Solution of the Probabilistic Lambert Problem: Optimal Transport Approach

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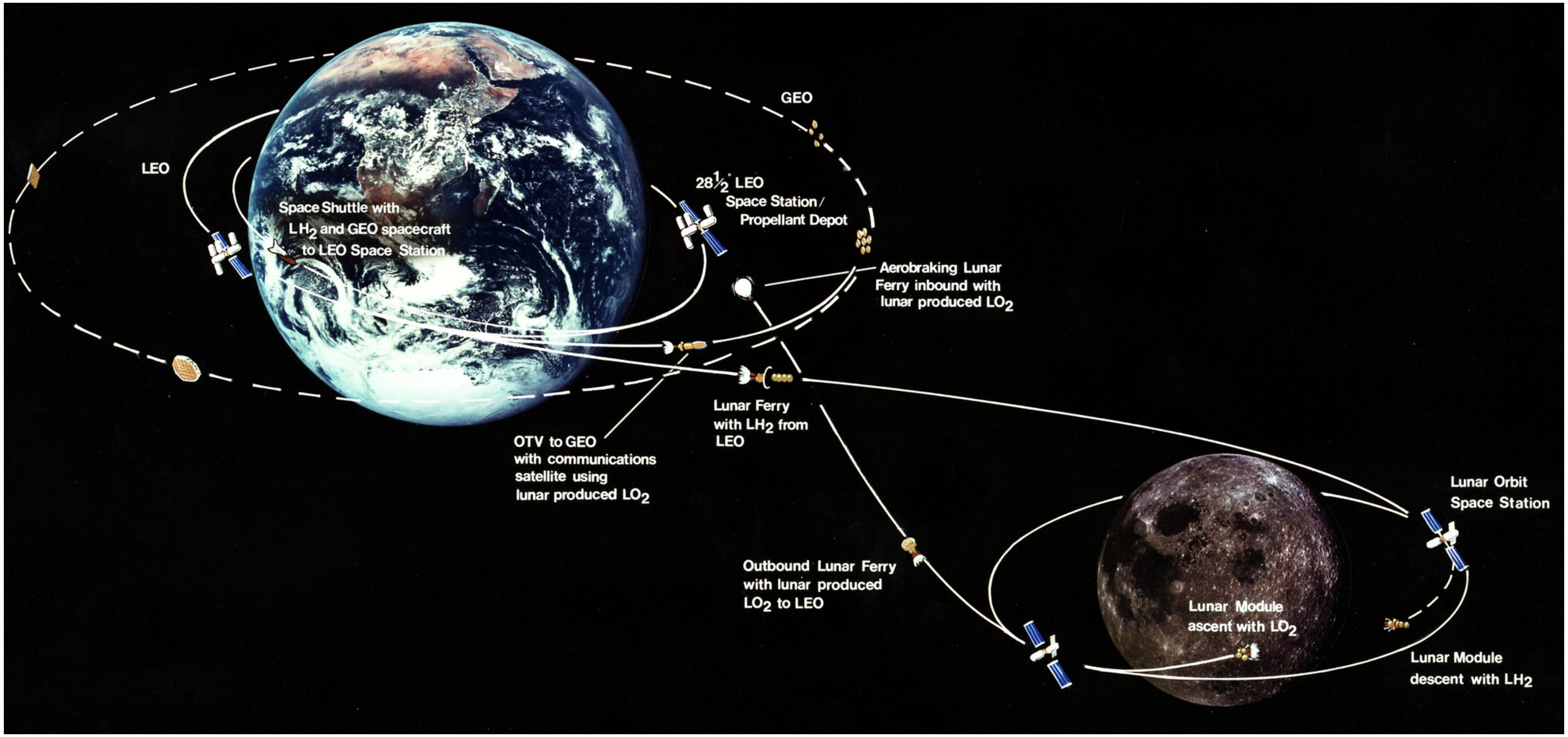
Lambert's Problem

