Certifying the Intersection of Reach Sets of Integrator Agents with Set-valued Input Uncertainties

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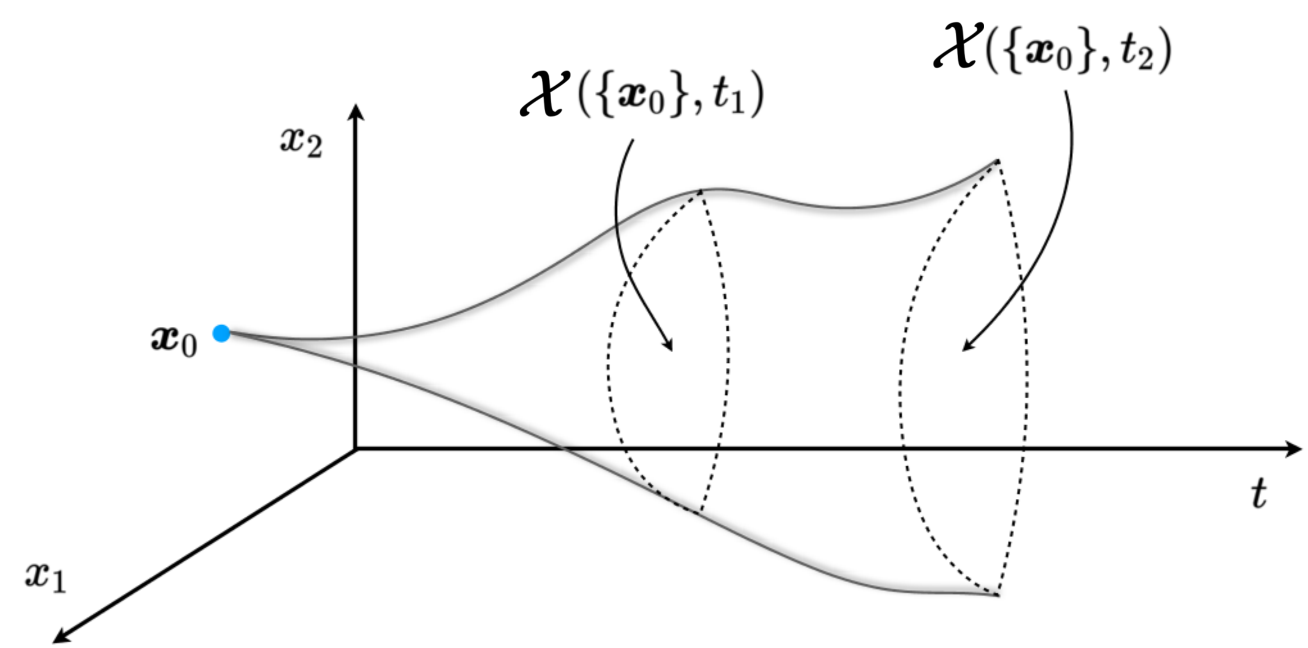
Reach Set

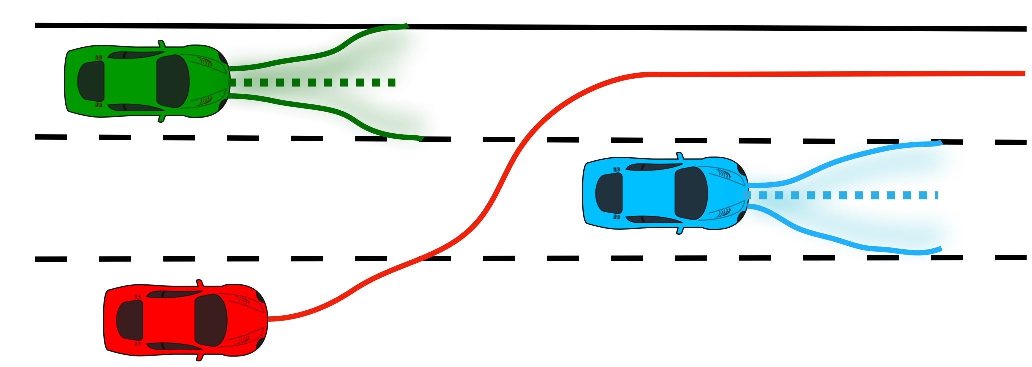
$$\mathcal{X}_t := \{\boldsymbol{x}(t) \in \mathbb{R}^n \mid \dot{x} = f(x) + g(x)\boldsymbol{u},$$

