{2}|\boldsymbol{v}|^2 + q(\boldsymbol{x})\right)\rho(\boldsymbol{x},t)\,d\boldsymbol{x}\,dt$$

$$\frac{\partial\rho}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v}) = \varepsilon\Delta_{\boldsymbol{x}}\rho$$

$$\boldsymbol{x}(t=t_0)\sim\rho_0\;(\text{given}),\quad \boldsymbol{x}(t=t_1)\sim\rho_1\;(\text{given})$$

Controlled sample path dynamics

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x},t)\mathrm{d}t + \sqrt{2\varepsilon}\mathrm{d}\boldsymbol{w}(t)$$

Solution for the SBP with State Cost

Thm. (informal)

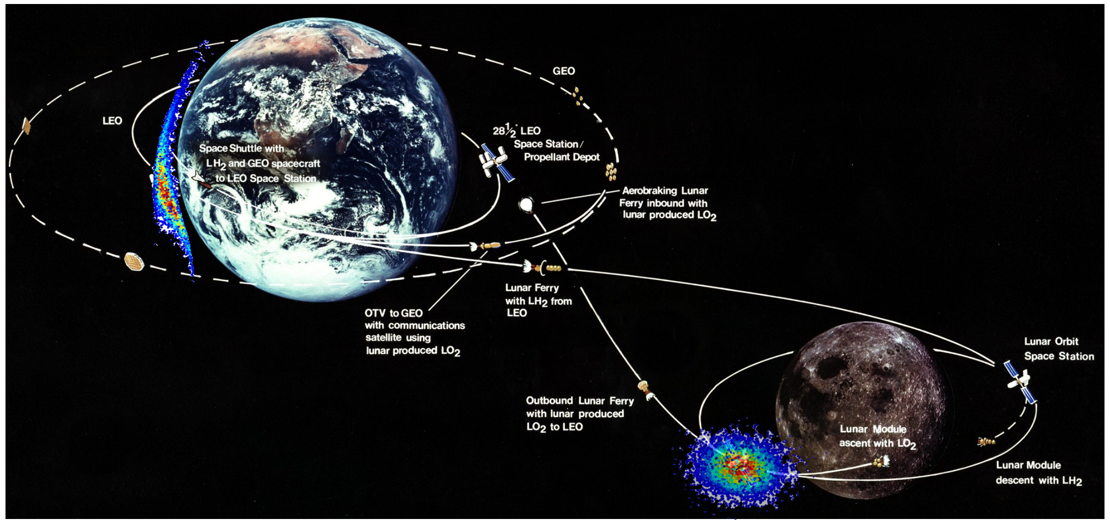
SBP with state cost admits a unique solution

Proof idea:

Reformulate as Kullback-Leibler minimization over path space:

$$\underbrace{\operatorname*{arg\,inf}_{\mathbb{P}\in\Pi_{01}} \quad D_{\mathrm{KL}}\left(\mathbb{P} \,\|\, \frac{\exp\left(-\frac{1}{2\varepsilon}\int_{t_0}^{t_1} q(\boldsymbol{x})dt\right)\mathbb{W}}{Z}\right)}_{\text{large deviation principle}}$$

State cost may arise due to…

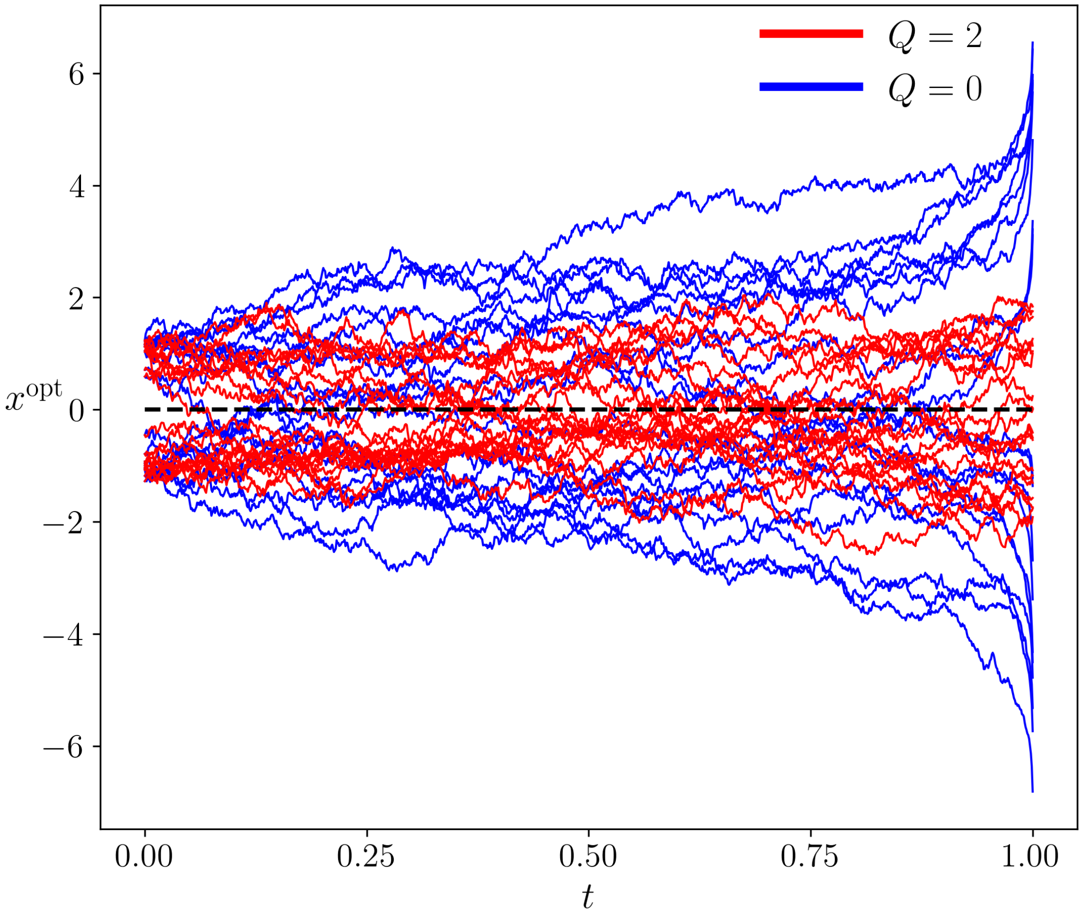


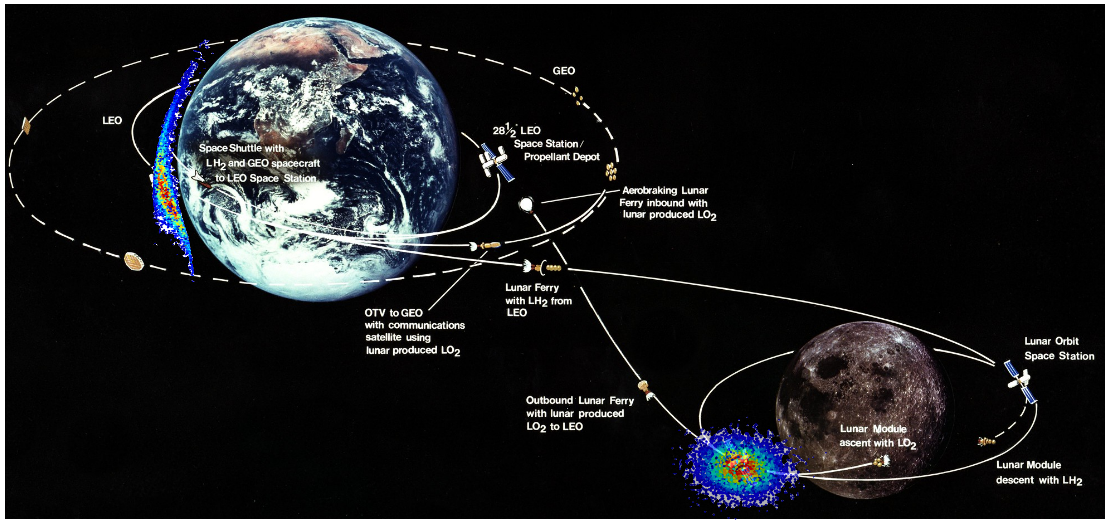
→ Pushing dynamical nonlinearity to Lagrangian

→ Application: Lambert's Problem

→ First principle modeling

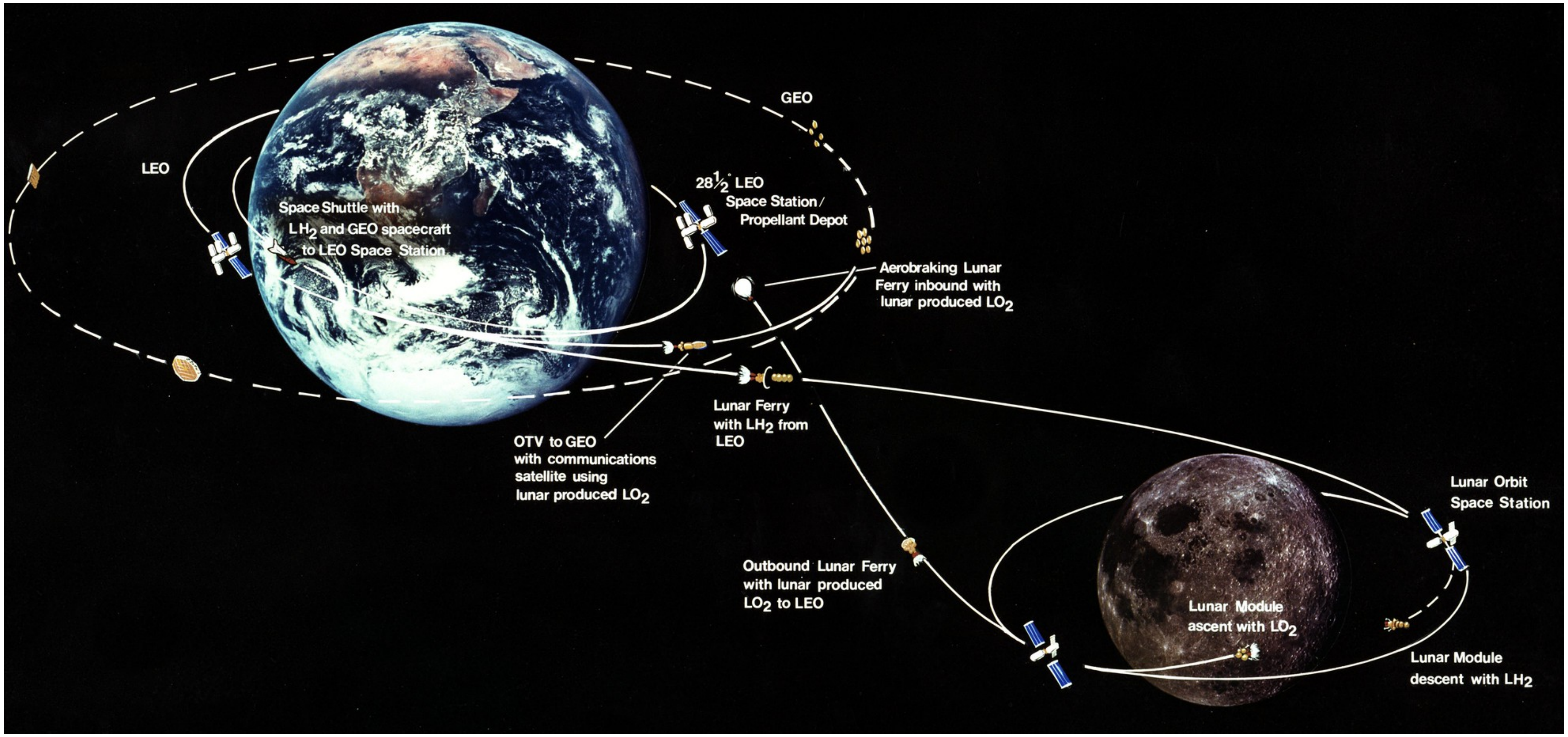
→ Example: A soft penalty from deviating from a desired value