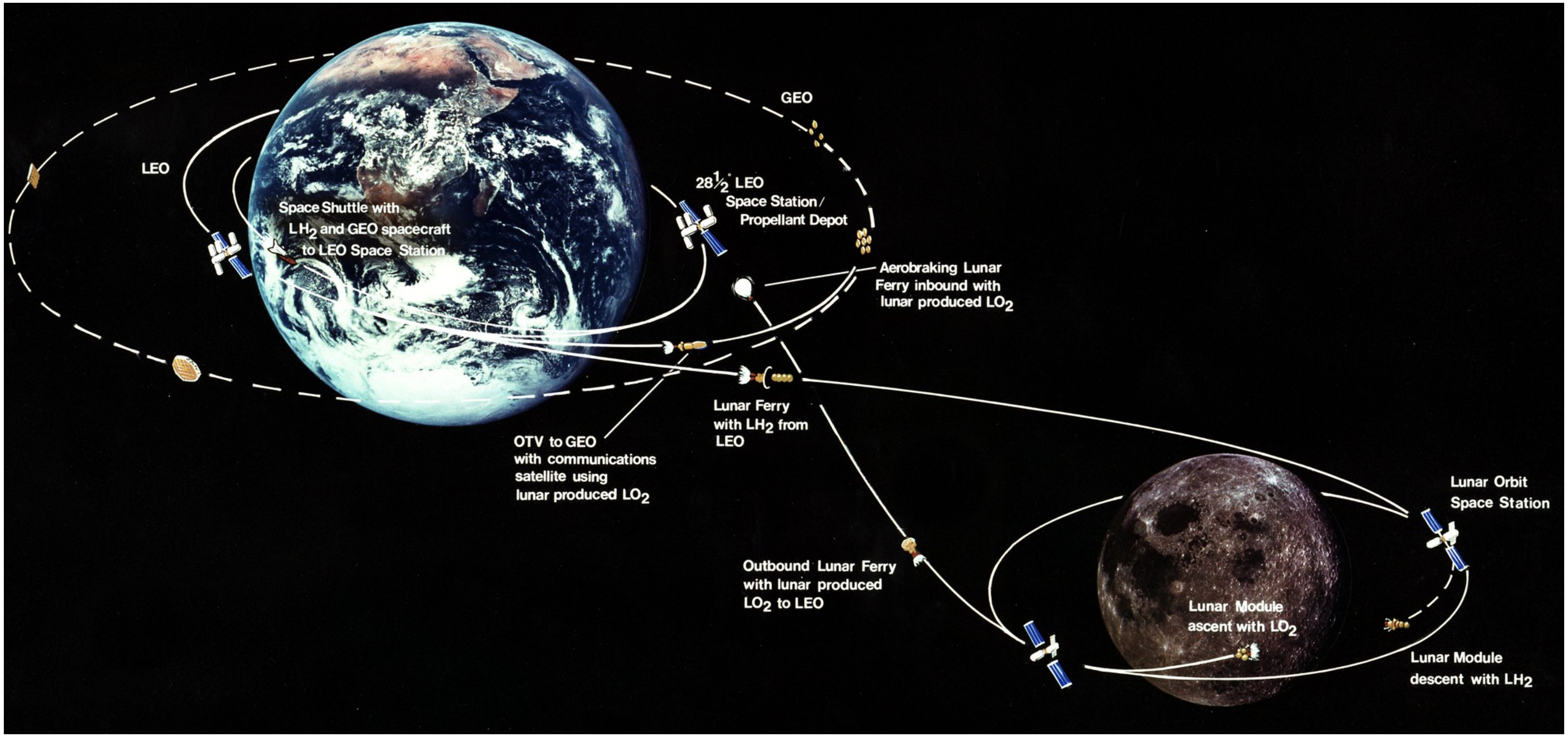


3D position coordinate $\boldsymbol{r} := \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

Find velocity control policy $\dot{\boldsymbol{r}} := \boldsymbol{v}(t, \boldsymbol{r})$ such that

$$\ddot{\boldsymbol{r}} = -\nabla_{\boldsymbol{r}} V(\boldsymbol{r}), \quad \boxed{\boldsymbol{r}(t = t_0) = \boldsymbol{r}_0(\text{ given }), \quad \boldsymbol{r}(t = t_1) = \boldsymbol{r}_1(\text{ given })}$$

Lambert's Problem



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ODE is 2nd order but endpoint boundary conditions are first order

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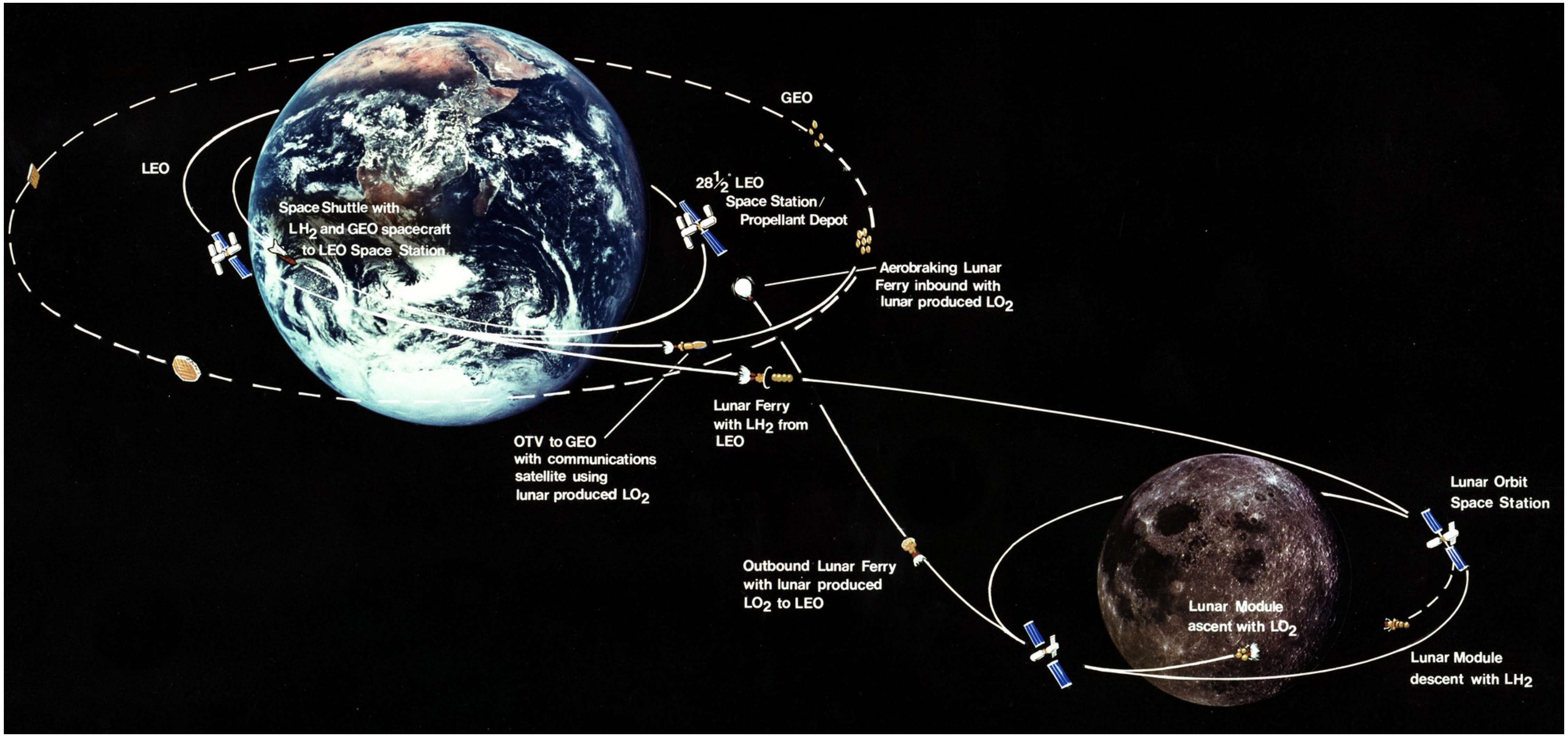
Lambert's Problem



