Schrodinger Meets Kuramoto via Feynman-Kac: Minimum Effort Distribution Steering for Noisy Nonuniform Kuramoto Oscillators

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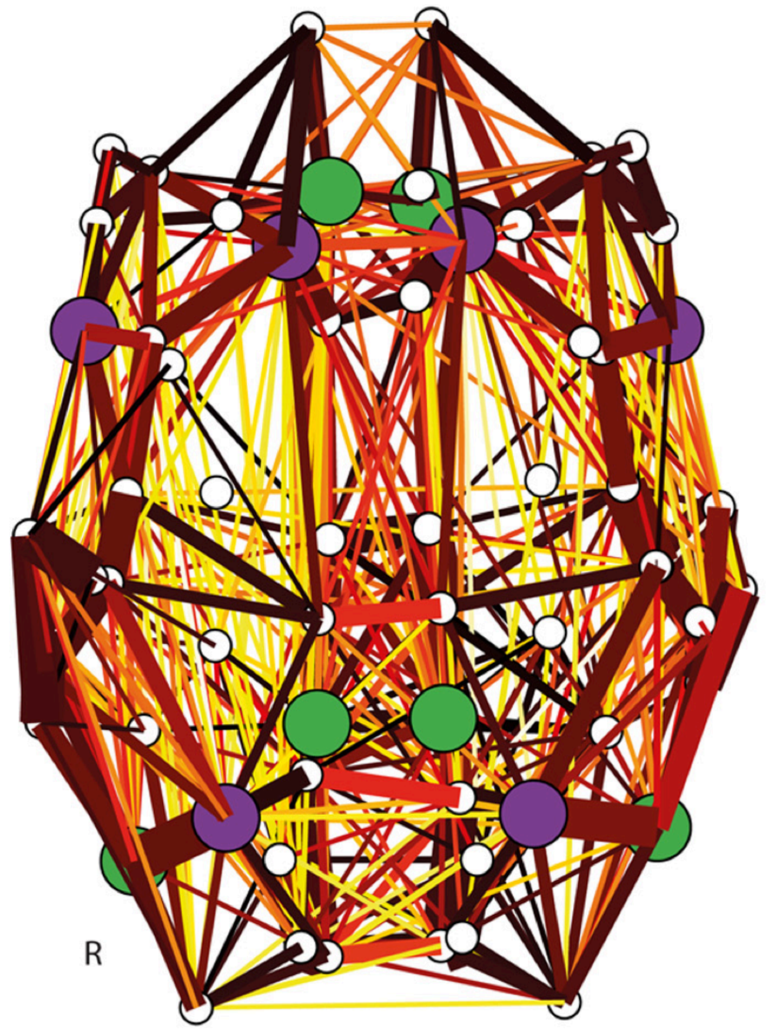
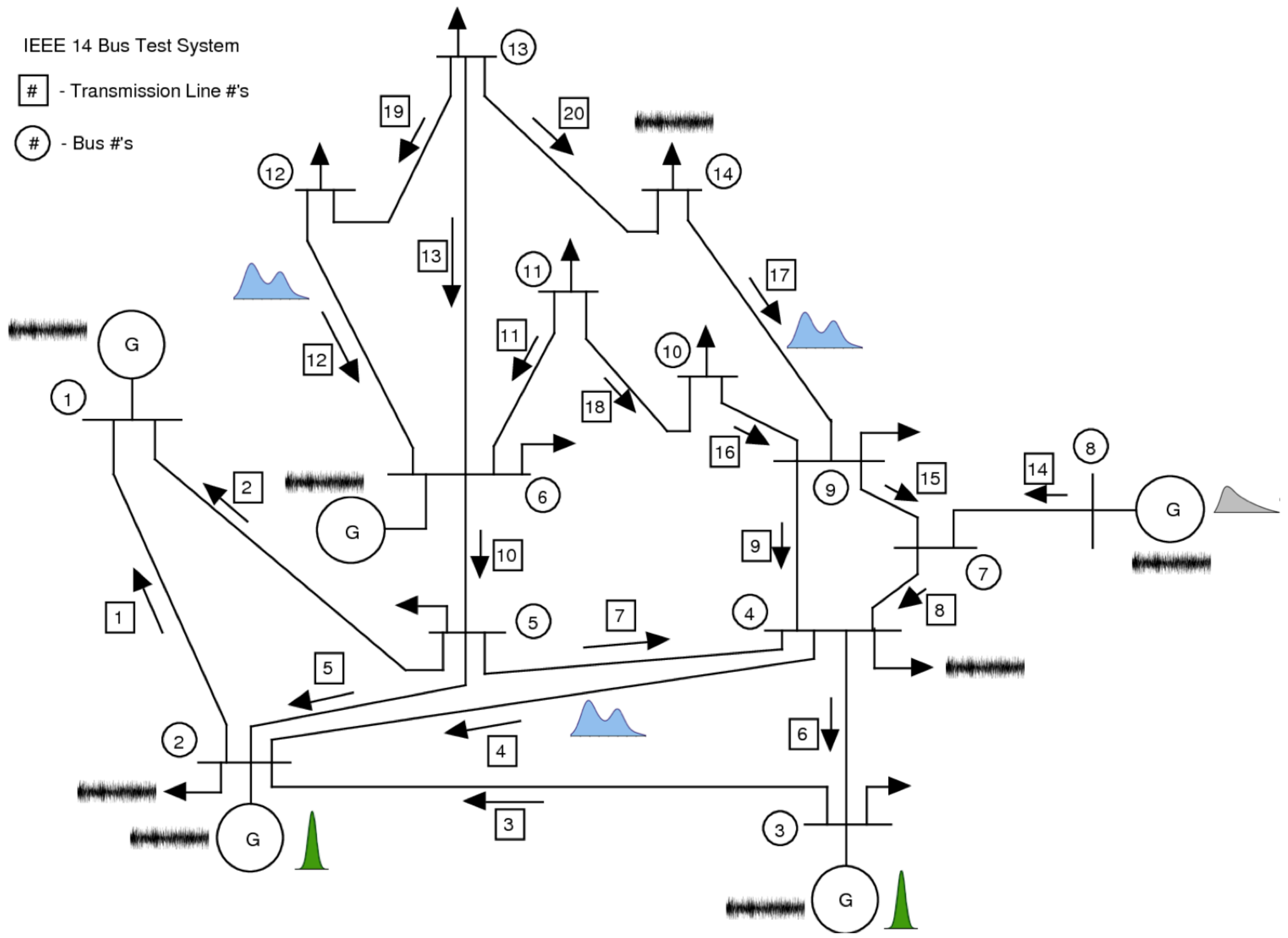
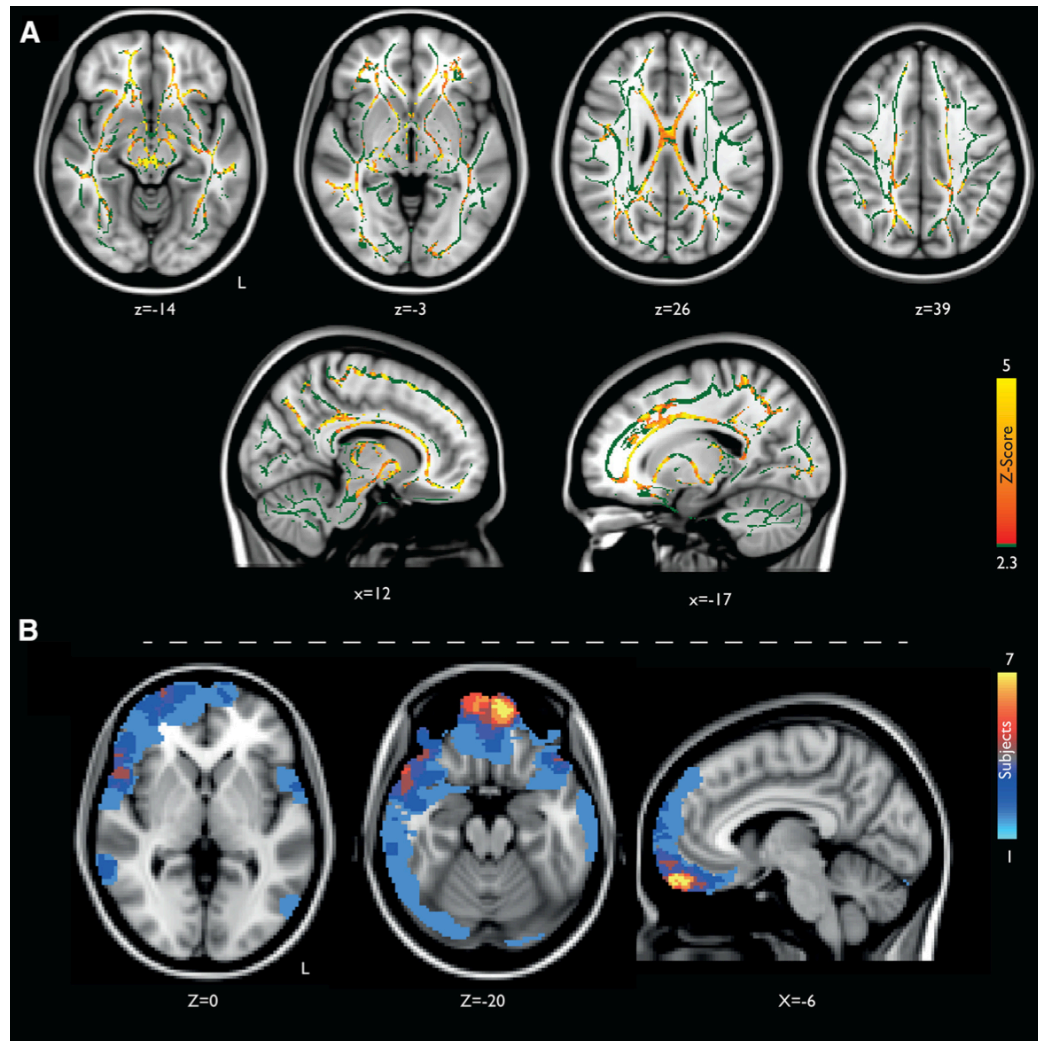
Kuramoto Oscillators