Probabilistic Lambert's Problem

Lambert's Problem

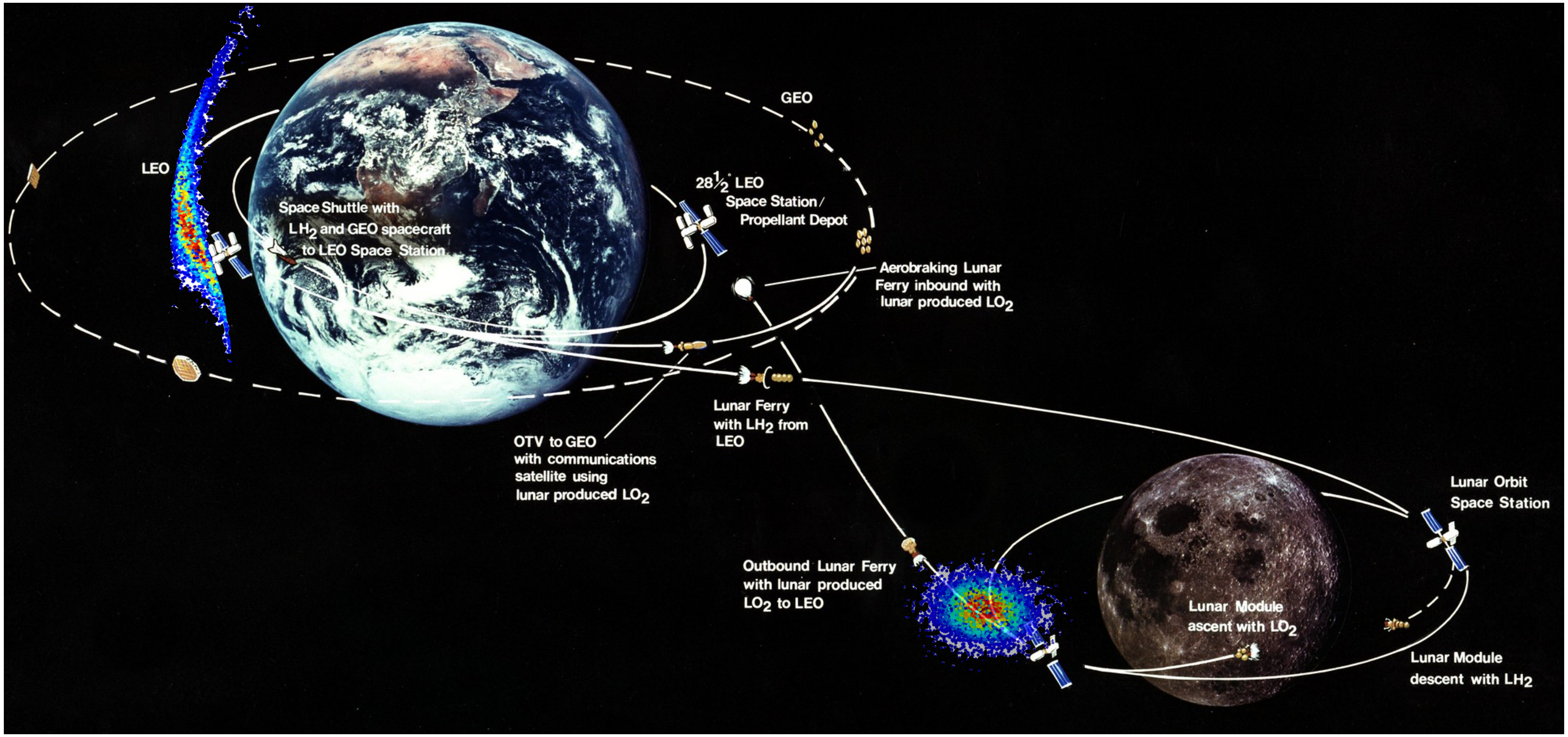


3D position coordinate $\boldsymbol{x} \in \mathbb{R}^3$

Find velocity control policy $\dot{\boldsymbol{x}} := \boldsymbol{v}(\boldsymbol{x}, t)$ such that

$$\ddot{\boldsymbol{x}} = -\nabla_{\boldsymbol{x}} V(\boldsymbol{x}), \quad \boxed{\boldsymbol{x}(t = t_0) = \boldsymbol{x}_0 \, (\text{given}), \quad \boldsymbol{x}(t = t_1) = \boldsymbol{x}_1 \, (\text{given})}$$

Probabilistic Lambert's Problem

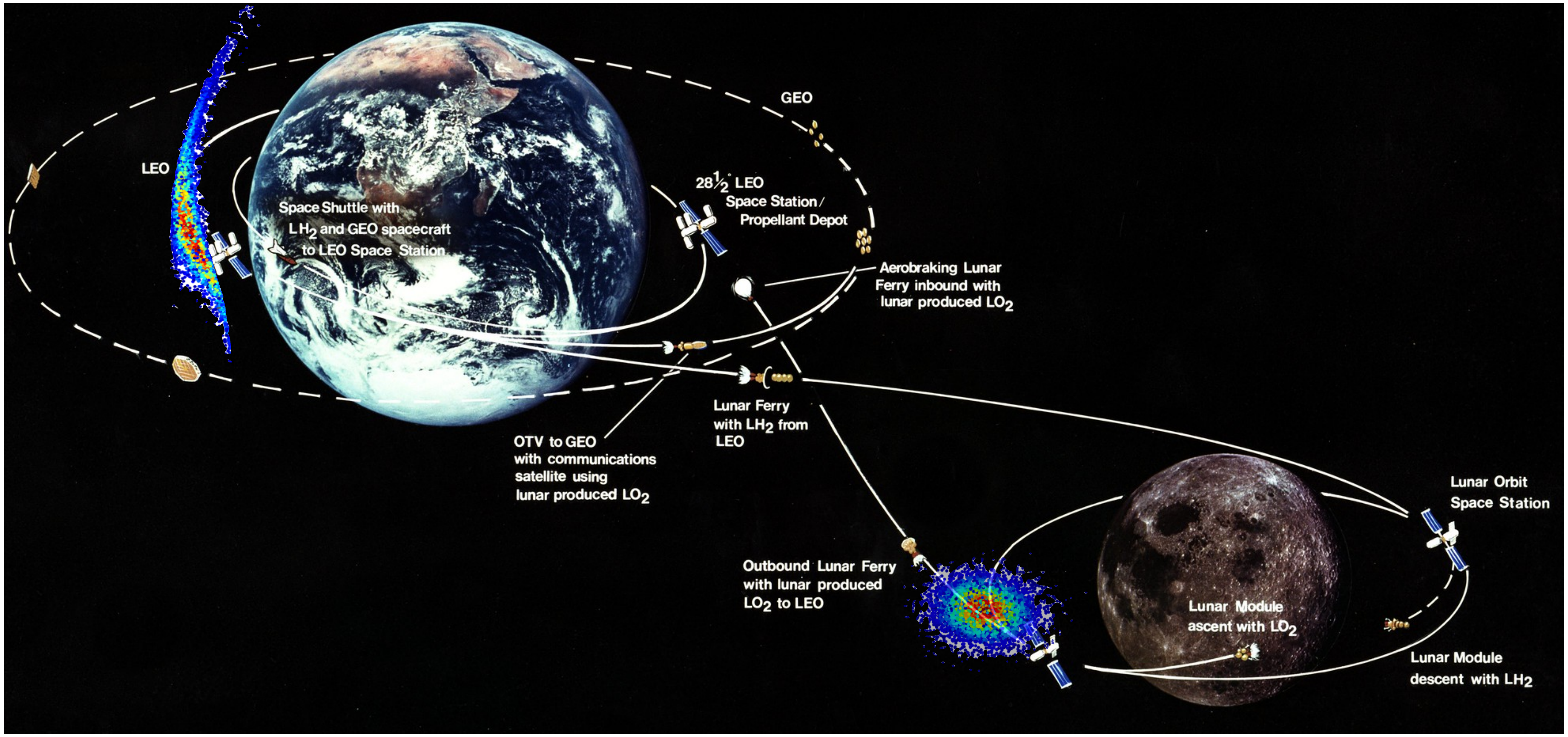


3D position coordinate $\boldsymbol{x} \in \mathbb{R}^3$

Find velocity control policy $\dot{\boldsymbol{x}} := \boldsymbol{v}(\boldsymbol{x}, t)$ such that

$$\begin{cases} \ddot{\boldsymbol{x}} = -\nabla_{\boldsymbol{x}} V(\boldsymbol{x}) \\ \boldsymbol{x}(t = t_0) \sim \rho_0 \quad \text{(given)} \\ \boldsymbol{x}(t = t_1) \sim \rho_1 \quad \text{(given)} \end{cases}$$

Probabilistic Lambert's Problem



3D position coordinate $\boldsymbol{x} \in \mathbb{R}^3$

Find velocity control policy $\dot{\boldsymbol{x}} := \boldsymbol{v}(\boldsymbol{x}, t)$ such that

$$\begin{cases} \ddot{\boldsymbol{x}} = -\nabla_{\boldsymbol{x}} V(\boldsymbol{x}) & \\ \boldsymbol{x}(t = t_0) \sim \rho_0 & \text{(given)} \\ \boldsymbol{x}(t = t_1) \sim \rho_1 & \text{(given)} \end{cases}$$

Motive: Allow for stochastic uncertainties, e.g.,

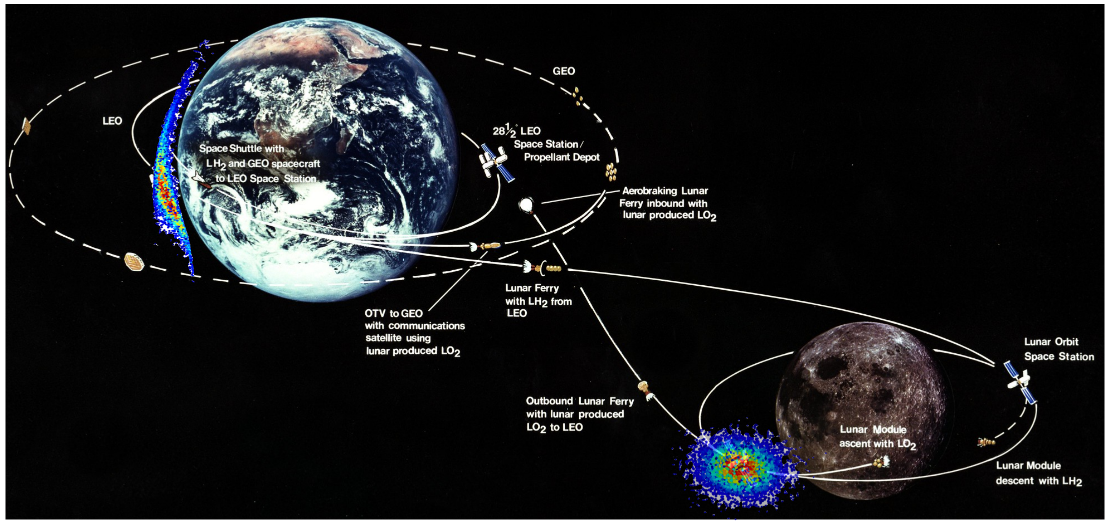
- → statistical estimation errors
- → statistical performance specification