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⇝ partially specified TPBVP

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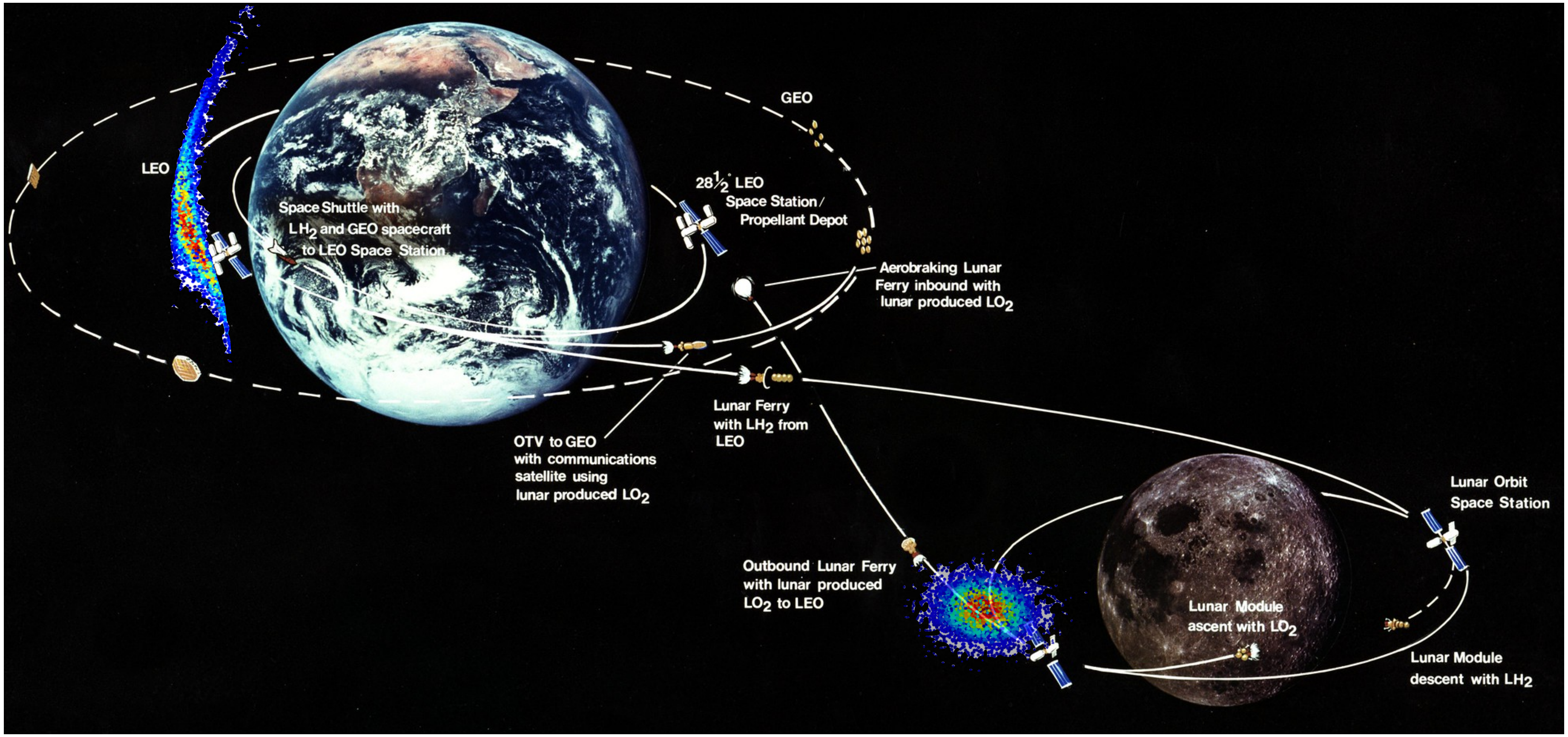
Probabilistic Lambert's Problem



3D position coordinate $\boldsymbol{r} := \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

Find velocity control policy $\dot{\boldsymbol{r}} := \boldsymbol{v}(t, \boldsymbol{r})$ such that

$$\ddot{\boldsymbol{r}} = -\nabla_{\boldsymbol{r}} V(\boldsymbol{r}), \quad \boldsymbol{r}(t = t_0) \sim \rho_0 \text{ (given)}, \quad \boldsymbol{r}(t = t_1) \sim \rho_1 \text{ (given)}$$

The Beginning of Lambert's Problem

Named after polymath **Johann Heinrich Lambert (1728 - 1777)**

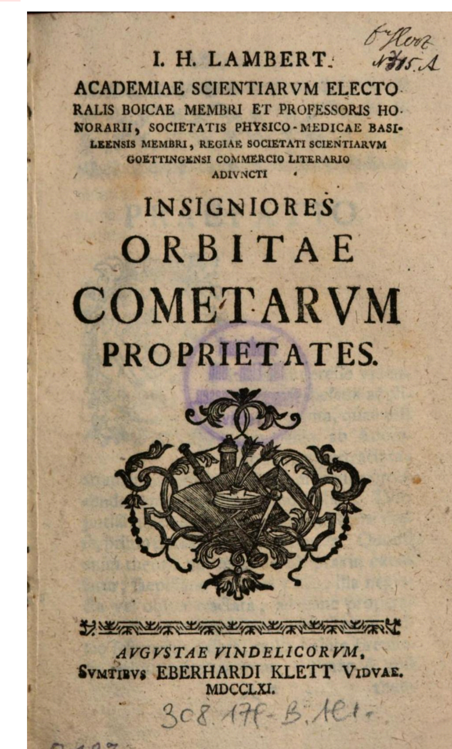
- known for first proof of irrationality of π, W function, area of a hyperbolic triangle

- special cases solved by Euler in 1743

- Lambert mentions this problem in letter to Euler in 1761

- solves the problem for parabolic, elliptic and hyperbolic Keplerian arcs in 1761 book

$$V(\boldsymbol{r}) = -\frac{\mu}{|\boldsymbol{r}|}$$



I. H. LAMBERT.
ACADEMIAE SCIENTIARVM ELECTORALIS BOICAE MEMBRI ET PROFESSORIS HONORARII, SOCIETATIS PHYSICO-MEDICAE BASILEENSIS MEMBRI, REGIAE SOCIETATI SCIENTIARVM GOETTINGENSI COMMERCIO LITERARIO ADIVNCTI

INSIGNIORES ORBITAE COMETARVM PROPRIETATES.

AVGVSTAE VINDELICORVM, SVMTIBVS EBERHARDI KLETT VIDVAE.
MDCCLXI.

- book receives high praise from Euler in 3 response letters

- alternative proofs by Lagrange (1780), Laplace (1798), Gauss (1809)

Modern History of Lambert's Problem

- Sustained interests for spacecraft guidance, missile interception

- 20th century astrodynamics research: fast computational algorithm, J2 effect in V

$$V(\boldsymbol{x}) = -\frac{\mu}{|\boldsymbol{x}|}\left(1 + \frac{J_2 R_{\mathrm{Earth}}^2}{2|\boldsymbol{x}|^2}\left(1 - \frac{3z^2}{|\boldsymbol{x}|^2}\right)\right) \longrightarrow \text{Bounded and negative for } |\boldsymbol{x}|^2 \geq \mathrm{R}_{\mathrm{Earth}}^2$$

- 21st century interests in aerospace community: probabilistic Lambert's problem

- Endpoint uncertainties due to estimation errors, statistical performance

- State-of-the-art: approx. dynamics (linearization) + approx. statistics (covariance)

- Our contribution: connections with OMT and SBP

- Formulation/computation: non-parametric, well-posedness, optimality certificate

Connection with Optimal Control Problem (OCP)