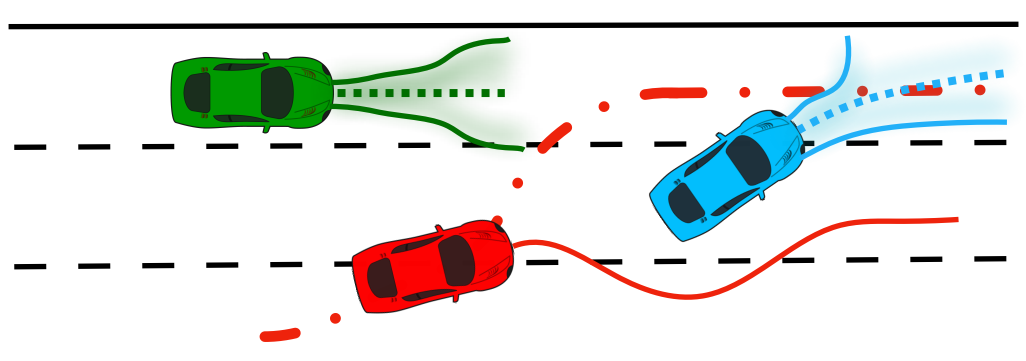
$$\boldsymbol{x}(t=0) = \boldsymbol{x}_0, \boldsymbol{u}(s) \in \mathcal{U}(s)$$

$$\text{for all } 0 \leq s \leq t\}$$

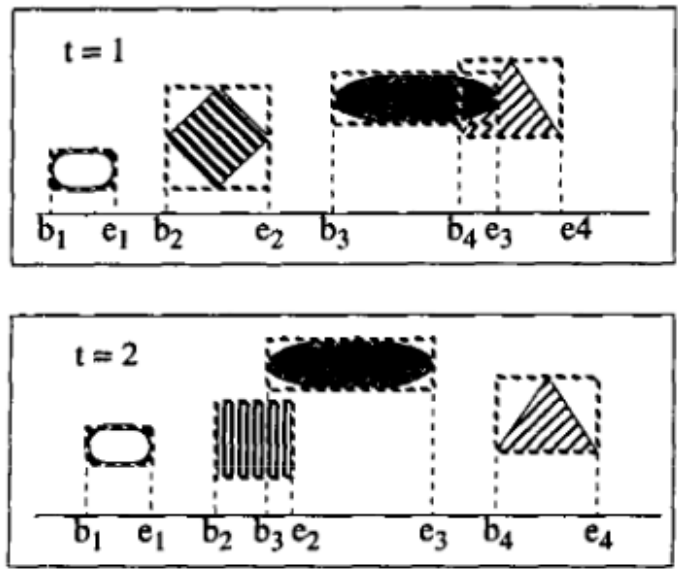


Safety critical applications such as motion planning & collision warning systems

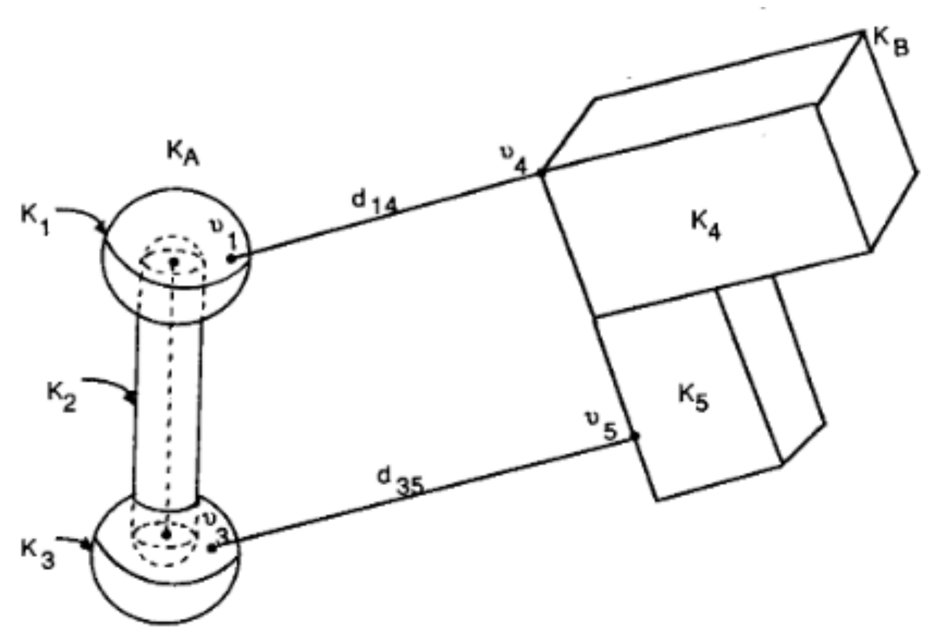




Existing Algorithms



[Cohen et. al, 1995]