Probabilistic Lambert's Problem

$$\underset{\dot{\boldsymbol{x}}=\boldsymbol{v}(\boldsymbol{x},t)}{\text{find}} \quad \boldsymbol{v}$$

$$\ddot{\boldsymbol{x}} = -\nabla_{\boldsymbol{x}} V(\boldsymbol{x}),$$

$$\boldsymbol{x}(t=t_0) \sim \rho_0, \quad \boldsymbol{x}(t=t_1) \sim \rho_1$$



feasibility problem ⟶ **optimization problem**

Lambertian OMT (L-OMT)

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf} \int_{t_0}^{t_1}\int_{\mathbb{R}^n}\left(\frac{1}{2}|\boldsymbol{v}|^2 - V(\boldsymbol{x})\right)\rho(\boldsymbol{x},t)\,d\boldsymbol{x}dt$$

$$\frac{\partial\rho}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v}) = 0,$$

$$\rho(\boldsymbol{x},t=t_0) = \rho_0(\boldsymbol{x}), \quad \rho(\boldsymbol{x},t=t_1) = \rho_1(\boldsymbol{x}),$$

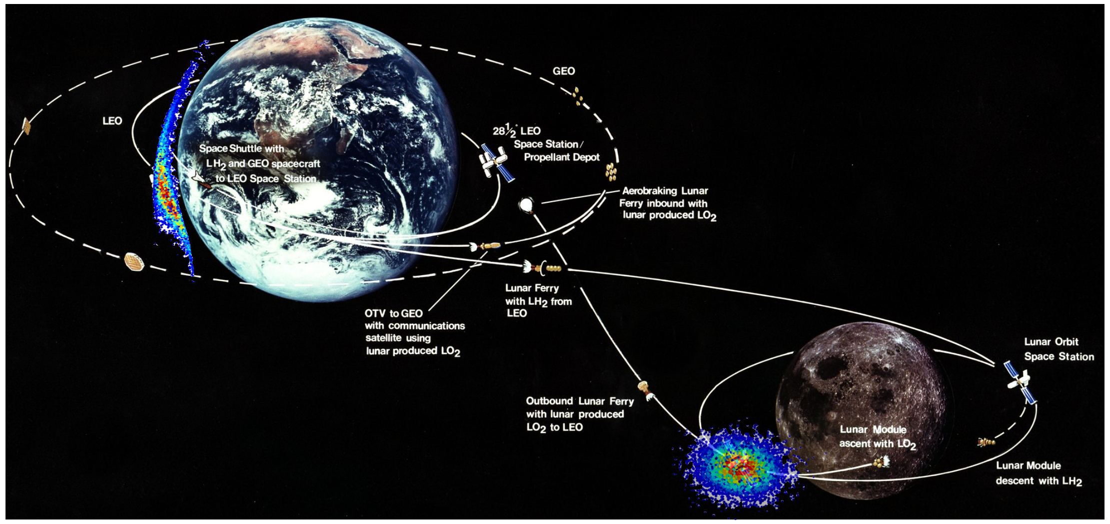
nonlinearity in dynamics pushed to Lagrangian

Probabilistic Lambert's Problem

$$\underset{\dot{\boldsymbol{x}}=\boldsymbol{v}(\boldsymbol{x},t)}{\text{find}}\quad \boldsymbol{v}$$

$$\ddot{\boldsymbol{x}} = -\nabla_{\boldsymbol{x}} V(\boldsymbol{x}),$$

$$\boldsymbol{x}(t=t_0) \sim \rho_0, \quad \boldsymbol{x}(t=t_1) \sim \rho_1$$



Generalize to velocity with additive process noise $\varepsilon > 0$

L-OMT

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x},t)$$

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x},t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}\boldsymbol{w}(t)$$

Lambertian SBP (L-SBP)

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf} \int_{t_0}^{t_1}\int_{\mathbb{R}^n}\left(\frac{1}{2}|\boldsymbol{v}|^2 - V(\boldsymbol{x})\right)\rho(\boldsymbol{x},t)\,d\boldsymbol{x}dt$$

$$\frac{\partial\rho}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v}) = \varepsilon\Delta_{\boldsymbol{x}}\rho,$$

$$\rho(\boldsymbol{x},t=t_0) = \rho_0(\boldsymbol{x}), \quad \rho(\boldsymbol{x},t=t_1) = \rho_1(\boldsymbol{x}),$$

L-SBP Solution

Thm. (informal) Existence and uniqueness of L-SBP is guaranteed

Gravitational Potential for LEO

$$V(\boldsymbol{x}) = -\frac{\mu}{|\boldsymbol{x}|}\left(1 + \frac{J_2 R_{\text{Earth}}^2}{2|\boldsymbol{x}|^2}\left(1 - \frac{3z^2}{|\boldsymbol{x}|^2}\right)\right) \longrightarrow$$

Bounded and negative for $|\boldsymbol{x}|^2 \geq \mathrm{R}_{\text{Earth}}^2$

Thm. (Necessary conditions of optimality for L-SBP)

$$\frac{\partial \psi_\varepsilon}{\partial t} + \frac{1}{2}|\nabla_{\boldsymbol{x}} \psi_\varepsilon|^2 + \varepsilon \Delta_{\boldsymbol{x}} \psi_\varepsilon = -V(\boldsymbol{x})$$

$$\frac{\partial \rho_\varepsilon^{\text{opt}}}{\partial t} + \nabla_{\boldsymbol{x}} \cdot \left(\rho_\varepsilon^{\text{opt}} \nabla_{\boldsymbol{x}} \psi_\varepsilon\right) = \varepsilon \Delta_{\boldsymbol{x}} \rho_\varepsilon^{\text{opt}}$$

$$\rho_\varepsilon^{\text{opt}}(t = t_0, \cdot) = \rho_0, \quad \rho_\varepsilon^{\text{opt}}(t = t_1, \cdot) = \rho_1$$

Structure of the solution

Boundary-coupled system of linear PDEs for the Schrödinger factors

***Reaction*-diffusion PDEs**

$$\frac{\partial \widehat{\varphi}_{\varepsilon}}{\partial t} = \left(\varepsilon \Delta_{\boldsymbol{x}} + \frac{1}{2\varepsilon} V(\boldsymbol{x})\right) \widehat{\varphi}_{\varepsilon} \leftarrow \mathcal{L}_{\text{forward}} \widehat{\varphi}$$

$$\frac{\partial \varphi_{\varepsilon}}{\partial t} = -\left(\varepsilon \Delta_{\boldsymbol{x}} + \frac{1}{2\varepsilon} V(\boldsymbol{x})\right) \varphi_{\varepsilon} \leftarrow \mathcal{L}_{\text{backward}} \varphi$$

$$\widehat{\varphi}_{\varepsilon}(\cdot, t = t_0)\varphi_{\varepsilon}(\cdot, t = t_0) = \rho_0$$

$$\widehat{\varphi}_{\varepsilon}(\cdot, t = t_1)\varphi_{\varepsilon}(\cdot, t = t_1) = \rho_1.$$

Optimally controlled joint state PDF

$$\rho_{\varepsilon}^{opt}(\cdot, t) = \hat{\varphi}_{\varepsilon}(\cdot, t)\varphi_{\varepsilon}(\cdot, t)$$

Optimal control

$$\boldsymbol{v}_{\varepsilon}^{\text{opt}}(\cdot, t) = 2\varepsilon \nabla_{\boldsymbol{x}} \log \varphi_{\varepsilon}(\cdot, t)$$

Algorithm

Recursion over pair $\left(\varphi_1, \hat{\varphi}_0\right)$

$$\begin{tikzcd}
\widehat{\varphi}_{\varepsilon,0}(\cdot) \arrow[r, "\int \mathcal{L}_{\text{forward}} \widehat{\varphi}"] & \widehat{\varphi}_{\varepsilon}(\cdot, t = t_1) \arrow[d, "\rho_1(\cdot)/\widehat{\varphi}_{\varepsilon}(\cdot, t = t_1)"] \\
\varphi_{\varepsilon}(\cdot, t = t_0) \arrow[u, "\rho_0(\cdot)/\varphi_{\varepsilon}(\cdot, t = t_0)"] & \varphi_{\varepsilon,1}(\cdot) \arrow[l, "\int \mathcal{L}_{\text{backward}} \varphi"']
\end{tikzcd}$$

Schrödinger system:

$$\rho_0(\boldsymbol{x}) = \widehat{\varphi}_{\varepsilon,0}(\boldsymbol{x}) \int_{\mathbb{R}^n} k(t_0, \boldsymbol{x}, t_1, \boldsymbol{y}) \varphi_{\varepsilon,1}(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}$$

$$\rho_1(\boldsymbol{x}) = \varphi_{\varepsilon,1}(\boldsymbol{x}) \int_{\mathbb{R}^n} k(t_0, \boldsymbol{y}, t_1, \boldsymbol{x}) \widehat{\varphi}_{\varepsilon,0}(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}$$

L-SBP Computation via Schrödinger Factors

Recursion over pair $(\varphi_1, \hat{\varphi}_0)$

$$\begin{tikzcd}
\widehat{\varphi}_{\varepsilon,0}(\cdot) \arrow[r, "\int \mathcal{L}_{\text{forward}}\widehat{\varphi}"] & \widehat{\varphi}_{\varepsilon}(\cdot, t = t_1) \arrow[d, "\rho_1(\cdot)/\widehat{\varphi}_{\varepsilon}(\cdot, t = t_1)"] \\
\varphi_{\varepsilon}(\cdot, t = t_0) \arrow[u, "\rho_0(\cdot)/\varphi_{\varepsilon}(\cdot, t = t_0)"] & \varphi_{\varepsilon,1}(\cdot) \arrow[l, "\int \mathcal{L}_{\text{backward}}\varphi"']
\end{tikzcd}$$

$$\frac{\partial \widehat{\varphi}_{\varepsilon}}{\partial t} = \left(\varepsilon \Delta_{\boldsymbol{x}} + \frac{1}{2\varepsilon} V(\boldsymbol{x})\right) \widehat{\varphi}_{\varepsilon} \quad \leftarrow \ \mathcal{L}_{\text{forward}}\widehat{\varphi}$$

$$\frac{\partial \varphi_{\varepsilon}}{\partial t} = -\left(\varepsilon \Delta_{\boldsymbol{x}} + \frac{1}{2\varepsilon} V(\boldsymbol{x})\right) \varphi_{\varepsilon} \quad \leftarrow \ \mathcal{L}_{\text{backward}}\varphi$$

$$\rho_{\varepsilon}^{\text{opt}}(t = t_0, \cdot) = \rho_0, \qquad \rho_{\varepsilon}^{\text{opt}}(t = t_1, \cdot) = \rho_1$$

Fredholm Integral Equation of 2nd Kind

Thm. (informal)

Solution of linear reaction-diffusion PDE IVP with state-dependent reaction rate:

$$\frac{\partial u}{\partial t} = a\Delta_{\boldsymbol{x}} u + q(\boldsymbol{x})u, \quad \boldsymbol{x} \in \mathbb{R}^n, \quad u(\boldsymbol{x}, t = t_0) = u_0(\boldsymbol{x}) \text{ given}$$

admits space-time Fredholm integral representation

$$u(\boldsymbol{x}, t) = \underbrace{\frac{1}{\sqrt{(4\pi a t)^n}} \int_{\mathbb{R}^n} \exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4at}\right) u_0(\boldsymbol{y}) d\boldsymbol{y}}_{\text{term 1}}$$

$$+ \underbrace{\int_{t_0}^{t} \frac{1}{\sqrt{(4\pi a(t-\tau))^n}} \int_{\mathbb{R}^n} \exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4a(t-\tau)}\right) q(\boldsymbol{y}) u(\boldsymbol{y}, \tau)\, d\boldsymbol{y}\, d\tau}_{\text{term 2}}$$