Lambert Problem ⇔ Deterministic OCP

Idea: use classical Hamiltonian mechanics to reformulate as deterministic OCP

$$\ddot{\boldsymbol{r}} = -\nabla_{\boldsymbol{r}} V(\boldsymbol{r}), \quad \boldsymbol{r}(t=t_0) = \boldsymbol{r}_0(\text{ given }), \quad \boldsymbol{r}(t=t_1) = \boldsymbol{r}_1(\text{ given })$$

$$\Updownarrow$$

$$\underset{\boldsymbol{v}}{\arg\inf} \int_{t_0}^{t_1} \left(\frac{1}{2}\|\boldsymbol{v}\|_2^2 - V(\boldsymbol{r})\right) \mathrm{d}t$$

Gravitational potential pushed from dynamics to Lagrangian

$$\dot{\boldsymbol{r}} = \boldsymbol{v},$$

$$\boldsymbol{r}(t=t_0) = \boldsymbol{r}_0 \text{ (given)}, \quad \boldsymbol{r}(t=t_1) = \boldsymbol{r}_1 \text{ (given)}$$

Lambertian OMT (L-OMT)

Probabilistic Lambert Problem $\Leftrightarrow$ Generalized OMT

$$\ddot{\boldsymbol{r}} = -\nabla_{\boldsymbol{r}} V(\boldsymbol{r}), \quad \boldsymbol{r}(t = t_0) \sim \rho_0 \text{ (given)}, \quad \boldsymbol{r}(t = t_1) \sim \rho_1 \text{ (given)}$$

$$\Updownarrow$$

$$\underset{(\rho,\boldsymbol{v})}{\arg\inf} \int_{t_0}^{t_1} \mathbb{E}_\rho \left[\frac{1}{2} \|\boldsymbol{v}\|_2^2 - V(\boldsymbol{r}) \right] \mathrm{d}t$$

$$\dot{\boldsymbol{r}} = \boldsymbol{v},$$

$$\boldsymbol{r}(t = t_0) \sim \rho_0 \text{ (given)}, \quad \boldsymbol{r}(t = t_1) \sim \rho_1 \text{ (given)}$$

Potential as state cost ($V = 0$ is OMT)

L-OMT as Density Steering

$$\underset{(\rho,\boldsymbol{v})}{\arg\inf} \int_{t_0}^{t_1} \mathbb{E}_\rho \left[\frac{1}{2}\|\boldsymbol{v}\|_2^2 - V(\boldsymbol{r})\right] \mathrm{d}t$$

$$\dot{\boldsymbol{r}} = \boldsymbol{v},$$

$$\boldsymbol{r}(t=t_0) \sim \rho_0 \text{ (given)}, \quad \boldsymbol{r}(t=t_1) \sim \rho_1 \text{ (given)}$$

$$\Updownarrow$$

$$\underset{(\rho,\boldsymbol{v})}{\arg\inf} \int_{t_0}^{t_1} \mathbb{E}_\rho \left[\frac{1}{2}\|\boldsymbol{v}\|_2^2 - V(\boldsymbol{r})\right] \mathrm{d}t$$

$$\frac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v}) = 0,$$ — **Liouville PDE**

$$\rho(t=t_0,\cdot) = \rho_0, \quad \rho(t=t_1,\cdot) = \rho_1$$

Existence-Uniqueness of L-OMT Solution

Thm. (informal)
Existence-uniqueness guaranteed for V bounded C^1,
and ρ_0, ρ_1 with finite second moments

Proof idea.
Figalli's theory for OMT with Tonelli Lagrangians that are
induced by action integrals

Connection to SBP with state cost

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\left(\frac{1}{2}|\boldsymbol{v}|^2 - V(\boldsymbol{x})\right)\rho(\boldsymbol{x},t)\,d\boldsymbol{x}dt$$

$$\frac{\partial\rho}{\partial t} + \nabla_{\boldsymbol{r}}\cdot(\rho\boldsymbol{v}) = 0,$$ — **Liouville PDE**

$$\rho(t=t_0,\cdot)=\rho_0,\quad \rho(t=t_1,\cdot)=\rho_1$$

Lambertian SBP (L-SBP)

$$\underset{(\rho,\boldsymbol{v})\in\mathcal{P}_{01}\times\mathcal{V}}{\arg\inf}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\left(\frac{1}{2}|\boldsymbol{v}|^2 - V(\boldsymbol{x})\right)\rho(\boldsymbol{x},t)\,d\boldsymbol{x}dt$$

Regularization > 0

$$\frac{\partial\rho}{\partial t} + \nabla_{\boldsymbol{r}}\cdot(\rho\boldsymbol{v}) = \varepsilon\Delta_{\boldsymbol{r}}\rho,$$ — **Fokker-Planck-Kolmogorov PDE**

$$\rho(t=t_0,\cdot)=\rho_0,\quad \rho(t=t_1,\cdot)=\rho_1$$

L-SBP Solution

Thm. (informal) Existence and uniqueness of L-SBP is guaranteed

$$V(\boldsymbol{x}) = -\frac{\mu}{|\boldsymbol{x}|}\left(1 + \frac{J_2 R_{\text{Earth}}^2}{2|\boldsymbol{x}|^2}\left(1 - \frac{3z^2}{|\boldsymbol{x}|^2}\right)\right) \longrightarrow$$

Bounded and negative for $|\boldsymbol{x}|^2 \geq \mathrm{R}_{\text{Earth}}^2$

Thm. (Necessary conditions of optimality for L-SBP)

$$\frac{\partial \psi_\varepsilon}{\partial t} + \frac{1}{2}|\nabla_{\boldsymbol{x}}\psi_\varepsilon|^2 + \varepsilon \Delta_{\boldsymbol{x}}\psi_\varepsilon = -V(\boldsymbol{x})$$

$$\frac{\partial \rho_\varepsilon^{\text{opt}}}{\partial t} + \nabla_{\boldsymbol{x}} \cdot \left(\rho_\varepsilon^{\text{opt}} \nabla_{\boldsymbol{x}}\psi_\varepsilon\right) = \varepsilon \Delta_{\boldsymbol{x}} \rho_\varepsilon^{\text{opt}}$$

$$\rho_\varepsilon^{\text{opt}}(t = t_0, \cdot) = \rho_0, \quad \rho_\varepsilon^{\text{opt}}(t = t_1, \cdot) = \rho_1$$

L-SBP Solution

Thm. (Hopf-Cole a.k.a. Fleming's log transform)