[Elmer et. al, 1988]

No specific algebraic results about the ground truth

Difficult to quantitatively compare performance

Our approach

Generic $\longrightarrow$ specific algorithm exploiting geometry of the true set

Contribution

Integrator reach set

Brunovsky normal form

$$\mathcal{X}_t := \{\boldsymbol{x}(t) \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u},\ \boldsymbol{x}(t=0) = x_0,\ \boldsymbol{u}(s) \in \mathcal{U}(s),\ \forall 0 \leq s \leq t\}$$

$$\mathcal{U}(s) := \{\boldsymbol{u}(s) \in \mathbb{R}^m \mid \|\boldsymbol{u}(s)\|_p \leq \ell(s)\}, \forall s \in [0, t]$$

$$\boldsymbol{A} := \text{blkdiag}(\boldsymbol{A}_1, ..., \boldsymbol{A}_m)\,, \quad \boldsymbol{B} := \text{blkdiag}(\boldsymbol{b}_1, ..., \boldsymbol{b}_m)\,,$$

$$\boldsymbol{A}_j = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \dots & \boldsymbol{e}_{r_j-1} \end{bmatrix}, \quad \boldsymbol{b}_j = \boldsymbol{e}_{r_j}$$

$$\boldsymbol{r} = (r_1, r_2, \dots, r_m)^\top \in \mathbb{Z}_+^m$$

Relative degree vector

Benchmarking the performance of over-approximation algorithms

Predicting the intersection between differentially flat nonlinear systems

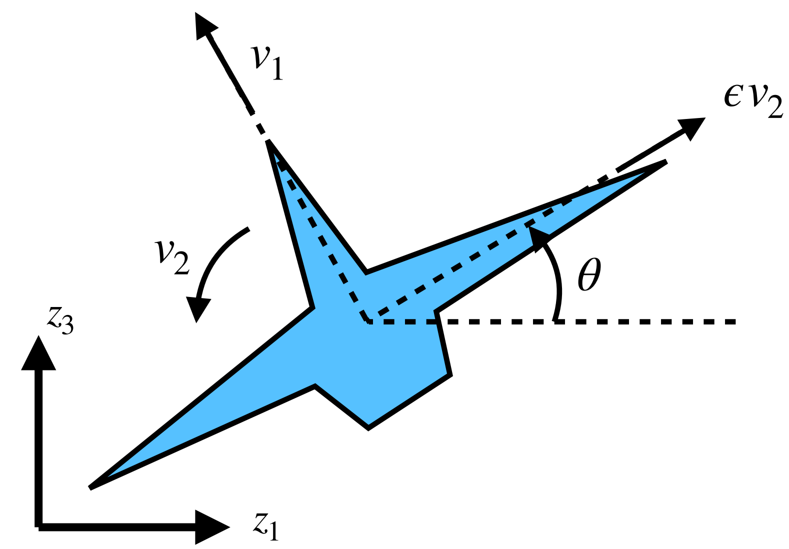
Motivation

Dynamics of VTOL aircraft

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -v_1 \sin(z_5) + \epsilon v_2 \cos(z_5) \\ \dot{z}_3 &= z_4 \end{aligned} \qquad \begin{aligned} \dot{z}_4 &= v_1 \cos(z_5) + \epsilon v_2 \sin(z_5) - g \\ \dot{z}_5 &= z_6 \\ \dot{z}_6 &= v_2 \end{aligned}$$