L-SBP Computation via Schrödinger Factors

Recursion over pair $(\varphi_1, \hat{\varphi}_0)$

$$\begin{tikzcd}
\widehat{\varphi}_{\varepsilon,0}(\cdot) \arrow[r, "\int \mathcal{L}_{\text{forward}}\widehat{\varphi}"] & \widehat{\varphi}_{\varepsilon}(\cdot, t = t_1) \arrow[d, "\rho_1(\cdot)/\widehat{\varphi}_{\varepsilon}(\cdot, t = t_1)"] \\
\varphi_{\varepsilon}(\cdot, t = t_0) \arrow[u, "\rho_0(\cdot)/\varphi_{\varepsilon}(\cdot, t = t_0)"] & \varphi_{\varepsilon,1}(\cdot) \arrow[l, "\int \mathcal{L}_{\text{backward}}\varphi"']
\end{tikzcd}$$

Thm. (Fredholm Integral Representation)

$$\widehat{\varphi}_{\varepsilon}(\boldsymbol{x}, t) = \frac{1}{\sqrt{(4\pi\varepsilon t)^3}} \int_{\mathbb{R}^3} \exp\left(-\frac{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|^2}{4\varepsilon t}\right) \widehat{\varphi}_{\varepsilon,0}(\tilde{\boldsymbol{x}})\, d\tilde{\boldsymbol{x}}$$

$$- \int_{t_0}^{t} \frac{1}{2\varepsilon\sqrt{(4\pi\varepsilon(t-\tau))^3}} \int_{\mathbb{R}^3} \exp\left(-\frac{|\boldsymbol{x} - \tilde{\boldsymbol{x}}|^2}{4\varepsilon(t-\tau)}\right) V(\tilde{\boldsymbol{x}}) \widehat{\varphi}_{\varepsilon}(\tilde{\boldsymbol{x}}, \tau)\, d\tilde{\boldsymbol{x}}\, d\tau$$

Solution: Computation

Recursion over pair $\left(\varphi_1, \hat{\varphi}_0\right)$

$$\begin{tikzcd}
\widehat{\varphi}_{\varepsilon,0}(\cdot) \arrow[r, "\int \mathcal{L}_{\text{forward}} \widehat{\varphi}"] & \widehat{\varphi}_{\varepsilon}(\cdot, t = t_1) \arrow[d, "\rho_1(\cdot)/\widehat{\varphi}_{\varepsilon}(\cdot, t = t_1)"] \\
\varphi_{\varepsilon}(\cdot, t = t_0) \arrow[u, "\rho_0(\cdot)/\varphi_{\varepsilon}(\cdot, t = t_0)"] & \varphi_{\varepsilon,1}(\cdot) \arrow[l, "\int \mathcal{L}_{\text{backward}} \varphi"]
\end{tikzcd}$$

Idea:

Left Riemann Approximation of Second Term in Fredholm Integral Representation

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^n} f(\widetilde{\boldsymbol{x}}, \boldsymbol{x}, \tau, t) d\widetilde{\boldsymbol{x}} d\tau$$

$$\approx \sum_{q=0}^{k-1} \sum_{m=0}^{N_x} \sum_{n=0}^{N_y} \sum_{j=0}^{N_z} f(\widetilde{\boldsymbol{x}}_{(m,n,j)}, \boldsymbol{x}, t_0 + k\Delta t, t) \Delta z \Delta y \Delta x \Delta t$$

where $\widetilde{\boldsymbol{x}}_{(m,n,j)} = (x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$

Numerical Case Study

Prescribed time horizon $\left[t_0, t_1\right] \equiv \left[0,1\right]$ hours

Endpoint joint PDFs $\boldsymbol{x}_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$ and $\boldsymbol{x}_1 \sim \mathcal{N}(\mu_1, \Sigma_1)$

where $\mu_0 = (5000 \quad 10000 \quad 2100)^\top$, $\mu_1 = (-14600 \quad 2500 \quad 7000)^\top$

and $\Sigma_0 = \dfrac{1}{100}\operatorname{diag}\left(\mu_0^2\right)$, $\Sigma_1 = \dfrac{1}{100}\operatorname{diag}\left(\mu_1^2\right)$.

Numerical Case Study

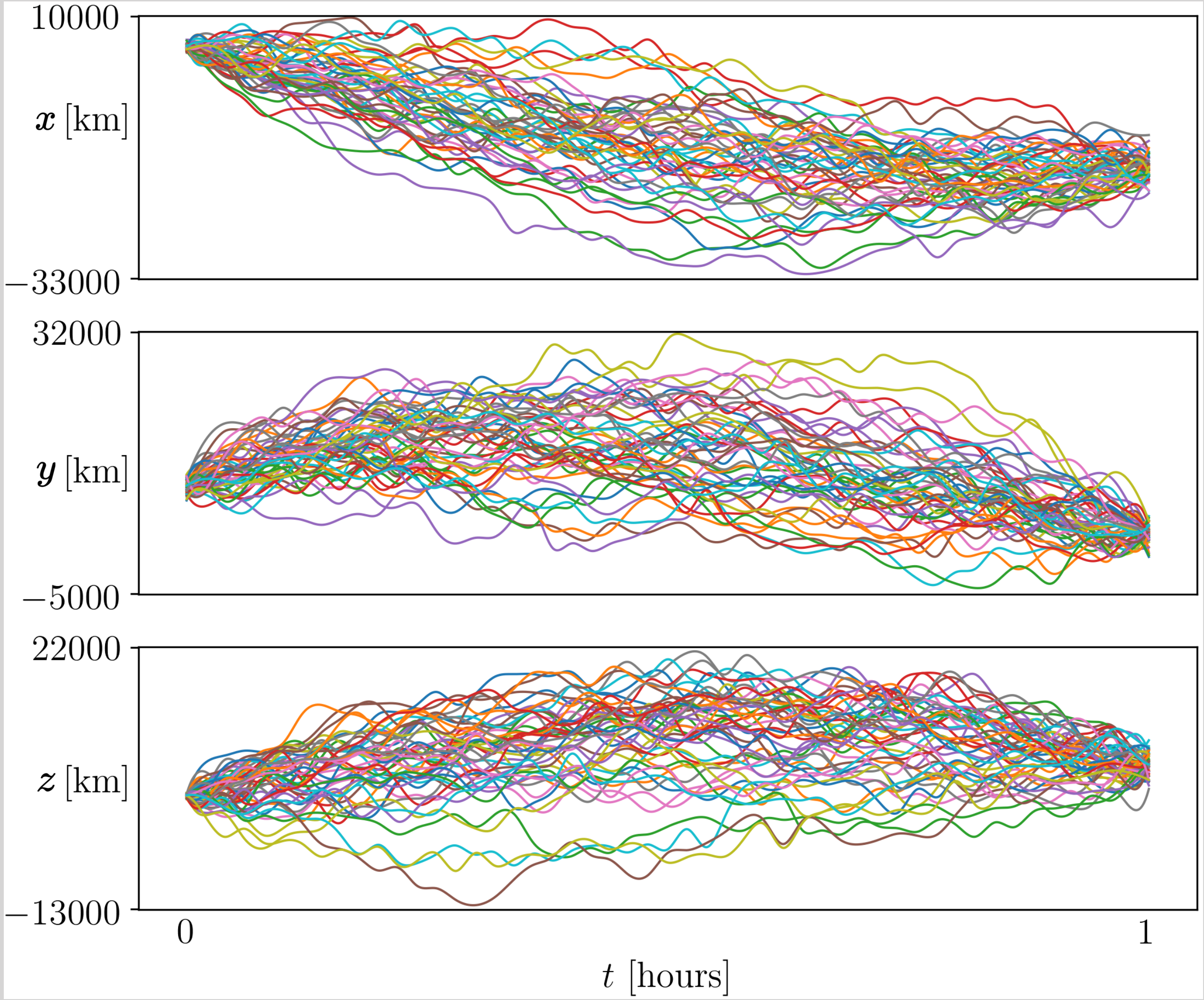
Prescribed time horizon $[t_0, t_1] \equiv [0,1]$ hours

Endpoint joint PDFs $\boldsymbol{x}_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$ and $\boldsymbol{x}_1 \sim \mathcal{N}(\mu_1, \Sigma_1)$

where $\mu_0 = (5000 \quad 10000 \quad 2100)^\top, \quad \mu_1 = (-14600 \quad 2500 \quad 7000)^\top$

and $\Sigma_0 = \dfrac{1}{100}\text{diag}\left(\mu_0^2\right), \; \Sigma_1 = \dfrac{1}{100}\text{diag}\left(\mu_1^2\right).$

Optimally controlled closed loop state sample paths



Numerical Case Study (cont.)

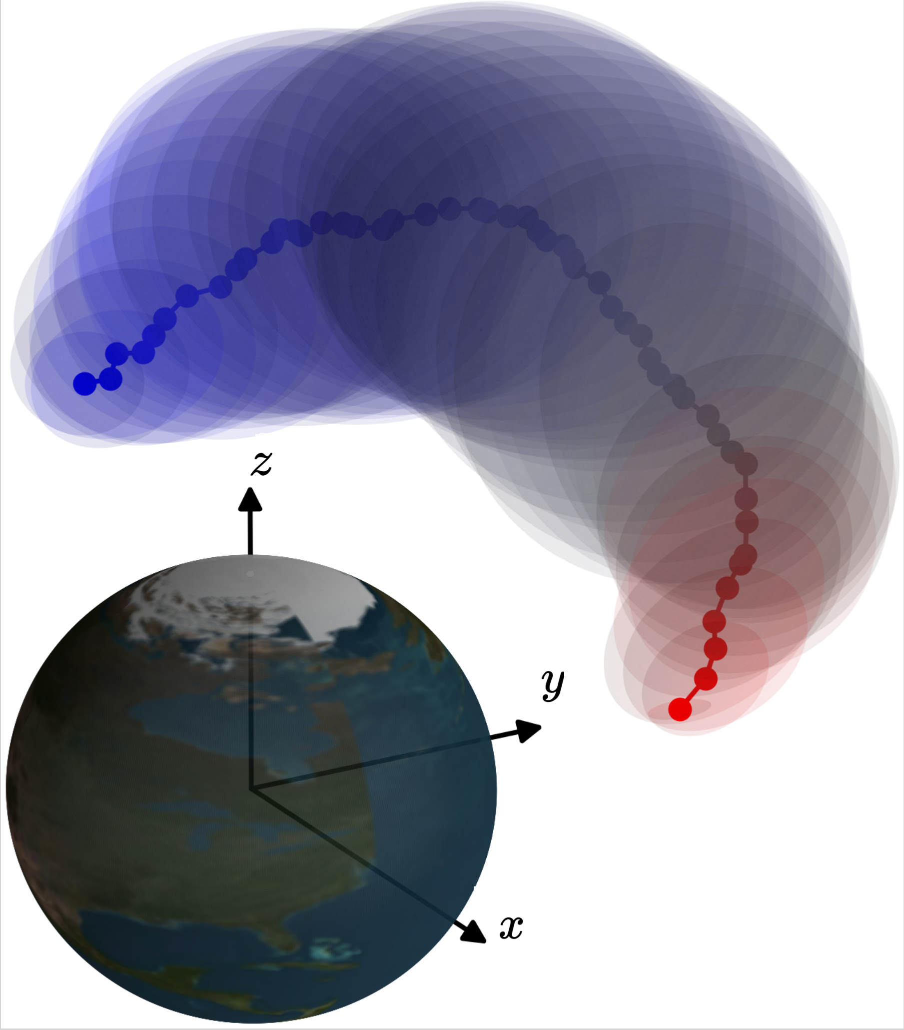
Prescribed time horizon $\left[t_0, t_1\right] \equiv \left[0,1\right]$ hours

Endpoint joint PDFs $\boldsymbol{x}_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$ and $\boldsymbol{x}_1 \sim \mathcal{N}(\mu_1, \Sigma_1)$

where $\mu_0 = (5000 \quad 10000 \quad 2100)^\top$, $\mu_1 = (-14600 \quad 2500 \quad 7000)^\top$

and $\Sigma_0 = \dfrac{1}{100}\mathrm{diag}\left(\mu_0^2\right)$, $\Sigma_1 = \dfrac{1}{100}\mathrm{diag}\left(\mu_1^2\right)$.