Change of variable $\left(\rho_{\varepsilon}^{\text{opt}}, \psi\right) \mapsto (\widehat{\varphi}, \varphi)$ — **Schrödinger factors**

$$\widehat{\varphi}(t, \boldsymbol{r}) = \rho_{\varepsilon}^{\text{opt}}(t, \boldsymbol{r}) \exp\left(-\frac{\psi(t, \boldsymbol{r})}{2\varepsilon}\right)$$

$$\varphi(t, \boldsymbol{r}) = \exp\left(\frac{\psi(t, \boldsymbol{r})}{2\varepsilon}\right)$$

results in a boundary-coupled system of forward-backward reaction-diffusion PDEs

$$\frac{\partial \widehat{\varphi}}{\partial t} = (\varepsilon \Delta_{\boldsymbol{r}} + V(\boldsymbol{r}))\widehat{\varphi} \longleftarrow \mathcal{L}_{\text{forward}}\widehat{\varphi}$$

$$\frac{\partial \varphi}{\partial t} = -(\varepsilon \Delta_{\boldsymbol{r}} + V(\boldsymbol{r}))\varphi \longleftarrow \mathcal{L}_{\text{backward}}\varphi$$

$$\widehat{\varphi}(t = t_0, \cdot)\varphi(t = t_0, \cdot) = \rho_0, \quad \widehat{\varphi}(t = t_1, \cdot)\varphi(t = t_1, \cdot) = \rho_1$$

Optimally controlled joint state PDF: $\rho_{\varepsilon}^{\text{opt}}(t, \boldsymbol{r}) = \widehat{\varphi}(t, \boldsymbol{r})\varphi(t, \boldsymbol{r})$

Optimal control: $\boldsymbol{v}_{\varepsilon}^{\text{opt}}(t, \boldsymbol{r}) = 2\varepsilon \nabla_{\boldsymbol{r}} \log \varphi(t, \boldsymbol{r})$

L-SBP Computation via Schrödinger Factors

Recursion over pair $(\varphi_1, \hat{\varphi}_0)$

$$
\begin{array}{ccc}
\widehat{\varphi}_{\varepsilon,0}(\cdot) & \xrightarrow{\quad \int \quad} & \widehat{\varphi}_{\varepsilon}(\cdot, t=t_1) \\
\rho_0(\cdot)/\varphi_{\varepsilon}(\cdot, t=t_0) \Big\uparrow & & \Big\downarrow \rho_1(\cdot)/\widehat{\varphi}_{\varepsilon}(\cdot, t=t_1) \\
\varphi_{\varepsilon}(\cdot, t=t_0) & \xleftarrow[\quad \int \quad]{} & \varphi_{\varepsilon,1}(\cdot)
\end{array}
$$

$$
\frac{\partial \widehat{\varphi}_{\varepsilon}}{\partial t} = \left(\varepsilon \Delta_{\boldsymbol{r}} + \frac{1}{2\varepsilon} V(\boldsymbol{r})\right) \widehat{\varphi}_{\varepsilon}
$$

$$
\frac{\partial \varphi_{\varepsilon}}{\partial t} = -\left(\varepsilon \Delta_{\boldsymbol{r}} + \frac{1}{2\varepsilon} V(\boldsymbol{r})\right) \varphi_{\varepsilon}
$$

$$
\rho_{\varepsilon}^{\text{opt}}(t=t_0, \cdot) = \rho_0, \quad \rho_{\varepsilon}^{\text{opt}}(t=t_1, \cdot) = \rho_1
$$