[Peter J. Hellyer,, et al., 2015]



[Halder et al., 2022]



[Joyce et al., 2019]

[Yufeng et al., 2021]

Outline of this talk

1. State Feedback Density Steering

2. Optimal Steering of Distributions for the Nonuniform Noisy Kuramoto Oscillators

3. Proximal Recursion

4. Feynman-Kac Path Integral Formulation

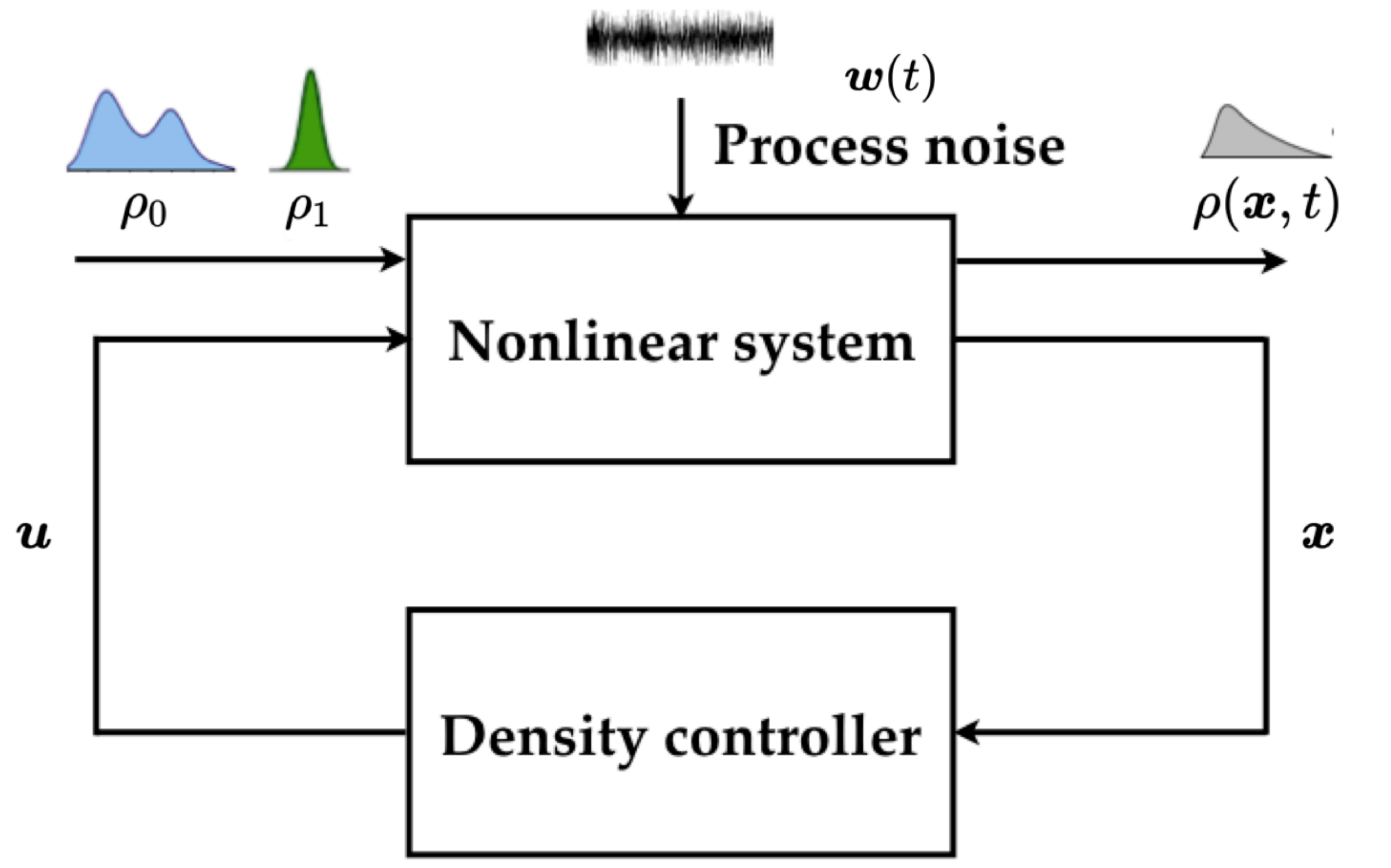
5. Numerical Example

State Feedback Density Steering

$$\inf_{\boldsymbol{u}\in\mathcal{U}} \quad \mathbb{E}_{\mu^u}\left[\int_0^T \frac{1}{2}\|\boldsymbol{u}(\boldsymbol{x},t)\|_2^2 \,\mathrm{d}t\right]$$

$$\text{s.t} \quad \mathrm{d}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x},t)\mathrm{d}t + \boldsymbol{B}(t)\boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}t + \sqrt{2}\epsilon\boldsymbol{B}(t)\mathrm{d}\boldsymbol{w}(t)$$

$$\boldsymbol{x}(t=0) \sim \mu_0(\boldsymbol{x}), \quad \boldsymbol{x}(t=T) \sim \mu_T(\boldsymbol{x})$$