Mean position snapshots for 50 optimally controlled sample paths in $\mathbb{R}^3$



Numerical Case Study (cont.)

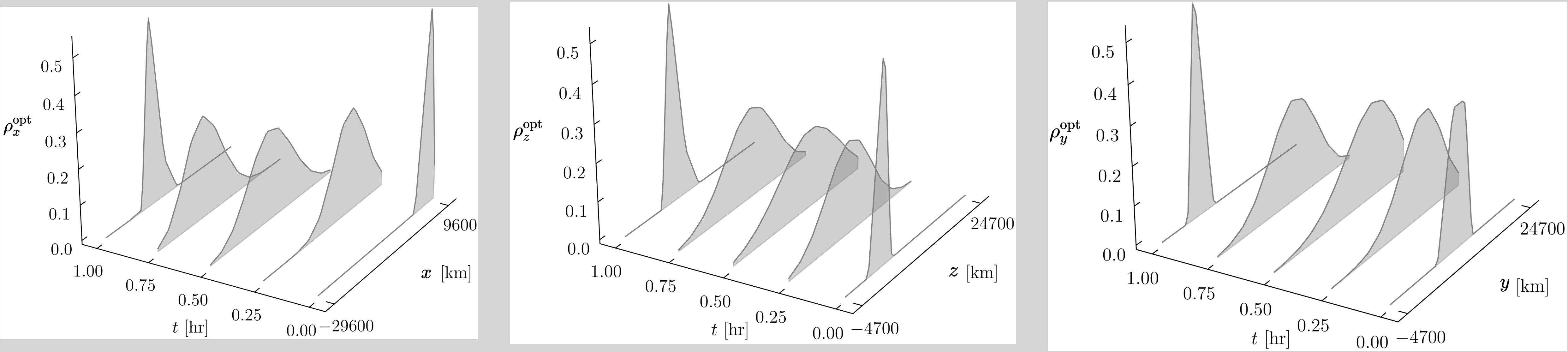
Prescribed time horizon $\left[t_0, t_1\right] \equiv \left[0, 1\right]$ hours

Endpoint joint PDFs $\boldsymbol{x}_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$ and $\boldsymbol{x}_1 \sim \mathcal{N}(\mu_1, \Sigma_1)$

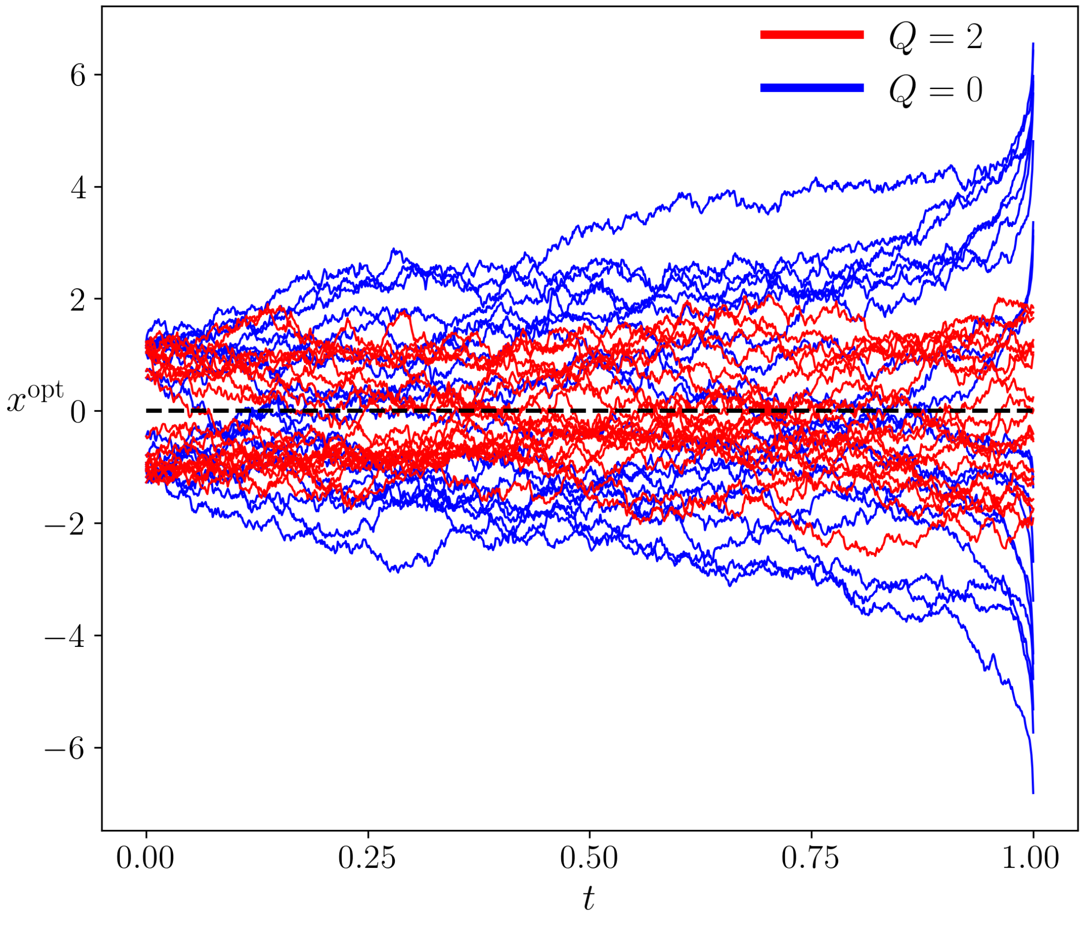
where $\mu_0 = \begin{pmatrix} 5000 & 10000 & 2100 \end{pmatrix}^{\top}$, $\mu_1 = \begin{pmatrix} -14600 & 2500 & 7000 \end{pmatrix}^{\top}$

and $\Sigma_0 = \dfrac{1}{100}\text{diag}\left(\mu_0^2\right)$, $\Sigma_1 = \dfrac{1}{100}\text{diag}\left(\mu_1^2\right)$.

Univariate marginals for optimally controlled joint PDFs



SBP with *Quadratic* State Cost



SBP with Quadratic State Cost

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\left(\frac{1}{2}|\boldsymbol{v}|^2+\frac{1}{2}\boldsymbol{x}^\top\boldsymbol{Q}\boldsymbol{x}\right)\rho(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$\frac{\partial\rho}{\partial t}+\nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v})=\Delta_{\boldsymbol{x}}\rho,$$

$$\boldsymbol{x}(t=t_0)\sim\rho_0\;(\text{given}),\;\boldsymbol{x}(t=t_1)\sim\rho_1\;(\text{given})$$

where $\boldsymbol{Q}\succeq\boldsymbol{0}\longrightarrow\frac{1}{2}\boldsymbol{Q}=\boldsymbol{V}^\top\boldsymbol{\Lambda}\boldsymbol{V}$

Hopf-Cole + additional change of variable

$$\boldsymbol{y}:=\boldsymbol{V}\boldsymbol{x}$$

$$\hat{\nu}(\boldsymbol{y},\,t):=\widehat{\varphi}(\boldsymbol{x}=\boldsymbol{V}^\top\boldsymbol{y},\,t)$$

SBP with Quadratic State Cost

Hopf-Cole + additional change of variable

$$\boldsymbol{y} := \boldsymbol{V}\boldsymbol{x}$$

$$\hat{\nu}(\boldsymbol{y},\ t) := \widehat{\varphi}(\boldsymbol{x} = \boldsymbol{V}^\top \boldsymbol{y},\ t)$$

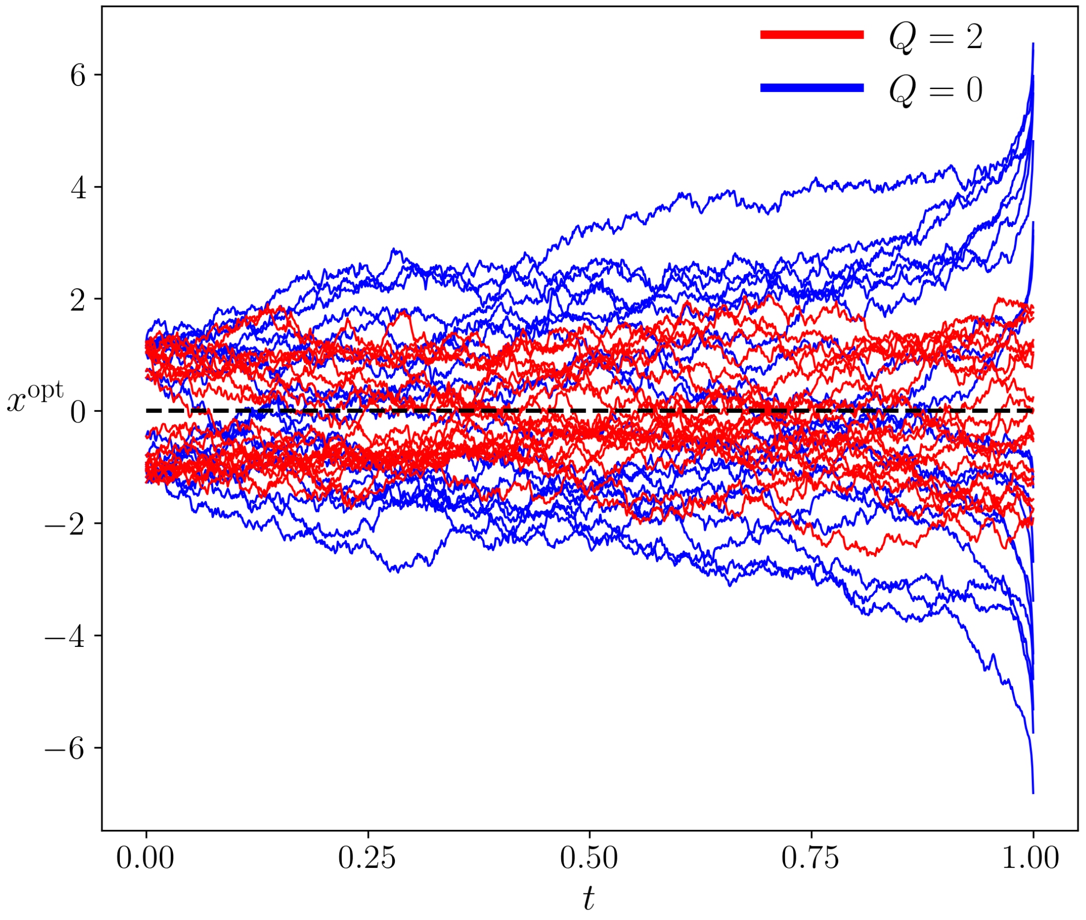
New PDE

$$\begin{aligned}\frac{\partial \hat{\nu}}{\partial t} &= \left(\Delta_{\boldsymbol{y}} - \boldsymbol{y}^\top \boldsymbol{\Lambda} \boldsymbol{y}\right)\hat{\nu} \\ &= \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial y_k^2} - \lambda_k y_k^2 \right) \hat{\nu}\end{aligned}$$

where

$$\hat{\nu}_0(\boldsymbol{y}) = \hat{\nu}(\boldsymbol{y}, 0) = \widehat{\varphi}(\boldsymbol{x} = \boldsymbol{V}^\top \boldsymbol{y}, 0)$$