Normal form

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_2 \end{pmatrix}$$

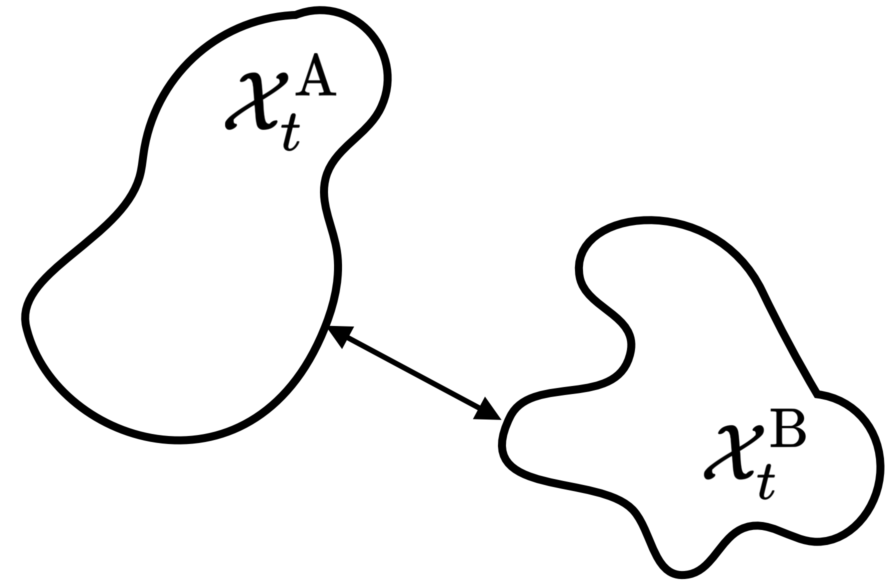


The VTOL aircraft

Intersection in normal coordinate $\boldsymbol{x}$ $\rightleftarrows$ **Intersection in original coordinates $\boldsymbol{z}$**

Integrator reach sets are in general, compact and convex

Intersection Detection



General formula for the collision detection

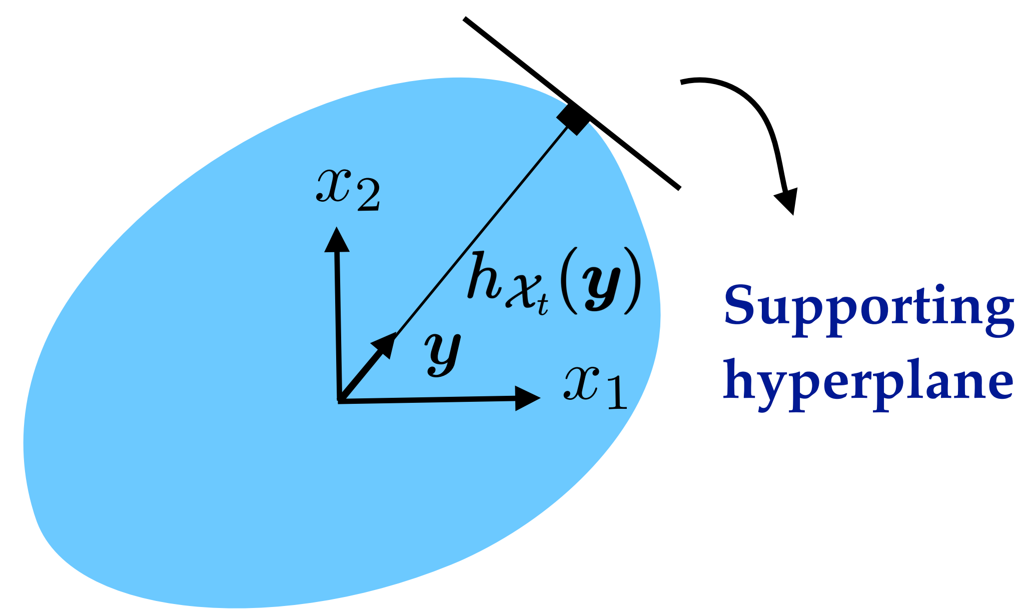
$$\operatorname{dist}(\mathrm{A},\mathrm{B}) := \min_{\boldsymbol{x}^{\boldsymbol{A}} \in \mathcal{X}_t^{\mathrm{A}}, \boldsymbol{x}^{\mathrm{B}} \in \mathcal{X}_t^{\mathrm{B}}} \left\| \boldsymbol{x}^{\mathrm{A}} - \boldsymbol{x}^{\mathrm{B}} \right\|_2^2$$

Lack of analytical handle on the boundary!!!