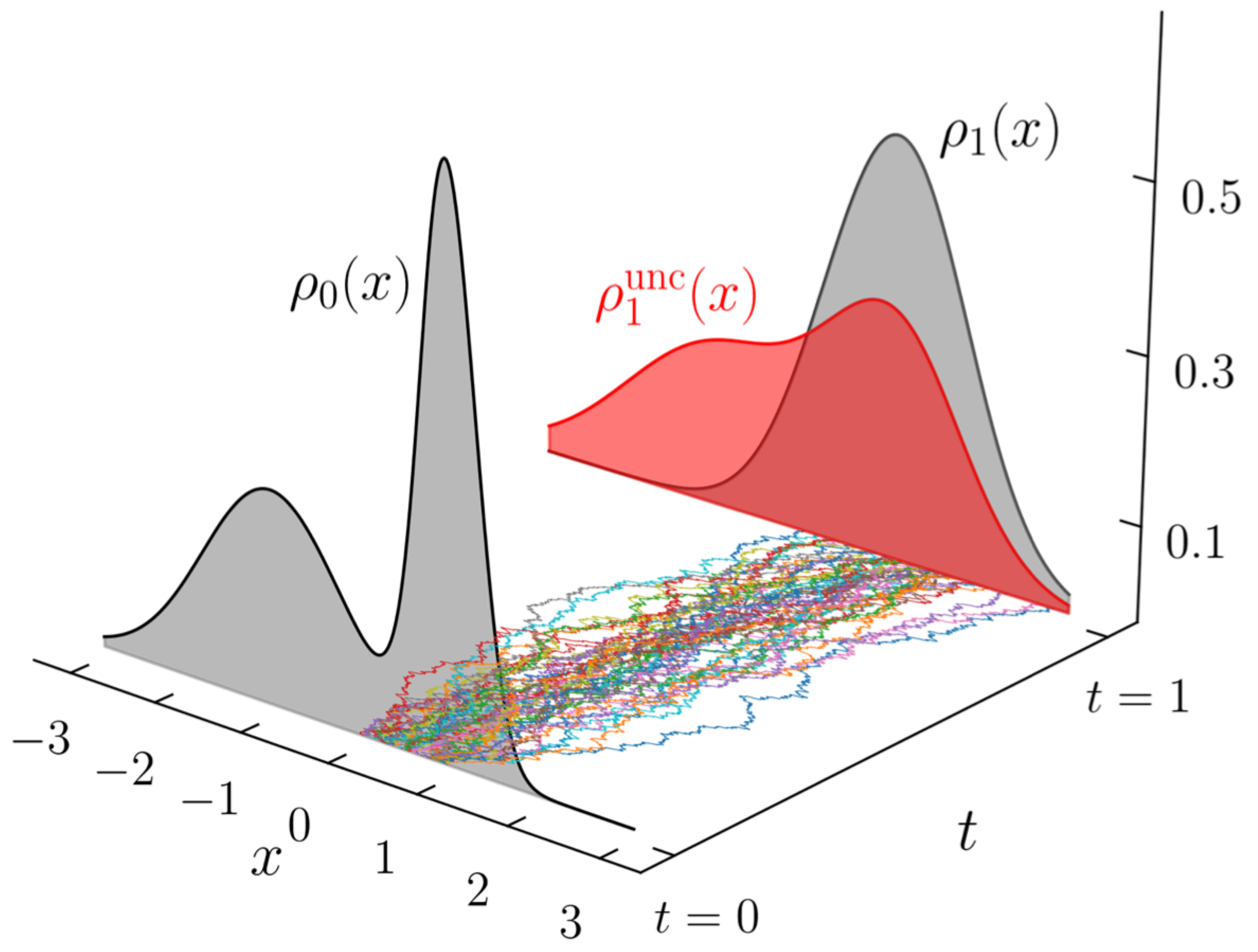


Fluid dynamic form:

$$\inf_{(\rho,\boldsymbol{u})} \frac{1}{2}\int_0^T \int_{\mathbb{R}^n} \|\boldsymbol{u}(\boldsymbol{x},t)\|_2^2 \,\rho(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t$$

$$\text{s.t} \quad \frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\,(\boldsymbol{f}+\boldsymbol{B}(t)\boldsymbol{u})\right) = \epsilon\left\langle \boldsymbol{D}(t), \mathrm{Hess}\,(\rho)\right\rangle$$

$$\rho(\boldsymbol{x},0) = \rho_0(\boldsymbol{x}), \quad \rho(\boldsymbol{x},T) = \rho_T(\boldsymbol{x})$$



Necessary Conditions for Optimality

Controlled Fokker-Planck or Kolmogorov's forward PDE

$$\frac{\partial}{\partial t}\rho^{\text{opt}} + \nabla \cdot \left(\rho^{\text{opt}}\left(\boldsymbol{f} + \boldsymbol{B}(t)^{\top}\nabla\psi\right)\right) = \epsilon\langle \boldsymbol{D}(t), \text{Hess}\left(\rho^{\text{opt}}\right)\rangle$$

Hamilton-Jacobi-Bellman PDE:

Value function

Coupled in the equation level !!!

$$\frac{\partial\psi}{\partial t} + \frac{1}{2}\left\|\boldsymbol{B}(t)^{\top}\nabla\psi\right\|_2^2 + \langle \nabla\psi, \boldsymbol{f}\rangle = -\epsilon\langle \boldsymbol{D}(t), \text{Hess}\ (\psi)\rangle$$

Initial and terminal conditions: $\rho^{\text{opt}}(\boldsymbol{x},0) = \rho_0(\boldsymbol{x}), \quad \rho^{\text{opt}}(\boldsymbol{x},T) = \rho_T(\boldsymbol{x})$

Optimal control: $\boldsymbol{u}^{\text{opt}}(\boldsymbol{x},t) = \boldsymbol{B}(t)^{\top}\nabla\psi(\boldsymbol{x},t)$

Feedback Synthesis via the Schrödinger System

Hopf-Cole a.k.a. Fleming's logarithmic transform:

$$\left(\rho^{\text{opt}}, \psi\right) \mapsto \underbrace{(\hat{\varphi}, \varphi)}_{\text{Schrödinger factors}}$$

$$\varphi(\boldsymbol{x}, t) = \exp\left(\frac{\psi(\boldsymbol{x}, t)}{2\epsilon}\right)$$

$$\hat{\varphi}(\boldsymbol{x}, t) = \rho^{\text{opt}}(\boldsymbol{x}, t) \exp\left(-\frac{\psi(\boldsymbol{x}, t)}{2\epsilon}\right)$$

Feedback Synthesis via the Schrödinger System

Coupled only in the constraints level :)

Uncontrolled forward-backward Fokker-Planck PDEs

$$\frac{\partial \varphi}{\partial t} = -\langle \nabla \varphi, \boldsymbol{f} \rangle - \epsilon \langle \boldsymbol{D}(t), \mathrm{Hess}(\varphi) \rangle$$

$$\frac{\partial \hat{\varphi}}{\partial t} = -\nabla \cdot (\hat{\varphi} \boldsymbol{f}) + \epsilon \langle \boldsymbol{D}(t), \mathrm{Hess}(\hat{\varphi}) \rangle$$

Initial and Terminal conditions

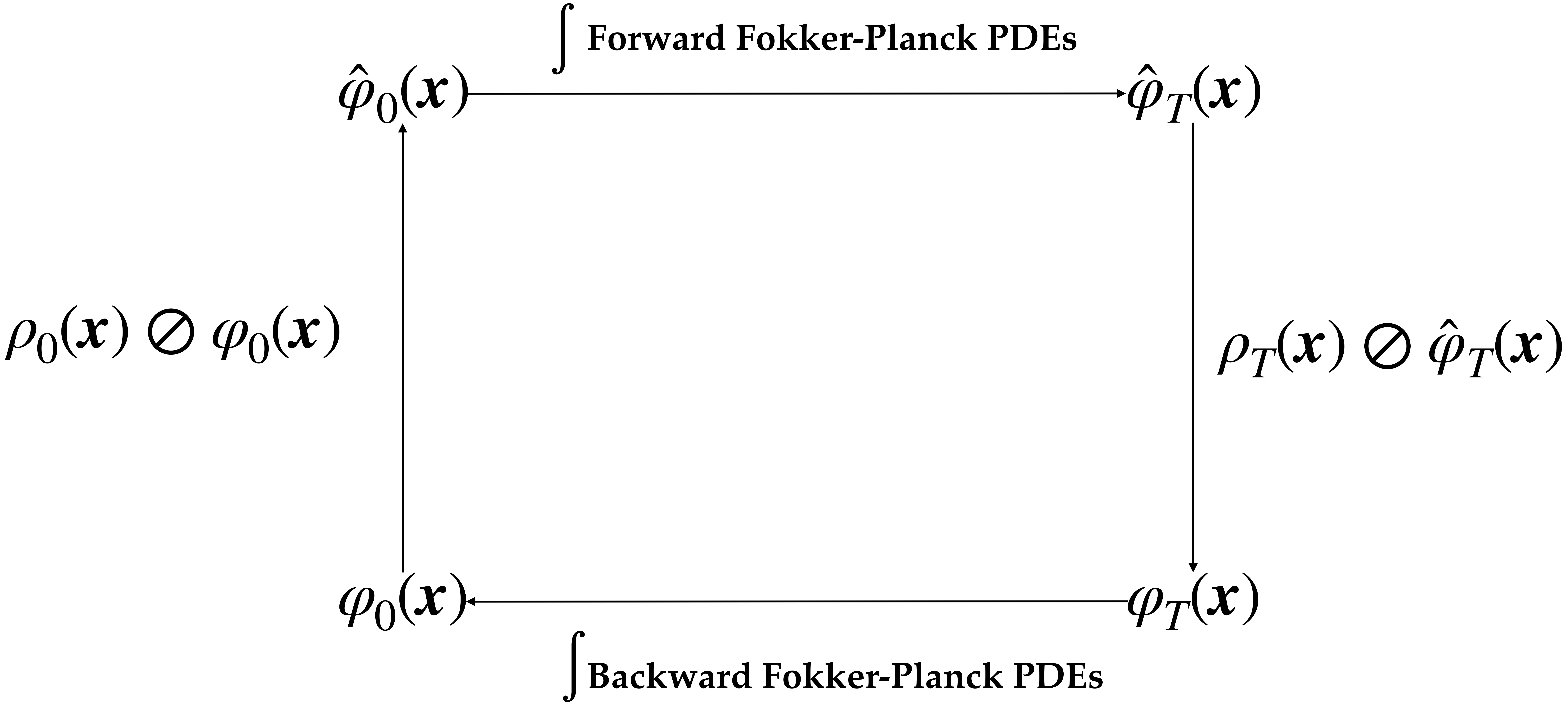
$$\varphi(\boldsymbol{x}, 0)\hat{\varphi}(\boldsymbol{x}, 0) = \rho_0(\boldsymbol{x})$$

$$\varphi(\boldsymbol{x}, T)\hat{\varphi}(\boldsymbol{x}, T) = \rho_T(\boldsymbol{x})$$