SBP with *Quadratic* State Cost via Separation-of-Variables



Separation-of-variables approach

$$\begin{aligned}\frac{\partial \hat{\nu}}{\partial t} &= (\Delta_{\boldsymbol{y}} - \boldsymbol{y}^\top \boldsymbol{\Lambda} \boldsymbol{y})\hat{\nu} \\ &= \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial y_k^2} - \lambda_k y_k^2 \right) \hat{\nu}\end{aligned}$$

$$\hat{\nu}(\boldsymbol{y}, t) = T(t) \prod_{i=1}^{n} Y_i(y_i)$$

$$\begin{cases} \dfrac{\mathrm{d}T}{\mathrm{d}t} = cT, \\ \dfrac{\mathrm{d}^2 Y_1}{\mathrm{d}y_1^2} - (\lambda_1 y_1^2 + c_1) Y_1 = 0, \\ \vdots \\ \dfrac{\mathrm{d}^2 Y_n}{\mathrm{d}y_n^2} - (\lambda_n y_n^2 + c_n) Y_n = 0 \end{cases}$$

Separation-of-variables approach

Solutions to PDE

$$\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} - (\lambda y^2 + c)Y = 0$$

are of the form

$$Y = a \exp\left(-\frac{y^2\sqrt{\lambda}}{2}\right) H_n\left(\lambda^{1/4} y\right)$$

with degree

$$n = -\frac{c}{2\sqrt{\lambda}} - \frac{1}{2} \in \mathbb{N}_0, \ a \in \mathbb{R}$$

Result for $\boldsymbol{Q} \succ \mathbf{0}$

Schrödinger factor in transformed coordinates

$$\hat{\nu}(\boldsymbol{y},\ t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \kappa_{++}(t_0, \boldsymbol{y}, t, \boldsymbol{z})\ \hat{\nu}_0(\boldsymbol{z})\ \mathrm{d}z_1 \dots \mathrm{d}z_n$$

for

$$\kappa_{++}(t_0, \boldsymbol{y}, t, \boldsymbol{z}) = \frac{(\det(\boldsymbol{M}_{tt_0}))^{1/4} \times \exp\left(-\frac{1}{2} (\boldsymbol{y}^\top \quad \boldsymbol{z}^\top) \boldsymbol{M}_{tt_0} \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} \right)}{(2\pi)^{n/2} \sqrt{\det(\sinh(2(t - t_0)\sqrt{\boldsymbol{\Lambda}}))}}$$

where

$$\boldsymbol{M}_{tt_0} = \begin{bmatrix} \sqrt{\boldsymbol{\Lambda}} \coth\left(2(t - t_0)\sqrt{\boldsymbol{\Lambda}}\right) & -\sqrt{\boldsymbol{\Lambda}} \operatorname{csch}\left(2(t - t_0)\sqrt{\boldsymbol{\Lambda}}\right) \\ -\sqrt{\boldsymbol{\Lambda}} \operatorname{csch}\left(2(t - t_0)\sqrt{\boldsymbol{\Lambda}}\right) & \sqrt{\boldsymbol{\Lambda}} \coth\left(2(t - t_0)\sqrt{\boldsymbol{\Lambda}}\right) \end{bmatrix}.$$

Result for $\boldsymbol{Q} \succeq \mathbf{0}$

Schrödinger factor in transformed coordinates

$$\hat{\nu}(\boldsymbol{y},\ t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \kappa_{+}(t_0, \boldsymbol{y}, t, \boldsymbol{z})\, \hat{\nu}_0(\boldsymbol{z})\, \mathrm{d}z_1 \dots \mathrm{d}z_n$$

where

$$\begin{aligned}\kappa_{+}(t_0, \boldsymbol{y}, t, \boldsymbol{z}) = & \kappa_{++}\left(t_0, \boldsymbol{y}_{[i_1:i_{n-p}]}, t, \boldsymbol{z}_{[i_1:i_{n-p}]}\right) \\ & \times \kappa_0\left(t_0, \boldsymbol{y}_{[i_{n-p+1}:i_n]}, t, \boldsymbol{z}_{[i_{n-p+1}:i_n]}\right)\end{aligned}$$

Recover Schrödinger factor in original coordinates:

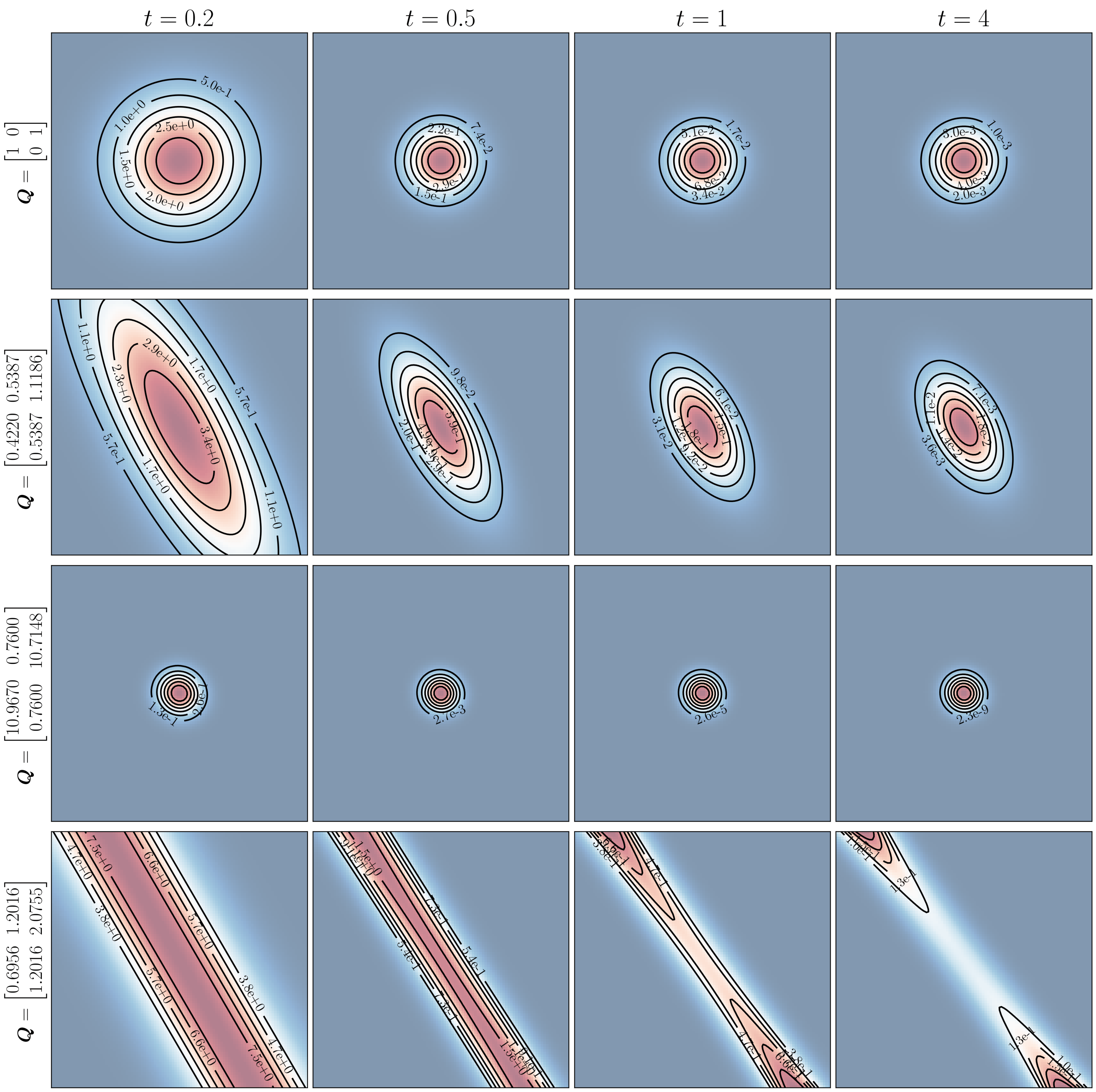
$$\hat{\nu}(\boldsymbol{y} = \boldsymbol{V}\boldsymbol{x},\ t) := \widehat{\varphi}(\boldsymbol{x},\ t)$$

Numerical simulation

$$\frac{\partial \widehat{\varphi}}{\partial t} = \left(\Delta_{\boldsymbol{x}} - \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x}\right)\widehat{\varphi}$$

Initial condition

$$\widehat{\varphi}_0(\boldsymbol{x}) = \mathbf{1}$$

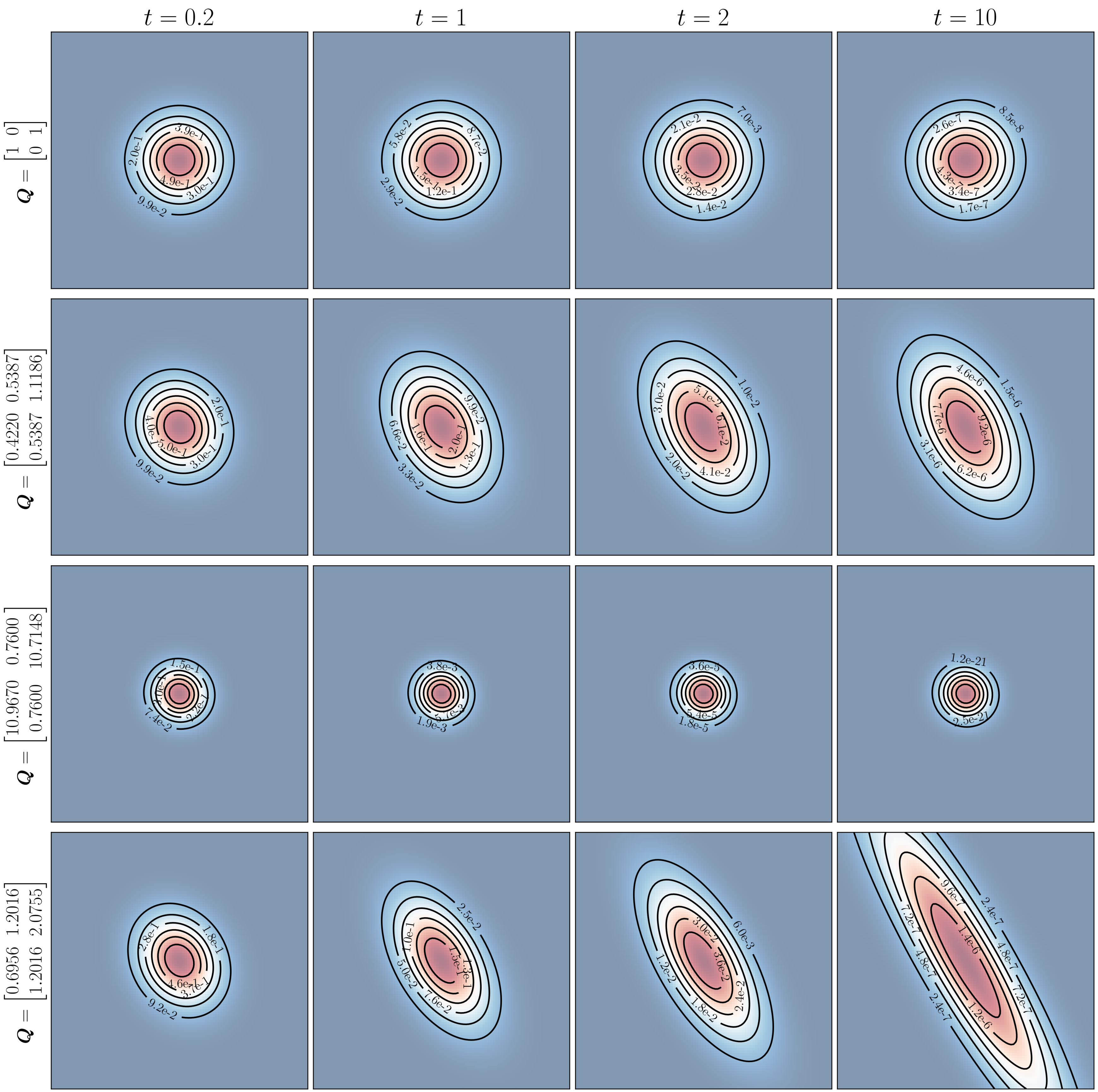


Numerical simulation

$$\frac{\partial \widehat{\varphi}}{\partial t} = (\Delta_{\boldsymbol{x}} - \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x})\widehat{\varphi}$$