Numerical Case Study

Prescribed time horizon $[t_0, t_1] \equiv [0,1]$ hours

Endpoint joint PDFs

$$\boldsymbol{x}_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$$

$$\boldsymbol{x}_1 \sim \mathcal{N}(\mu_1, \Sigma_1)$$

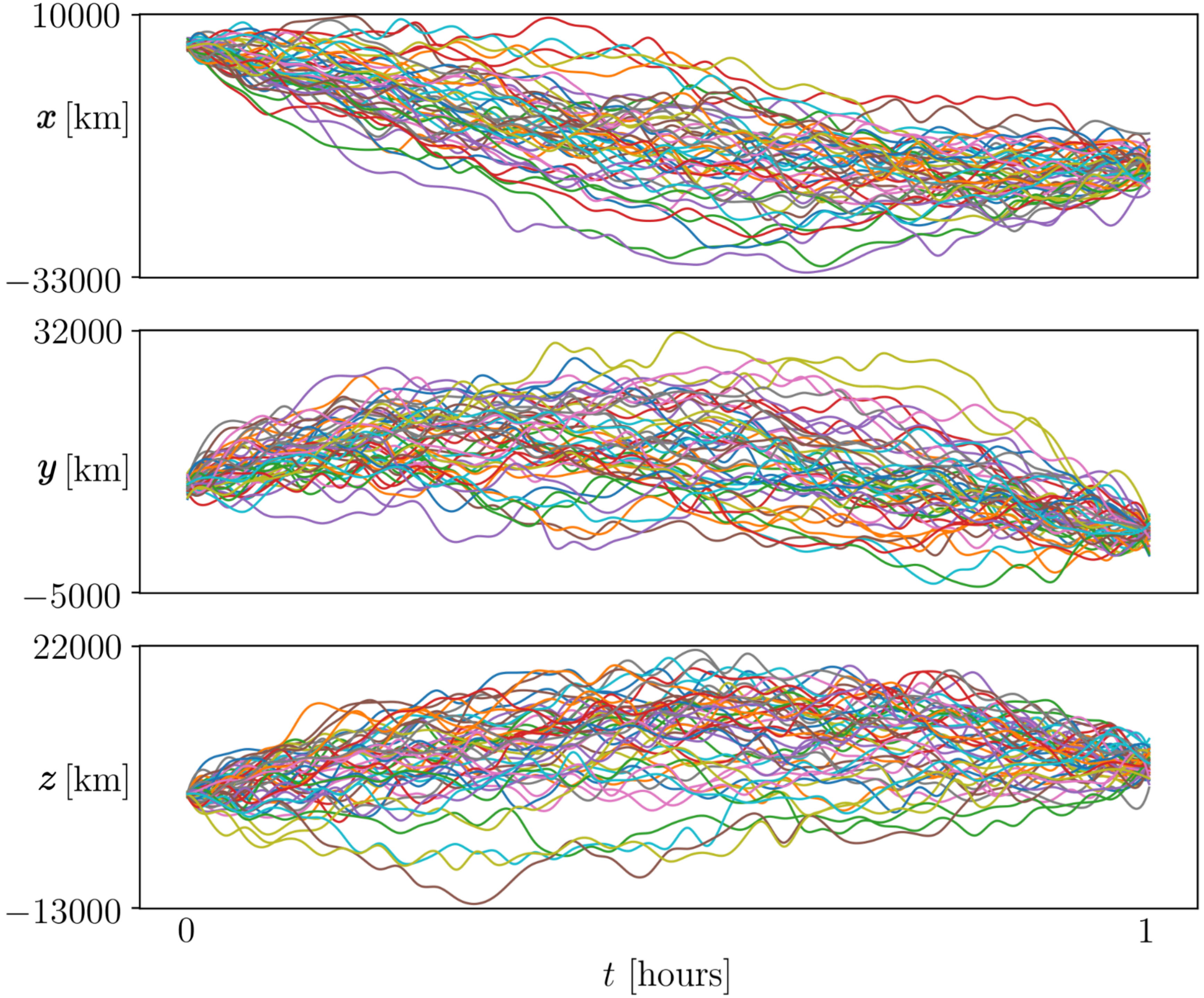
where

$$\mu_0 = \begin{pmatrix} 5000 \\ 10000 \\ 2100 \end{pmatrix}, \quad \mu_1 = \begin{pmatrix} -14600 \\ 2500 \\ 7000 \end{pmatrix}$$

$$\Sigma_0 = \frac{1}{100}\mathrm{diag}(\mu_0^2), \quad \Sigma_1 = \frac{1}{100}\mathrm{diag}(\mu_1^2),$$

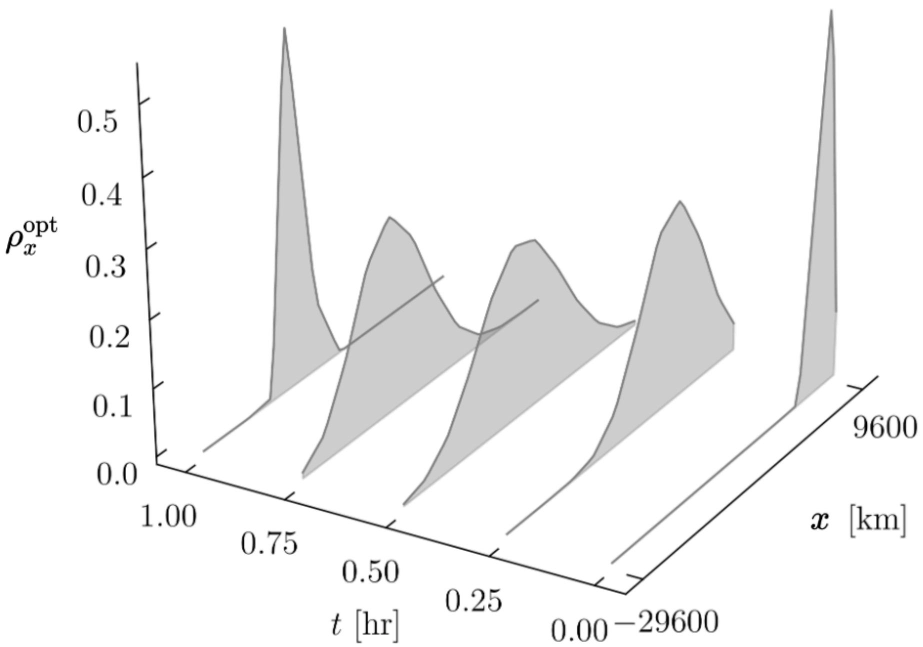
Numerical Case Study (cont.)

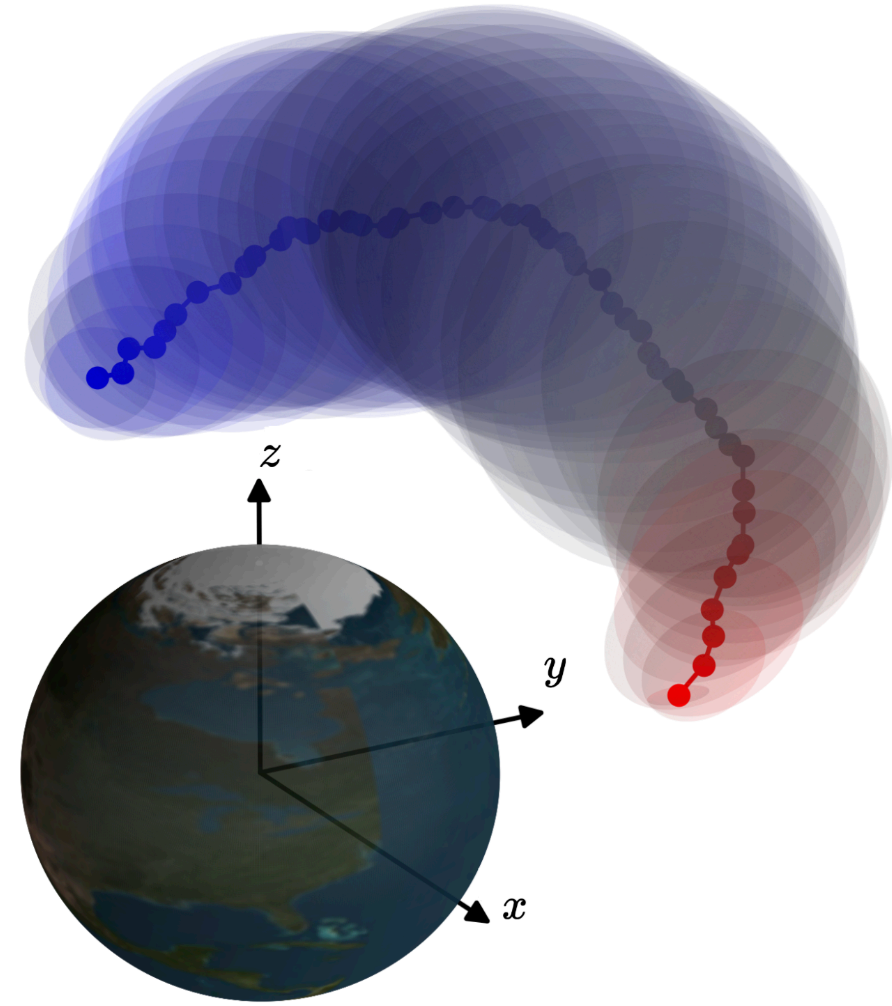
Optimally controlled closed loop state sample paths