Optimal controlled joint state PDF: $\rho^{\mathrm{opt}}(\boldsymbol{x}, t) = \varphi(\boldsymbol{x}, t)\hat{\varphi}(\boldsymbol{x}, t)$

Optimal control: $\boldsymbol{u}^{\mathrm{opt}}(\boldsymbol{x}, t) = 2\epsilon \boldsymbol{B}(t)^\top \nabla \log \varphi$

Fixed Point Recursion over Pair $(\varphi, \hat{\varphi})$



This recursion is contractive in the Hilbert metric

Nonuniform Noisy Kuramoto Oscillators

First order $$\mathrm{d}\boldsymbol{\theta} = \left(-\nabla_{\boldsymbol{\theta}} V(\boldsymbol{\theta}) + \boldsymbol{S}\boldsymbol{u}\right)\mathrm{d}t + \sqrt{2}\boldsymbol{S}\mathrm{d}\boldsymbol{w}$$

Second order $$\begin{pmatrix}\mathrm{d}\boldsymbol{\theta}\\ \mathrm{d}\boldsymbol{\omega}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\omega}\\ -\boldsymbol{M}^{-1}\nabla_{\boldsymbol{\theta}} V(\boldsymbol{\theta}) - \boldsymbol{M}^{-1}\boldsymbol{\Gamma}\boldsymbol{\omega} + \boldsymbol{M}^{-1}\boldsymbol{S}\boldsymbol{u}\end{pmatrix}\mathrm{d}t + \begin{pmatrix}\boldsymbol{0}_{n\times 1}\\ \sqrt{2}\boldsymbol{M}^{-1}\boldsymbol{S}\mathrm{d}\boldsymbol{w}\end{pmatrix}$$

Potential function $$V(\boldsymbol{\theta}) := \sum_{i<j} k_{ij}(1-\cos(\theta_i - \theta_j - \varphi_{ij})) - \sum_{i=1}^{n} P_i\theta_i$$

Coupling > 0

Phase difference $\in [0, \pi/2)$

Linear coeff. > 0

Positive diagonal matrices $\boldsymbol{M}, \boldsymbol{\Gamma}, \boldsymbol{S}$

Nonuniform Noisy Kuramoto Oscillators

$$\inf_{(\rho,u)} \int_0^T \int_{\mathcal{X}} \|\boldsymbol{u}(\boldsymbol{x},t)\|_2^2 \; \rho(\boldsymbol{x},t) \; \mathrm{d}\boldsymbol{x}\mathrm{d}t$$

First order, $\mathcal{X} \equiv \mathbb{T}^n$

$$\text{s.t} \quad \frac{\partial \rho}{\partial t} = -\nabla_{\boldsymbol{\theta}} \cdot \left(\rho\left(\boldsymbol{S}\boldsymbol{u} - \nabla_{\boldsymbol{\theta}} V\right)\right) + \left\langle \boldsymbol{D}, \mathbf{Hess}\left(\rho\right)\right\rangle$$

($\boldsymbol{D} \to \boldsymbol{S}\boldsymbol{S}^\top$)

Second order, $\mathcal{X} \equiv \mathbb{T}^n \times \mathbb{R}^n$

$$\text{s.t} \quad \frac{\partial \rho}{\partial t} = \nabla_{\boldsymbol{\omega}} \cdot \left(\rho\left(\boldsymbol{M}^{-1}\nabla_{\boldsymbol{\theta}} V(\boldsymbol{\theta}) + \boldsymbol{M}^{-1}\boldsymbol{\Gamma}\boldsymbol{\omega} - \boldsymbol{M}^{-1}\boldsymbol{S}\boldsymbol{u} + \boldsymbol{M}^{-1}\boldsymbol{D}\boldsymbol{M}^{-1}\nabla_{\boldsymbol{\omega}}\log\rho\right) - \left\langle \boldsymbol{\omega}, \nabla_{\boldsymbol{\theta}}\rho\right\rangle\right.$$

Initial and Terminal conditions $\quad \rho(\boldsymbol{x}, t=0) = \rho_0, \qquad \rho(\boldsymbol{x}, t=T) = \rho_T$

Isotropic Degenerate Diffusion

The First Order Case

$$\boldsymbol{\theta} \mapsto \boldsymbol{\xi} := \boldsymbol{S}^{-1}\boldsymbol{\theta}$$

$$\mathrm{d}\boldsymbol{\xi} = \left(\boldsymbol{u} - \boldsymbol{\Upsilon}\,\nabla_{\boldsymbol{\xi}}\tilde{V}(\boldsymbol{\xi})\right)\mathrm{d}t + \sqrt{2}\ \mathrm{d}\boldsymbol{w}$$

$$\boldsymbol{\Upsilon} := \left(\prod_{i=1}^{n}\sigma_i^2\right)\boldsymbol{S}^{-2} = \operatorname{diag}\left(\prod_{j\neq i}\sigma_j^2\right) \succ \boldsymbol{0}$$

$$\tilde{V}(\boldsymbol{\xi}) := \left(\frac{1}{2}\sum_{i<j}k_{ij}\left(1-\cos\left(\sigma_i\xi_i - \sigma_j\xi_j - \varphi_{ij}\right)\right) - \sum_{i=1}^{n}\sigma_i P_i \xi_i\right) / \left(\prod_{i=1}^{n}\sigma_i^2\right)$$

Isotropic Degenerate Diffusion

The Second Order Case

$$\begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix} := \left(\boldsymbol{I}_2 \otimes \left(\boldsymbol{M}\boldsymbol{S}^{-1} \right) \right) \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{d}\boldsymbol{\xi} \\ \mathrm{d}\boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\eta} \\ \boldsymbol{u} - \widetilde{\boldsymbol{\Upsilon}}\, \nabla_{\boldsymbol{\xi}} U(\boldsymbol{\xi}) - \nabla_{\boldsymbol{\eta}} F(\boldsymbol{\eta}) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \boldsymbol{0}_{n\times n} \\ \boldsymbol{I}_n \end{pmatrix} \mathrm{d}\boldsymbol{w}$$

$$\widetilde{\Upsilon} := \left(\prod_{i=1}^{n} \sigma_i^2 m_i^{-2} \right) MS^{-2} \qquad F(\boldsymbol{\eta}) := \frac{1}{2} \left\langle \boldsymbol{\eta}, \boldsymbol{S}^{-1} \boldsymbol{\Gamma} \boldsymbol{\eta} \right\rangle$$

$$U(\boldsymbol{\xi}) := \left(\frac{1}{2} \sum_{i<j} k_{ij} \left(1 - \cos\left(\frac{\sigma_i}{m_i}\xi_i - \frac{\sigma_j}{m_j}\xi_j - \varphi_{ij} \right) \right) - \sum_{i=1}^{n} \frac{\sigma_i}{m_i} P_i \xi_i \right) \left(\prod_{i=1}^{n} \left(\frac{m_i}{\sigma_i} \right)^2 \right)$$