Initial condition

$$\widehat{\varphi}_0(\boldsymbol{x}) \sim \mathcal{N}(0, \boldsymbol{I})$$



Numerical simulation

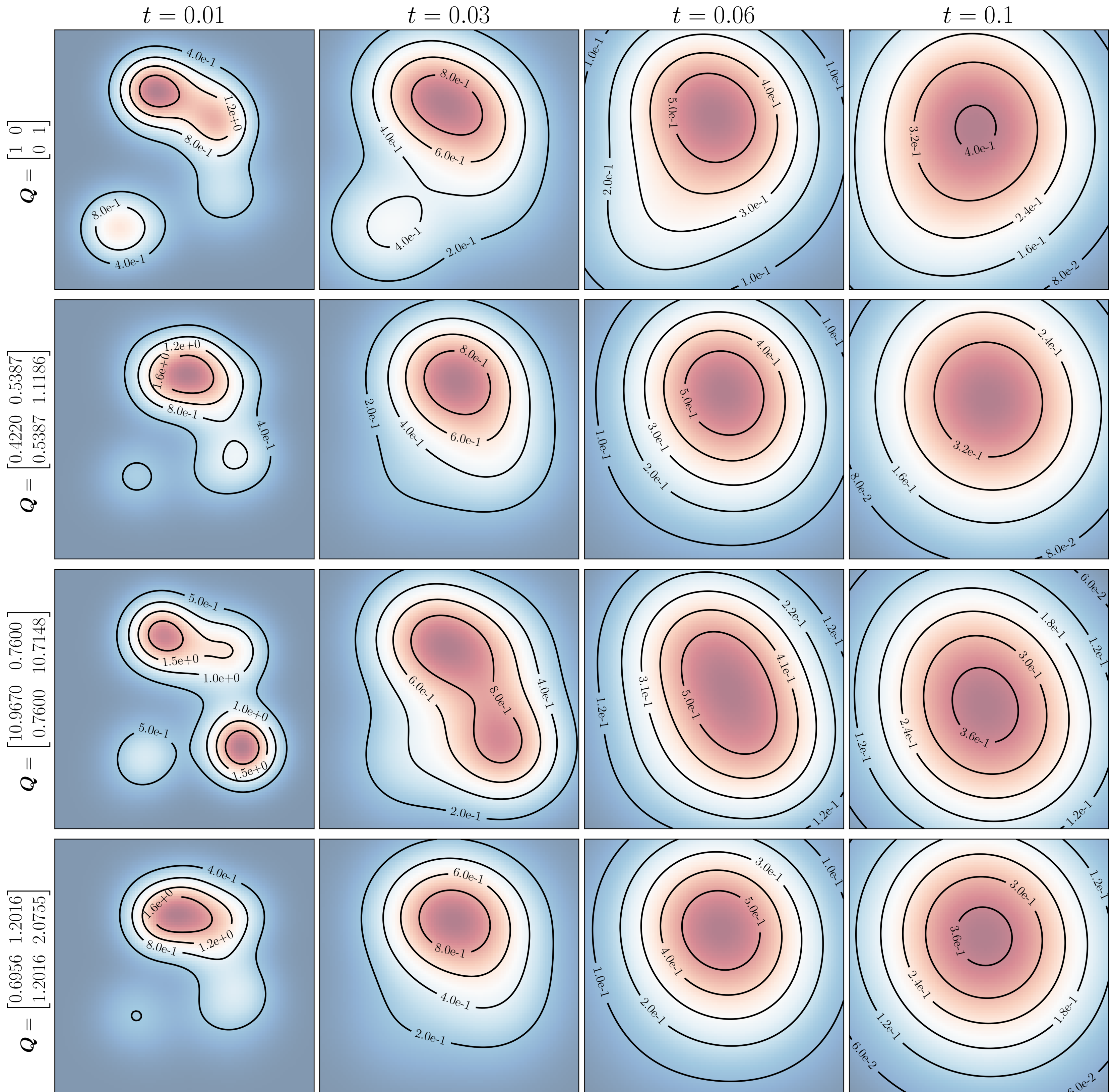
scaled Himmelblau function

$$\widehat{\varphi}_0(\boldsymbol{x}) \propto \exp\left(-\frac{\overset{\text{scaled Himmelblau function}}{f}(x_1, x_2)}{35}\right)$$

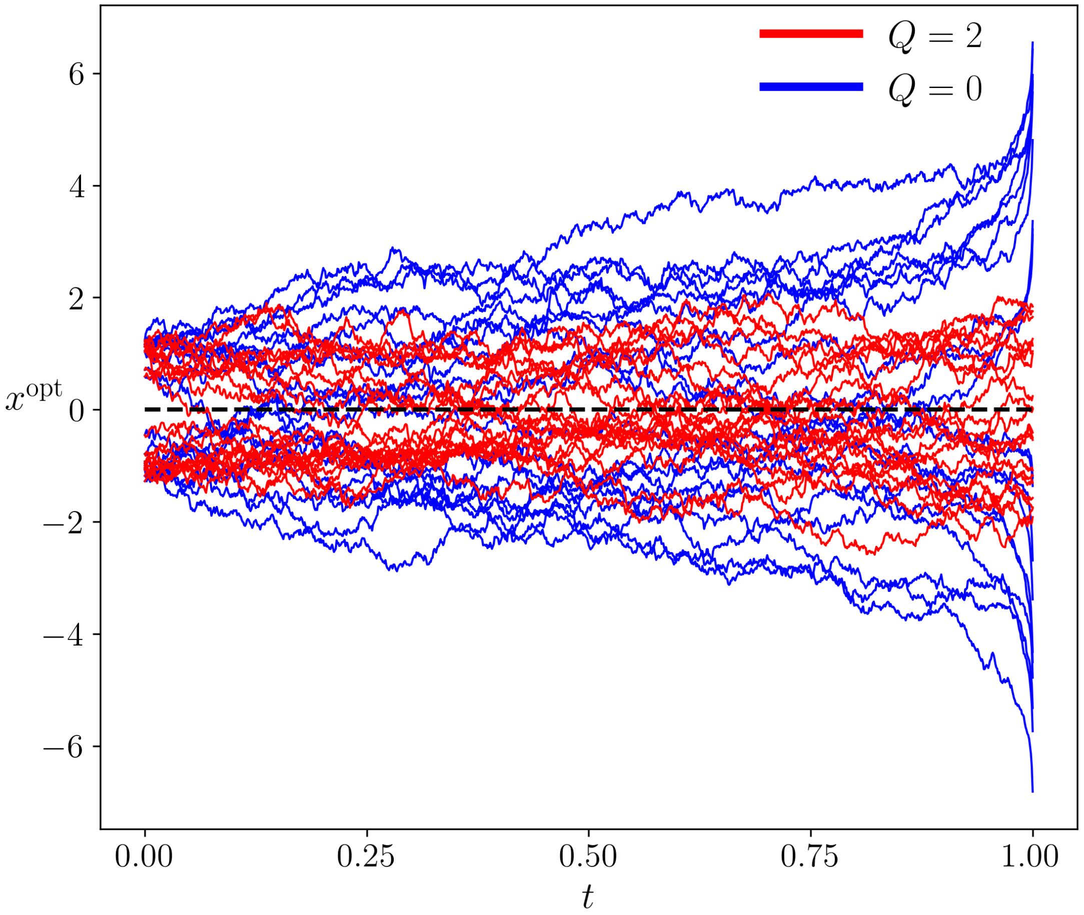
$$\frac{\partial \widehat{\varphi}}{\partial t} = \left(\Delta_{\boldsymbol{x}} - \boldsymbol{x}^\top \boldsymbol{Q} \boldsymbol{x}\right)\widehat{\varphi}$$

Initial condition

Non-Gaussian $\widehat{\varphi}_0(\boldsymbol{x})$



SBP with *Quadratic* State Cost
Weyl Calculus Approach



PDE to Weyl Operator

Operators

$$X_k := x_k \quad \forall k \in [n]$$

$$D_k := \frac{1}{\mathrm{i}} \frac{\partial}{\partial x_k} \quad \forall k \in [n]$$

Define

$$\boldsymbol{X} := \begin{pmatrix} X_1 & \dots & X_n \end{pmatrix}^\top, \quad \boldsymbol{D} := \begin{pmatrix} D_1 & \dots & D_n \end{pmatrix}^\top$$

Observe that

$$|\boldsymbol{D}|^2 := \langle \boldsymbol{D}, \boldsymbol{D} \rangle = (-\iota)^2 (\nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}}) = -\Delta_{\boldsymbol{x}}$$

PDE to Weyl Operator

$$\frac{\partial}{\partial t}\widehat{\varphi} = -\mathcal{L}\widehat{\varphi} \longrightarrow \widehat{\varphi} = \exp\left(-(t-t_0)\mathcal{L}\right)\widehat{\varphi}_0$$

Classical SBP

$$\frac{\partial}{\partial t}\widehat{\vartheta} = \Delta_{\boldsymbol{x}}\widehat{\vartheta}$$

$$\downarrow$$

$$\begin{aligned} &H_{\text{heat}}(\boldsymbol{X}, \boldsymbol{D}) \\ &\quad = \exp\left(-(t-t_0)|\boldsymbol{D}|^2\right) \end{aligned}$$

SBP with Quadratic State Cost

$$\begin{aligned} \frac{\partial \hat{\nu}}{\partial t} &= \Delta_{\boldsymbol{y}}\hat{\nu} - \left(\boldsymbol{y}^\top \boldsymbol{\Lambda} \boldsymbol{y}\right)\hat{\nu} \\ &= \sum_{k=1}^{n}\left(\frac{\partial^2}{\partial y_k^2} - \lambda_k y_k^2\right)\hat{\nu} \end{aligned}$$

$$\downarrow$$

$$H_{\boldsymbol{\Lambda}}(\boldsymbol{X}, \boldsymbol{D}) = \exp\left(-(t-t_0)Q_{\boldsymbol{\Lambda}}(\boldsymbol{X}, \boldsymbol{D})\right)$$

where

$$Q_{\boldsymbol{\Lambda}}(\boldsymbol{X}, \boldsymbol{D}) := |\boldsymbol{D}|^2 + \sum_{k=1}^{n} \lambda_k X_k^2$$