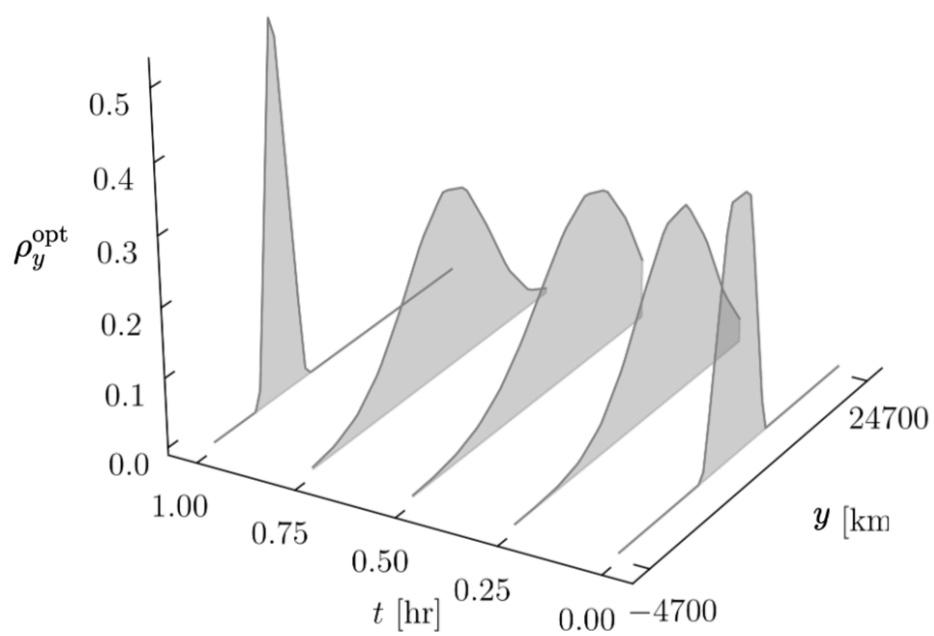


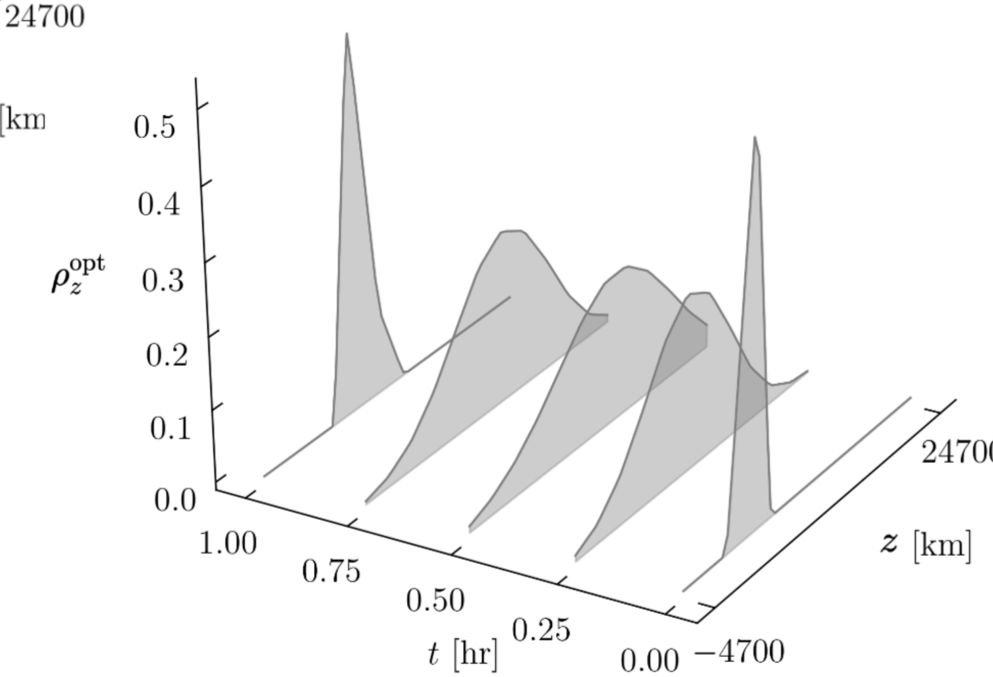
Numerical Case Study (cont.)

Univariate marginals for optimally controlled joint PDFs









Ongoing Efforts

- Find explicit Green's function for reaction-diffsion PDE with reaction rate equal to gravitational potential

- Connections with solution of time-dependent Schrödinger's equation in quantum mechanics for Hydrogen atom

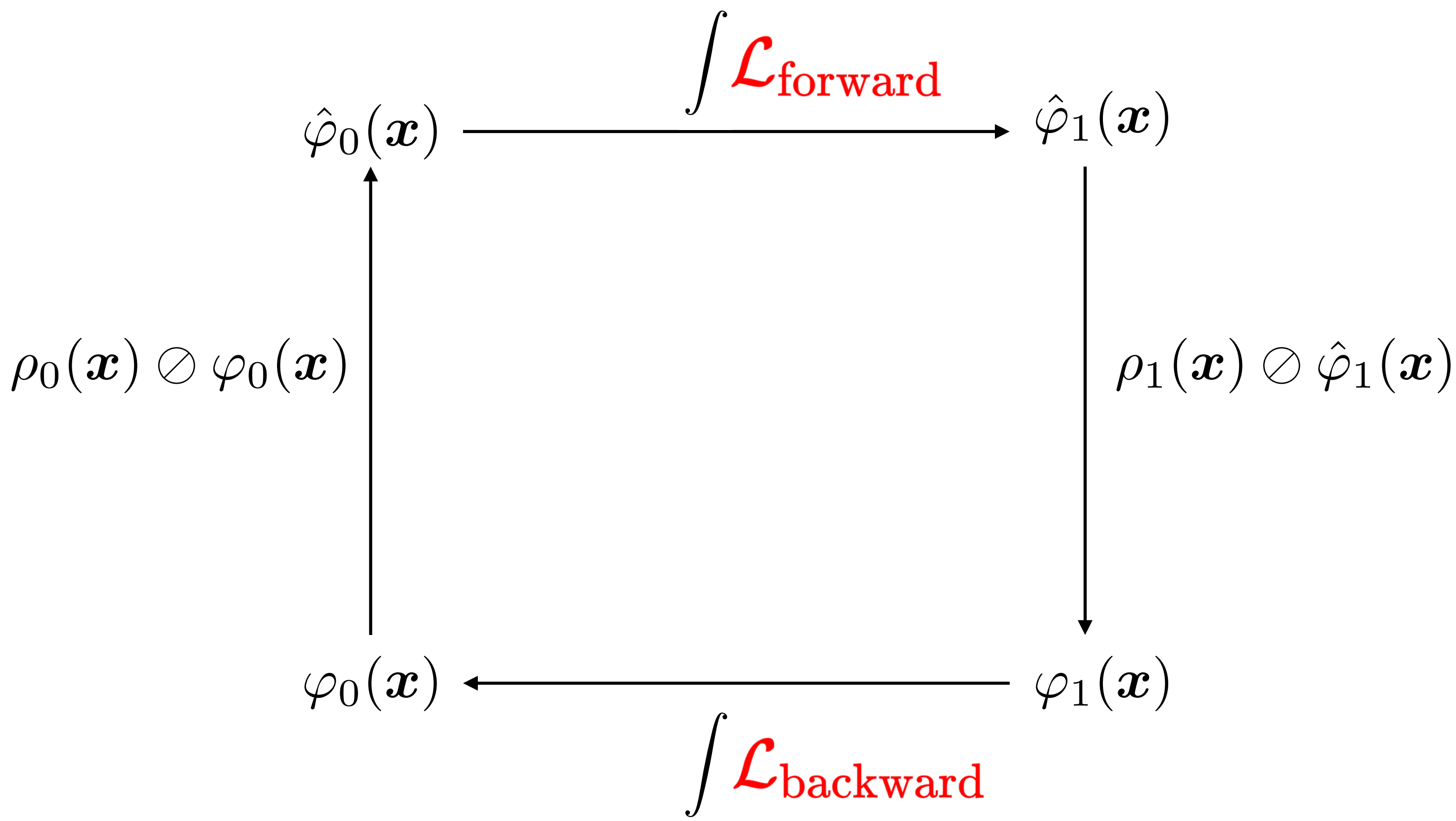
- Preprint with L-OMT and L-SBP details: arXiv:2401.07961