Feedback Synthesis via the Schrödinger System

The First Order Case

Uncontrolled forward-backward Fokker-Planck PDEs

$$\frac{\partial \hat{\varphi}}{\partial t} = \nabla_{\xi} \cdot \left(\hat{\varphi} \Upsilon \nabla_{\xi} \tilde{V} \right) + \Delta_{\xi} \hat{\varphi}$$

$$\frac{\partial \varphi}{\partial t} = \left\langle \nabla_{\xi} \varphi, \Upsilon \nabla_{\xi} \tilde{V} \right\rangle - \Delta_{\xi} \varphi$$

Initial and Terminal conditions

$$\hat{\varphi}_0(\boldsymbol{\xi}) \varphi_0(\boldsymbol{\xi}) = \rho_0(\boldsymbol{S}\boldsymbol{\xi}) \left(\prod_{i=1}^{n} \sigma_i \right)$$

$$\hat{\varphi}_T(\boldsymbol{\xi}) \varphi_T(\boldsymbol{\xi}) = \rho_T(\boldsymbol{S}\boldsymbol{\xi}) \left(\prod_{i=1}^{n} \sigma_i \right)$$

Optimal controlled joint state PDF: $\rho^{\mathrm{opt}}(\boldsymbol{\theta}, t) = \hat{\varphi}(\boldsymbol{S}^{-1}\boldsymbol{\theta}, t) \varphi(\boldsymbol{S}^{-1}\boldsymbol{\theta}, t) / \left(\prod_{i=1}^{n} \sigma_i \right)$

Optimal control: $\boldsymbol{u}^{\mathrm{opt}}(\boldsymbol{\theta}, t) = \boldsymbol{S} \nabla_{\boldsymbol{\theta}} \log \varphi \left(\boldsymbol{S}^{-1}\boldsymbol{\theta}, t \right)$

Feedback Synthesis via the Schrödinger System

The Second Order Case

Uncontrolled forward-backward Fokker-Planck PDEs

$$\frac{\partial \hat{\varphi}}{\partial t} = -\left\langle \boldsymbol{\eta}, \nabla_{\boldsymbol{\xi}} \hat{\varphi} \right\rangle + \nabla_{\boldsymbol{\eta}} \cdot \left(\hat{\varphi} \left(\widetilde{\boldsymbol{\Upsilon}}\, \nabla_{\boldsymbol{\xi}} \boldsymbol{U}(\boldsymbol{\xi}) + \nabla_{\boldsymbol{\eta}} \boldsymbol{F}(\boldsymbol{\eta}) \right) \right) + \Delta_{\boldsymbol{\eta}} \hat{\varphi}$$

$$\frac{\partial \varphi}{\partial t} = -\left\langle \boldsymbol{\eta}, \nabla_{\boldsymbol{\xi}} \varphi \right\rangle + \left\langle \widetilde{\boldsymbol{\Upsilon}}\, \nabla_{\boldsymbol{\xi}} \boldsymbol{U}(\boldsymbol{\xi}) + \nabla_{\boldsymbol{\eta}} \boldsymbol{F}(\boldsymbol{\eta}), \nabla_{\boldsymbol{\eta}} \varphi \right\rangle - \Delta_{\boldsymbol{\eta}} \varphi$$

Initial and Terminal conditions

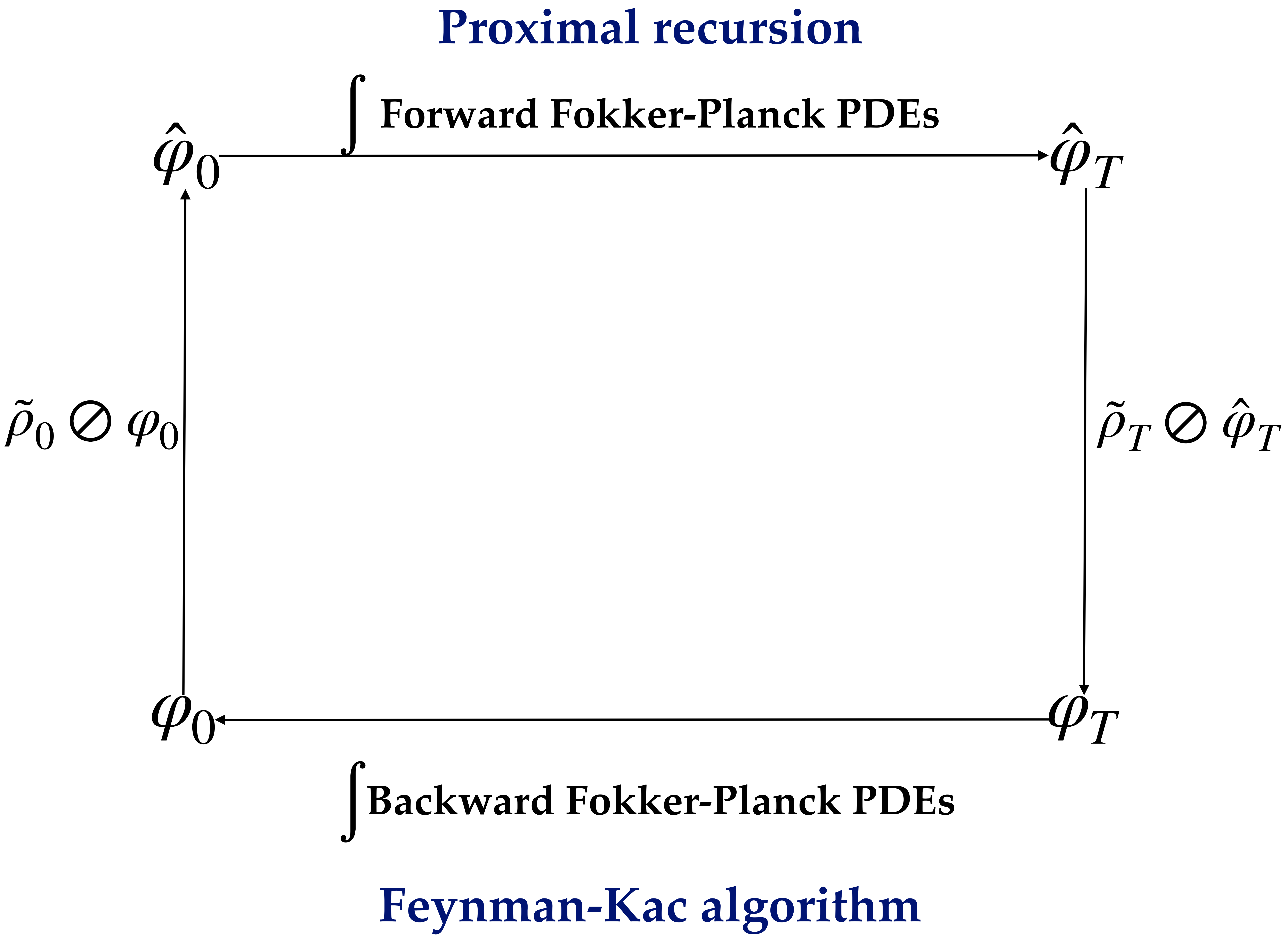
$$\hat{\varphi}_0(\boldsymbol{\xi})\varphi_0(\boldsymbol{\xi}) = \rho_0 \left(\left(\boldsymbol{I}_2 \otimes \boldsymbol{S}\boldsymbol{M}^{-1} \right) \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix} \right) \left(\prod_{i=1}^{n} \frac{\sigma_i^2}{m_i^2} \right)$$

$$\hat{\varphi}_T(\boldsymbol{\xi})\varphi_T(\boldsymbol{\xi}) = \rho_T \left(\left(\boldsymbol{I}_2 \otimes \boldsymbol{S}\boldsymbol{M}^{-1} \right) \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix} \right) \left(\prod_{i=1}^{n} \frac{\sigma_i^2}{m_i^2} \right)$$

Optimal controlled joint state PDF: $\rho^{\text{opt}}(\boldsymbol{\theta}, \boldsymbol{\omega}, t) = \hat{\varphi} \left(\left(\boldsymbol{I}_2 \otimes \boldsymbol{M}\boldsymbol{S}^{-1} \right) \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{pmatrix}, t \right) \varphi \left(\left(\boldsymbol{I}_2 \otimes \boldsymbol{M}\boldsymbol{S}^{-1} \right) \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{pmatrix}, t \right) \left(\prod_{i=1}^{n} \frac{m_i^2}{\sigma_i^2} \right)$