Weyl Operator to Weyl Symbol

1.) Rewrite $H(\boldsymbol{X}, \boldsymbol{D})$ **with**

Commutation Relation

$$[X_k, D_k] := X_k D_k - D_k X_k = \iota, \quad k \in [n]$$

2.) Let $H(\boldsymbol{X}, \boldsymbol{D}) = R(\boldsymbol{x}, \boldsymbol{\xi})$

3.) Calculate the Weyl symbol

$$\begin{aligned} h(\boldsymbol{x}, \boldsymbol{\xi}) &= \frac{1}{\pi^n} \int_{\mathbb{R}^{2n}} R(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{\xi}}) \exp\Big(2\mathrm{i}\langle \tilde{\boldsymbol{x}} - \boldsymbol{x}, \tilde{\boldsymbol{\xi}} - \boldsymbol{\xi}\rangle\Big) \mathrm{d}\tilde{\boldsymbol{x}}\, \mathrm{d}\tilde{\boldsymbol{\xi}}, \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\iota}{2}\right)^m \left(\frac{\partial^m}{\partial \boldsymbol{x}^m} \cdot \frac{\partial^m}{\partial \boldsymbol{\xi}^m}\right) R(\boldsymbol{x}, \boldsymbol{\xi}). \end{aligned}$$

Weyl Operator to Weyl Symbol

Classical SBP

$$H_{\text{heat}}(\boldsymbol{X}, \boldsymbol{D}) = \exp\left(-(t - t_0)|\boldsymbol{D}|^2\right)$$

$$h_{\text{heat}}(\boldsymbol{x}, \boldsymbol{\xi}) = \exp\left(-(t - t_0)|\boldsymbol{\xi}|^2\right)$$

SBP with Quadratic State Cost

$$H_{\boldsymbol{\Lambda}}(\boldsymbol{X}, \boldsymbol{D}) = \exp\left(-(t - t_0)Q_{\boldsymbol{\Lambda}}(\boldsymbol{X}, \boldsymbol{D})\right)$$

where

$$Q_{\boldsymbol{\Lambda}}(\boldsymbol{X}, \boldsymbol{D}) := |\boldsymbol{D}|^2 + \sum_{k=1}^{n} \lambda_k X_k^2$$

???

Product Rule of Weyl Calculus

1.) Weyl operator must satisfy PDE:

$$\frac{\partial}{\partial t} H_{\boldsymbol{\Lambda}}(\boldsymbol{X}, \boldsymbol{D}) = -Q_{\boldsymbol{\Lambda}}(\boldsymbol{X}, \boldsymbol{D}) H_{\boldsymbol{\Lambda}}(\boldsymbol{X}, \boldsymbol{D})$$

2.) Using

Product Rule

$$C(\boldsymbol{X}, \boldsymbol{D}) = A(\boldsymbol{X}, \boldsymbol{D}) B(\boldsymbol{X}, \boldsymbol{D}) \longrightarrow c(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{j=0}^{d_A \wedge d_B} \frac{1}{j!} \{a, b\}_j(\boldsymbol{x}, \boldsymbol{\xi})$$

rewrite RHS in terms of Weyl symbols

3.) Use

Generalized Poisson Bracket

$$\{f, g\}_j(\boldsymbol{x}, \boldsymbol{\xi}) := \left(\frac{1}{2\iota}\right)^j \left(\sum_{k=1}^{n} \left(\frac{\partial^2}{\partial y_k \partial \xi_k} - \frac{\partial^2}{\partial x_k \partial \nu_k} \right) \right)^j f(\boldsymbol{x}, \boldsymbol{\xi}) g(\boldsymbol{y}, \boldsymbol{\eta}) \Big|_{\boldsymbol{y}=\boldsymbol{x}, \boldsymbol{\eta}=\boldsymbol{\xi}}$$

get a solvable system of PDEs

Weyl Operator to Weyl Symbol

Classical SBP

$$H_{\text{heat}}(\boldsymbol{X}, \boldsymbol{D}) = \exp\left(-(t - t_0)|\boldsymbol{D}|^2\right)$$

$$h_{\text{heat}}(\boldsymbol{x}, \boldsymbol{\xi}) = \exp\left(-(t - t_0)|\boldsymbol{\xi}|^2\right)$$

SBP with Quadratic State Cost

$$H_{\boldsymbol{\Lambda}}(\boldsymbol{X}, \boldsymbol{D}) = \exp\left(-(t - t_0)Q_{\boldsymbol{\Lambda}}(\boldsymbol{X}, \boldsymbol{D})\right)$$

where

$$Q_{\boldsymbol{\Lambda}}(\boldsymbol{X}, \boldsymbol{D}) := |\boldsymbol{D}|^2 + \sum_{k=1}^{n} \lambda_k X_k^2$$

$$h_{\boldsymbol{\Lambda}}(\boldsymbol{x}, \boldsymbol{\xi}) = \left(\prod_{k=1}^{n} \frac{1}{\cosh\left(\sqrt{\lambda_k}(t - t_0)\right)}\right) \times \exp\left(-\sum_{k=1}^{n} \frac{\lambda_k x_k^2 + \xi_k^2}{\sqrt{\lambda_k}} \tanh\left(\sqrt{\lambda_k}(t - t_0)\right)\right)$$

Weyl Symbol to Kernel

$$\kappa(t_0, \boldsymbol{x}, t, \boldsymbol{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} h\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}, \boldsymbol{\xi}\right) e^{\mathrm{i}\langle \boldsymbol{x}-\boldsymbol{y}, \boldsymbol{\xi}\rangle} \mathrm{d}\boldsymbol{\xi}$$

Classical SBP

$$\begin{aligned} &h_{\text{heat}}(\boldsymbol{x}, \boldsymbol{\xi}) \\ &\qquad = \exp(-(t-t_0)|\boldsymbol{\xi}|^2) \end{aligned}$$

$$\begin{aligned} &\kappa_{\text{heat}}(t_0, \boldsymbol{x}, t, \boldsymbol{y}) \\ &\qquad = \frac{1}{(4\pi(t-t_0))^{n/2}} \exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4(t-t_0)}\right) \end{aligned}$$

Weyl Symbol to Kernel

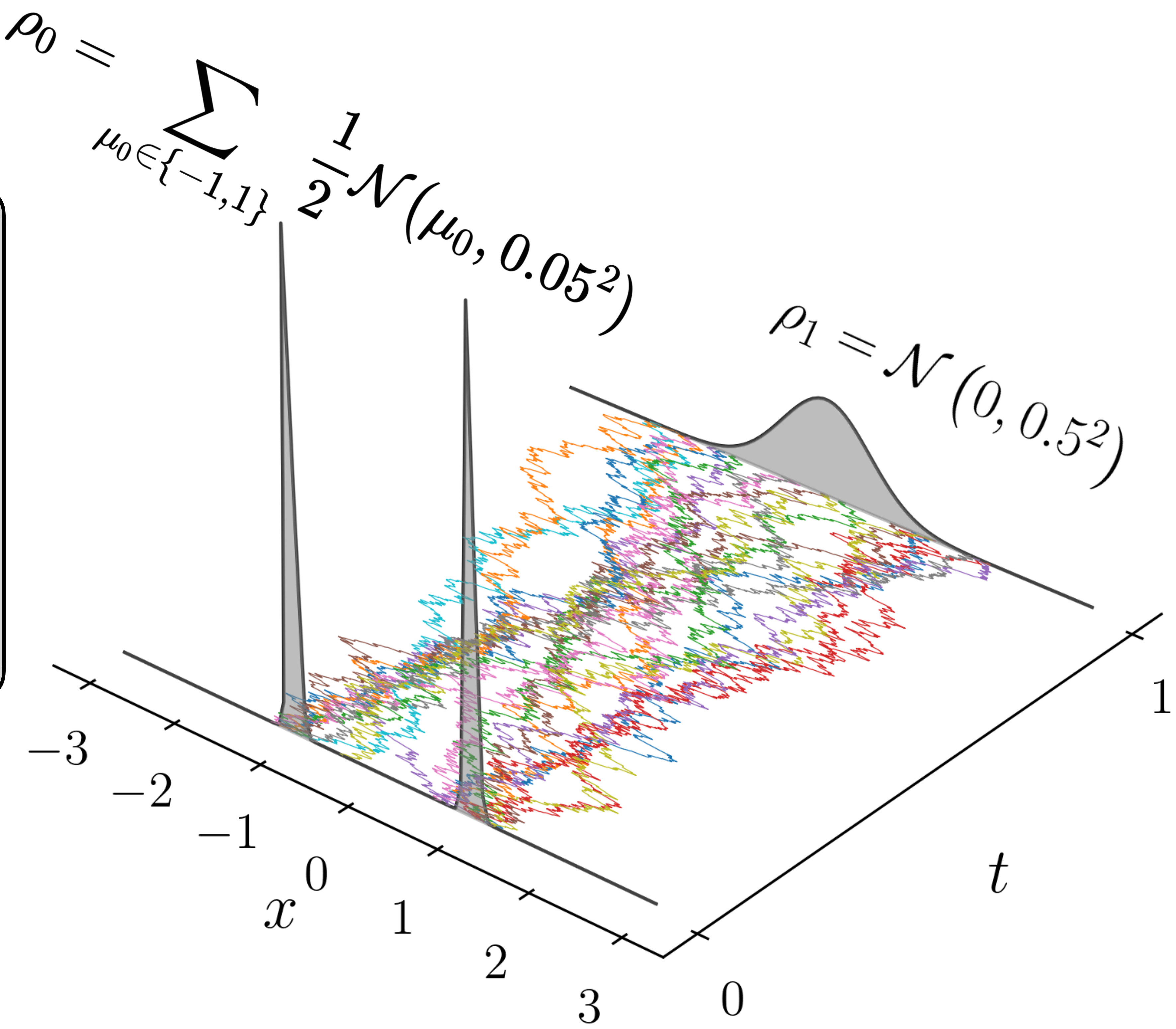
SBP with Quadratic State Cost

$$h_{\boldsymbol{\Lambda}}(\boldsymbol{x}, \boldsymbol{\xi}) = \left(\prod_{k=1}^{n} \frac{1}{\cosh\left(\sqrt{\lambda_k}(t - t_0)\right)} \right) \\ \times \exp\left(- \sum_{k=1}^{n} \frac{\lambda_k x_k^2 + \xi_k^2}{\sqrt{\lambda_k}} \tanh\left(\sqrt{\lambda_k}(t - t_0)\right) \right)$$

$$\kappa_{\boldsymbol{\Lambda}}(t_0, \boldsymbol{x}, t, \boldsymbol{y}) \\ = \left(\prod_{k=1}^{n} \frac{\lambda_k^{1/4}}{\sqrt{2\pi \sinh\left(2\sqrt{\lambda_k}(t - t_0)\right)}} \right) \\ \times \exp\left(- \sum_{k=1}^{n} \frac{\sqrt{\lambda_k}}{2} (x_k^2 + y_k^2) \frac{\cosh\left(2\sqrt{\lambda_k}(t - t_0)\right)}{\sinh\left(2\sqrt{\lambda_k}(t - t_0)\right)} \right) \\ \times \exp\left(\sum_{k=1}^{n} \sqrt{\lambda_k} x_k y_k \left(\frac{1}{\sinh\left(2\sqrt{\lambda_k}(t - t_0)\right)} \right) \right).$$

Numerical simulations

Optimally controlled sample paths for 1D Schrödinger bridge with quadratic state cost where $\boldsymbol{Q} = 2$



Additional avenues of research

↳ **Kernel for Keplerian SBP**

↳ **Kernel for LQ SBP**

$$\kappa = c(t, t_0)\exp\left(-\frac{1}{2}\mathrm{dist}^2_{tt_0}(\boldsymbol{x}, \boldsymbol{y})\right)$$

Thank You