Thank You

Backup Slides

L-SBP Solution: Computation

IDEA: Fixed point recursion over pair $\left(\varphi_1, \hat{\varphi}_0\right)$



Thm. (Existence-uniqueness-convergence) Proof by contraction mapping

Thm. (Fredholm Integral Representation)

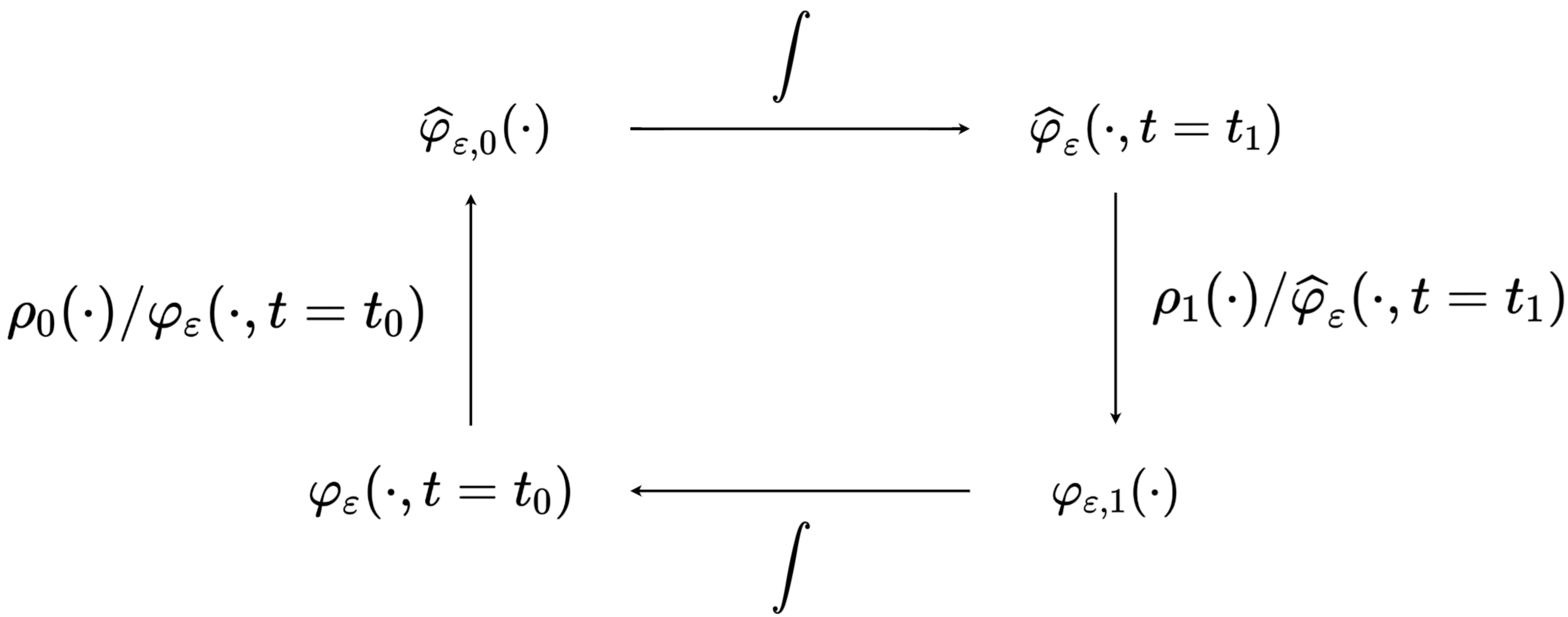
$$\widehat{\varphi}(t,\boldsymbol{x}) = \underbrace{\frac{1}{\sqrt{(4\pi\varepsilon t)^3}} \int_{\mathbb{R}^3} \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|_2^2}{4\varepsilon t}\right) \widehat{\varphi}_0(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}}_{\text{term 1}}$$

$$+ \underbrace{\int_0^t \frac{1}{\sqrt{(4\pi\varepsilon(t-\tau))^3}} \int_{\mathbb{R}^3} \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|_2^2}{4\varepsilon(t-\tau)}\right) V(\boldsymbol{y})\, \widehat{\varphi}(\tau,\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}\tau}_{\text{term 2}}$$

Likewise for $\varphi(t,\boldsymbol{x})$

Solution: Computation

IDEA: Fixed point recursion over pair $(\varphi_1, \hat{\varphi}_0)$



Idea:
Left Riemann Approximation of Second Term

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^n} f(\widetilde{\boldsymbol{x}}, \boldsymbol{x}, \tau, t) d\widetilde{\boldsymbol{x}} d\tau$$

$$\approx \sum_{q=0}^{k-1} \sum_{m=0}^{N_x} \sum_{n=0}^{N_y} \sum_{j=0}^{N_z} f(\widetilde{\boldsymbol{x}}_{(m,n,j)}, \boldsymbol{x}, t_0 + k\Delta t, t) \Delta z \Delta y \Delta x \Delta t$$

where $\widetilde{\boldsymbol{x}}_{(m,n,j)} = (x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$