Optimal control: $\boldsymbol{u}^{\text{opt}} \left(\left(\boldsymbol{I}_2 \otimes \boldsymbol{M}\boldsymbol{S}^{-1} \right) \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{pmatrix}, t \right) = \left(\boldsymbol{I}_2 \otimes \boldsymbol{S}\boldsymbol{M}^{-1} \right) \nabla_{\boldsymbol{\theta}} \log \varphi \left(\left(\boldsymbol{I}_2 \otimes \boldsymbol{M}\boldsymbol{S}^{-1} \right) \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{pmatrix}, t \right)$

Fixed Point Recursion Over Pair $(\varphi, \hat{\varphi})$



Proximal recursion

$$\hat{\phi}_k = \text{prox}^{d}_{h\Psi}\left(\hat{\phi}_{k-1}\right) := \arg\inf_{\hat{\phi}} \frac{1}{2}\left(d\left(\hat{\phi}, \hat{\phi}_{k-1}\right)\right)^2 + h\Psi(\hat{\phi})$$

Distance

Step size

Energy-like functional

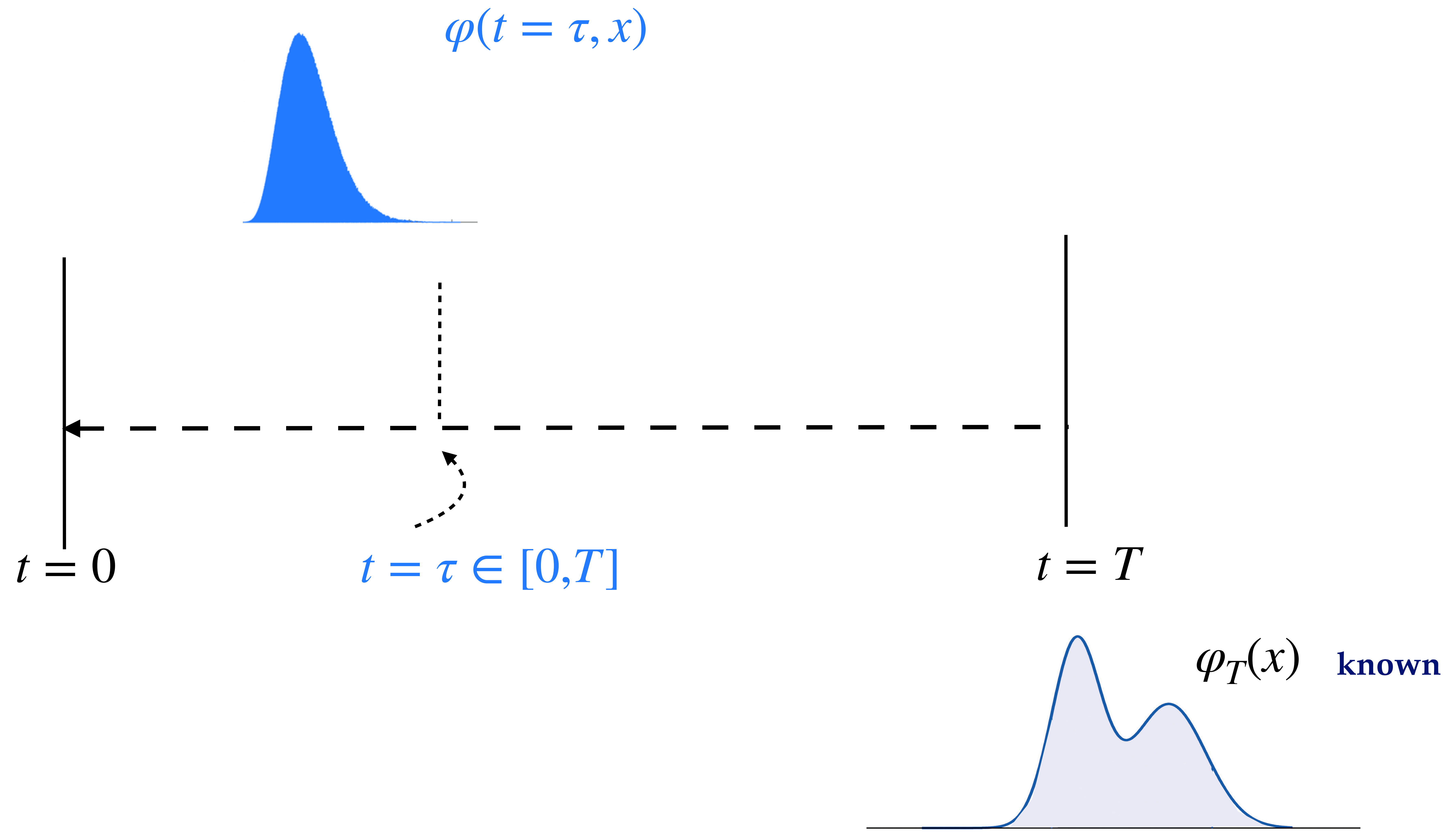
First order: $d \equiv W_{\Upsilon}$ $\quad \Psi(\hat{\phi}) \equiv \int_{\prod_{i=1}^{n}[0,2\pi/\sigma_i)} (\tilde{V} + \log\hat{\phi})\hat{\phi}\mathrm{d}\boldsymbol{\xi}$

Second order: $d \equiv W_{h,\tilde{\Upsilon}}$

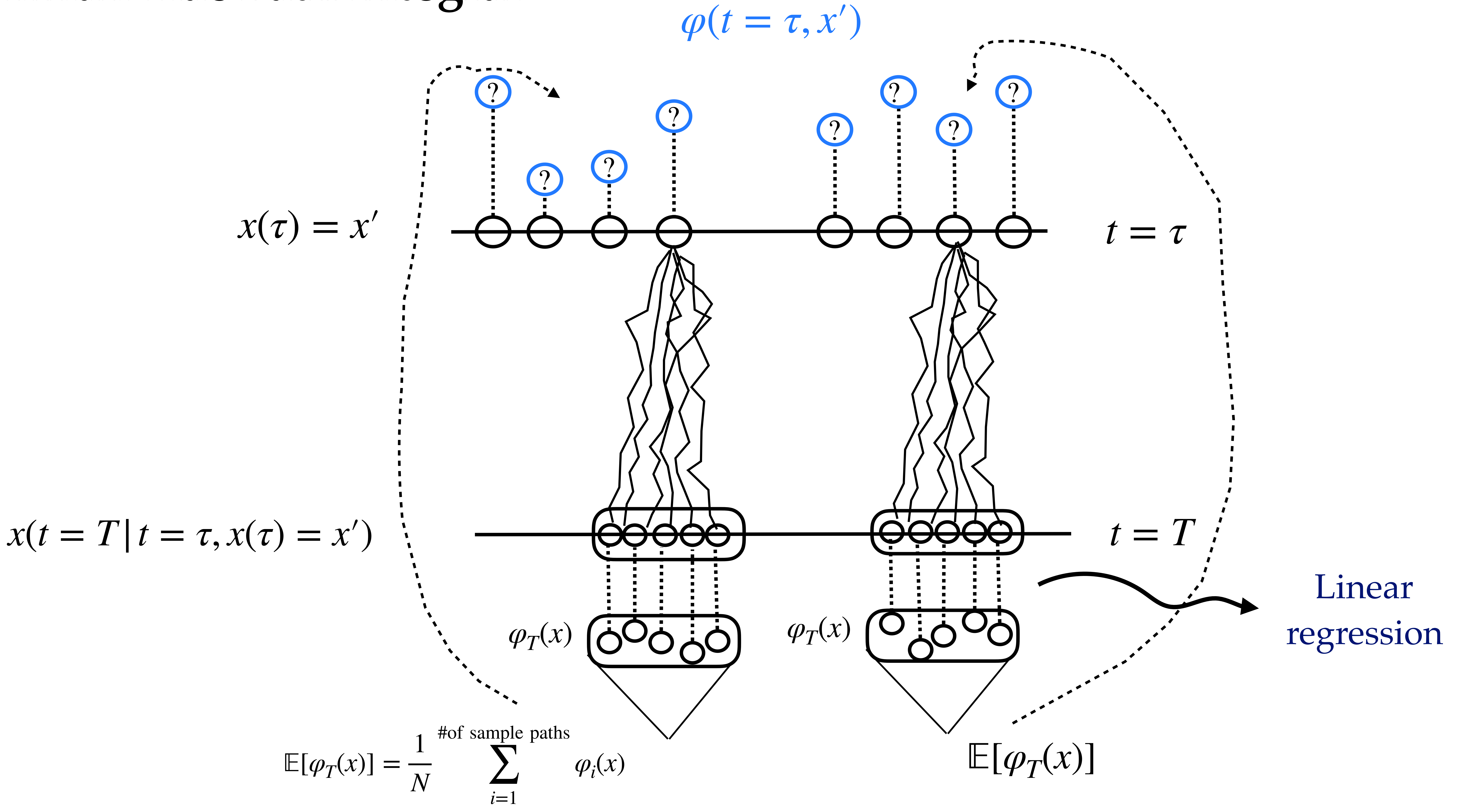
$$\Psi(\hat{\phi}) \equiv \int \left(\prod_{i=1}^{n}[0, 2\pi m_i/\sigma_i)\right) \times \mathbb{R}^n (F + \log\hat{\phi})\hat{\phi}\mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta}$$

Feynman-Kac Path Integral

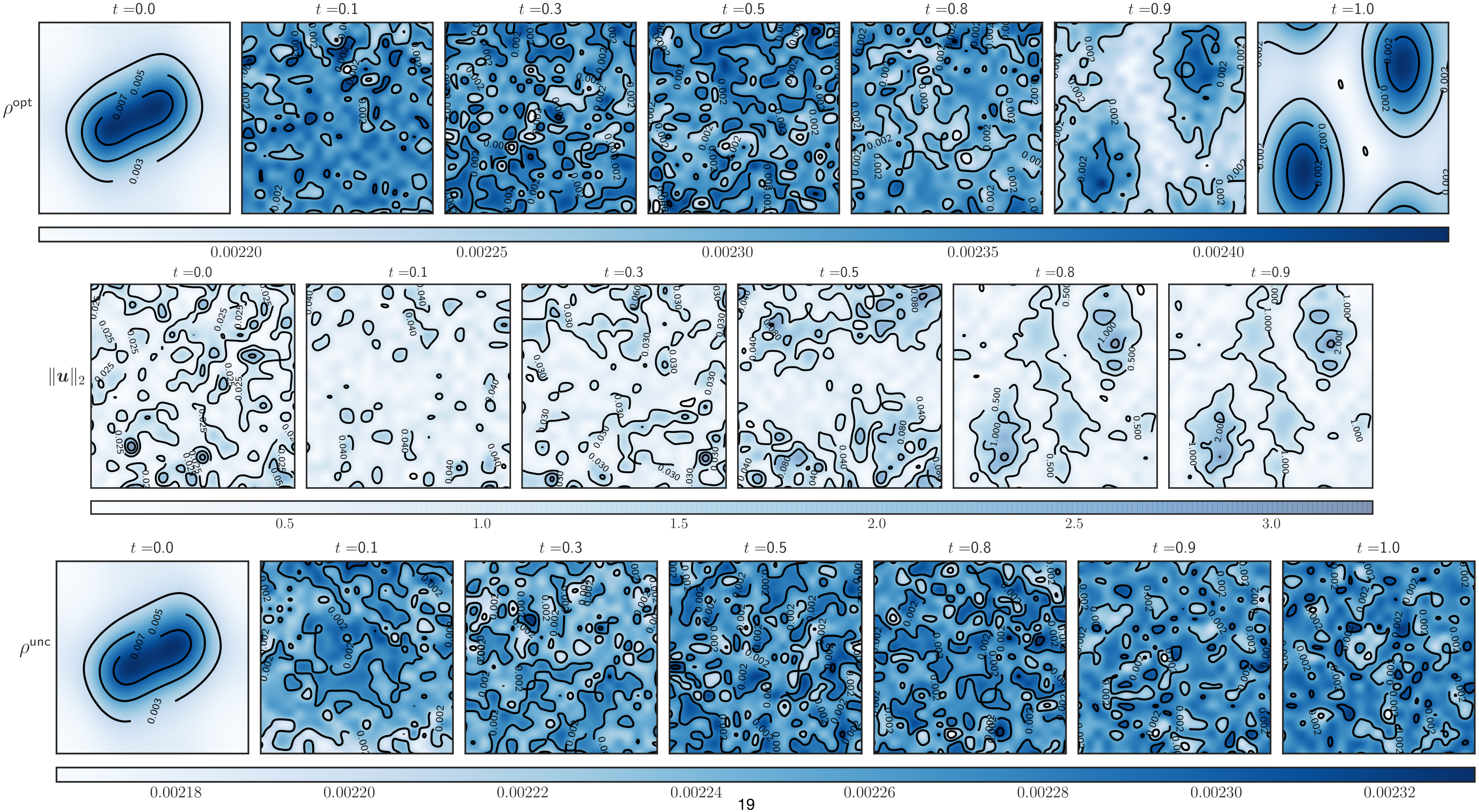
$$\frac{\partial \varphi}{\partial t} = L_{\text{Backward}}\varphi$$



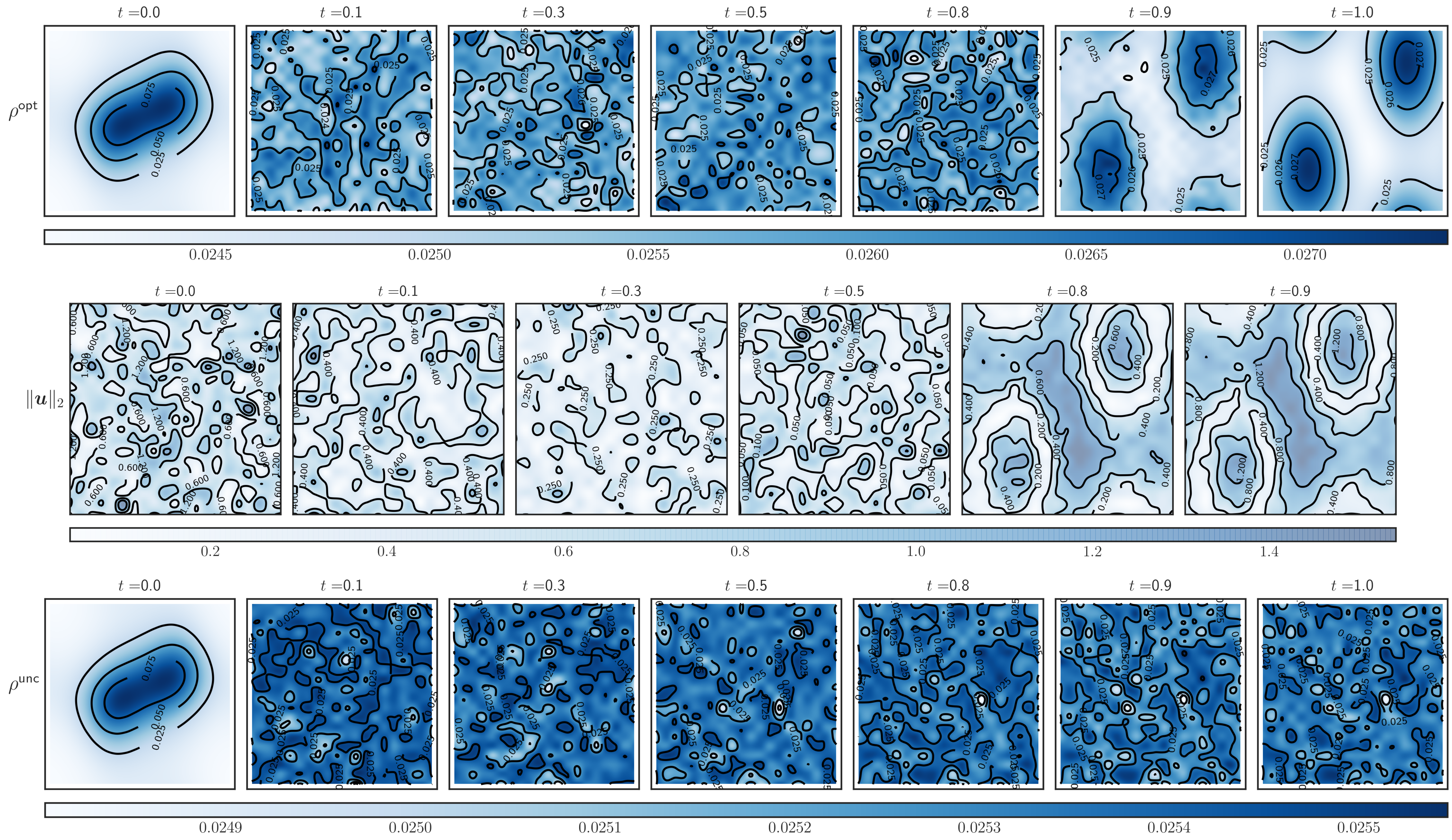
Feynman-Kac Path Integral



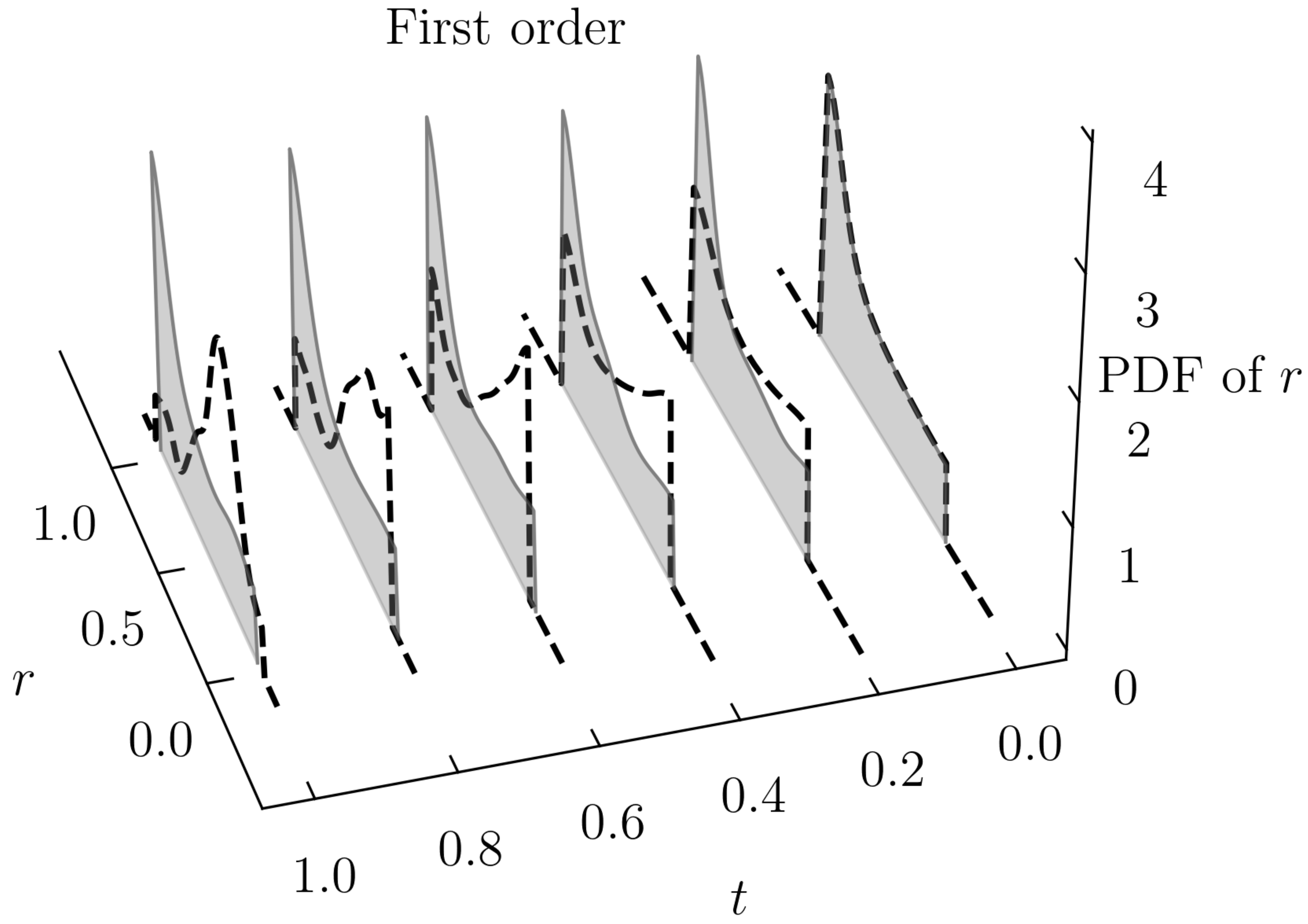
Numerical Example: First Order Case

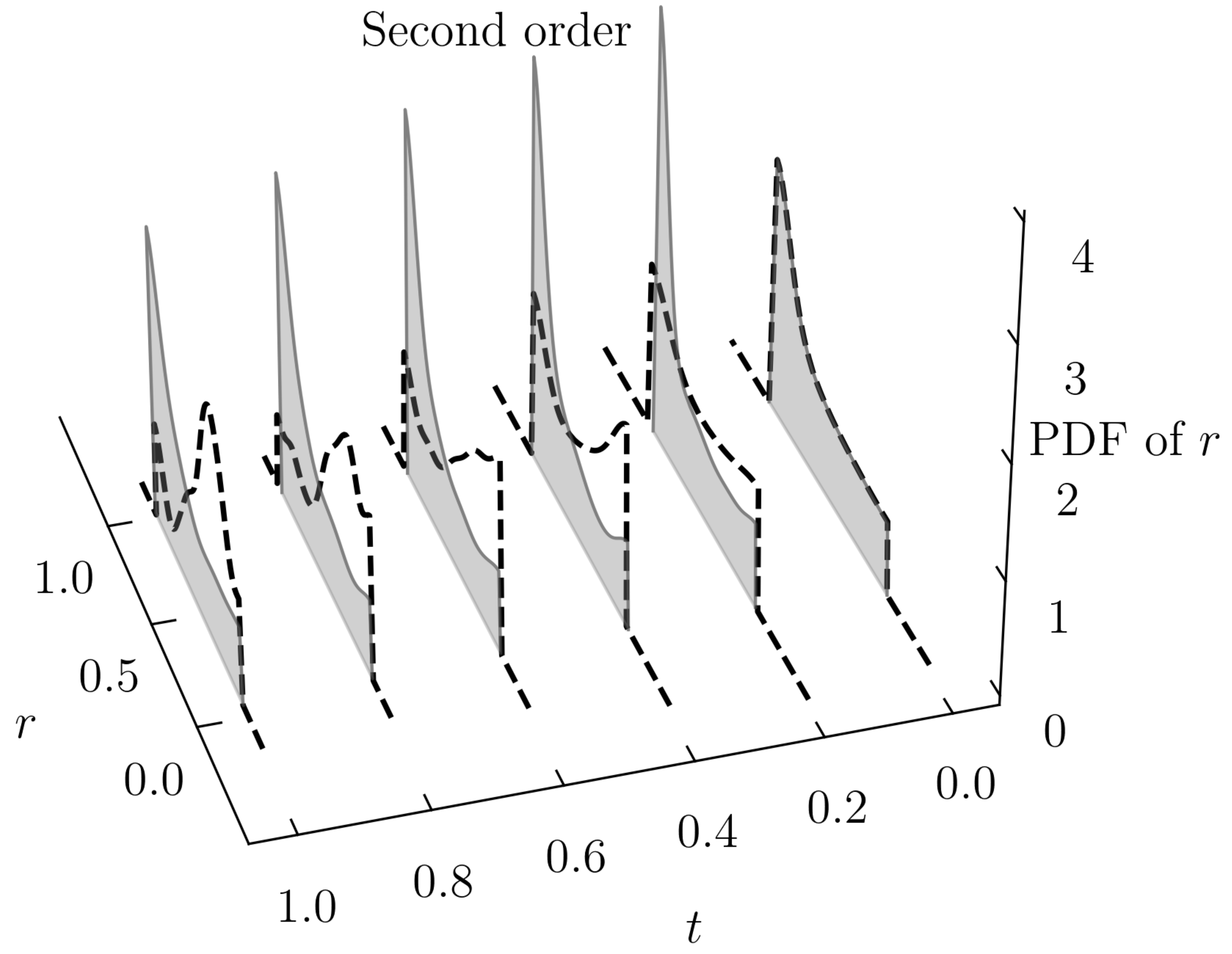


Numerical Example: Second Order Case



Numerical Example: Controlled Order Parameter PDFs





PDF of order parameter $$r := \frac{1}{n}\sqrt{\left(\sum_{i=1}^{n}\cos\theta_i\right)^2 + \left(\sum_{i=1}^{n}\sin\theta_i\right)^2}$$

Thank You