Minkowski Difference

Alternative approach, intersection occurs if $\mathbf{0} \in \{\mathcal{X}_t^{\mathrm{A}} \dot{-} \mathcal{X}_t^{\mathrm{B}}\}$

Support function $h_{\mathcal{X}_t}(\boldsymbol{y}) = \max_{\boldsymbol{x} \in \mathcal{X}} \langle \boldsymbol{x}^\top, \boldsymbol{y} \rangle$



$$\mathcal{X}_t^{\mathrm{A}} \cap \mathcal{X}_t^{\mathrm{B}} \neq \varnothing \Leftrightarrow \mathbf{0} \in \mathcal{X}_t^{\mathrm{A}} - \mathcal{X}_t^{\mathrm{B}} \Leftrightarrow$$

$$\forall \boldsymbol{y} \in \mathbb{S}^{n-1}, h_{\mathcal{X}_t^{\mathrm{A}} - \mathcal{X}_t^{\mathrm{B}}}(\boldsymbol{y}) \geq 0$$

Intersection Detection

Intersection Detection Oracle

$$\min_{\boldsymbol{y}\in\mathbb{S}^{n-1}} \{h_{\mathcal{X}_t^{\mathtt{A}}}(\boldsymbol{y}) + h_{\mathcal{X}_t^{\mathtt{B}}}(-\boldsymbol{y})\} \geq (<)\ 0 \iff \mathcal{X}_t^{\mathtt{A}} \cap \mathcal{X}_t^{\mathtt{B}} \neq (=)\ \varnothing$$

Theorem: The support function of $\mathcal{X}_t$

$$h_{\mathcal{X}_t}(\boldsymbol{y}) = \sum_{j=1}^{m} \langle \boldsymbol{y}_j, \exp(t\boldsymbol{A})\boldsymbol{x}_{j0}\rangle + \int_0^t \ell(s)\left\|(\exp(s\boldsymbol{A})\boldsymbol{B})^\top \boldsymbol{y}\right\|_q \mathrm{d}s,\ \ 1/p + 1/q = 1$$

Discretize $[0, t]$ **into** K **intervals:**

$$\int_0^t \left\|(\exp(s\boldsymbol{A})\boldsymbol{B})^\top \boldsymbol{y}\right\|_q \mathrm{d}s \approx$$

$$\frac{\Delta s}{2}\sum_{k=1}^{K}\left(\left\|(\exp(s_{k-1}\boldsymbol{A})\boldsymbol{B})^\top \boldsymbol{y}\right\|_q + \left\|(\exp(s_k\boldsymbol{A})\boldsymbol{B}(s_k))^\top \boldsymbol{y}\right\|_q\right)$$

Lossless Convexification

Nonconvex Optimization Problem

$$\overbrace{\begin{aligned} p^* &= \min_{\boldsymbol{\eta}\in\mathbb{R}^{n+K+1}} \langle \boldsymbol{\kappa}(t), \boldsymbol{\eta}\rangle \\ \textbf{s.}\ & \|\boldsymbol{M}_k\boldsymbol{\eta}\|_q - \langle \boldsymbol{e}_{n+k}^{n+K+1}, \boldsymbol{\eta}\rangle \le 0, \\ -\tilde{\boldsymbol{N}}\boldsymbol{\eta} &\le \boldsymbol{0},\ \ \|\boldsymbol{N}\boldsymbol{\eta}\|_2 = 1 \end{aligned}}$$

Nonconvex Constraint

$$\xrightarrow{\text{Lossless Convexification}}$$

Second-Order Cone Program

$$\overbrace{\begin{aligned} \tilde{p}^* &= \min_{\boldsymbol{\eta}\in\mathbb{R}^{n+K+1}} \langle \boldsymbol{\kappa}(t), \boldsymbol{\eta}\rangle \\ \textbf{s.}\ & \|\boldsymbol{M}_k\boldsymbol{\eta}\|_q - \langle \boldsymbol{e}_{n+k}^{n+K+1}, \boldsymbol{\eta}\rangle \le 0, \\ -\tilde{\boldsymbol{N}}\boldsymbol{\eta} &\le \boldsymbol{0},\ \ \|\boldsymbol{N}\boldsymbol{\eta}\|_2 \le 1 \end{aligned}}$$

(i) $\tilde{p}^* \le 0$

(ii) $\tilde{p}^* = 0 \Rightarrow 0 \le p^* \Leftrightarrow\ \mathcal{X}_t^{\mathrm{A}} \cap \mathcal{X}_t^{\mathrm{B}} \neq \varnothing$

(iii) $\tilde{p}^* < 0 \Rightarrow \tilde{p}^* = p^* < 0 \Leftrightarrow \mathcal{X}_t^{\mathrm{A}} \cap \mathcal{X}_t^{\mathrm{B}} = \varnothing$

Example

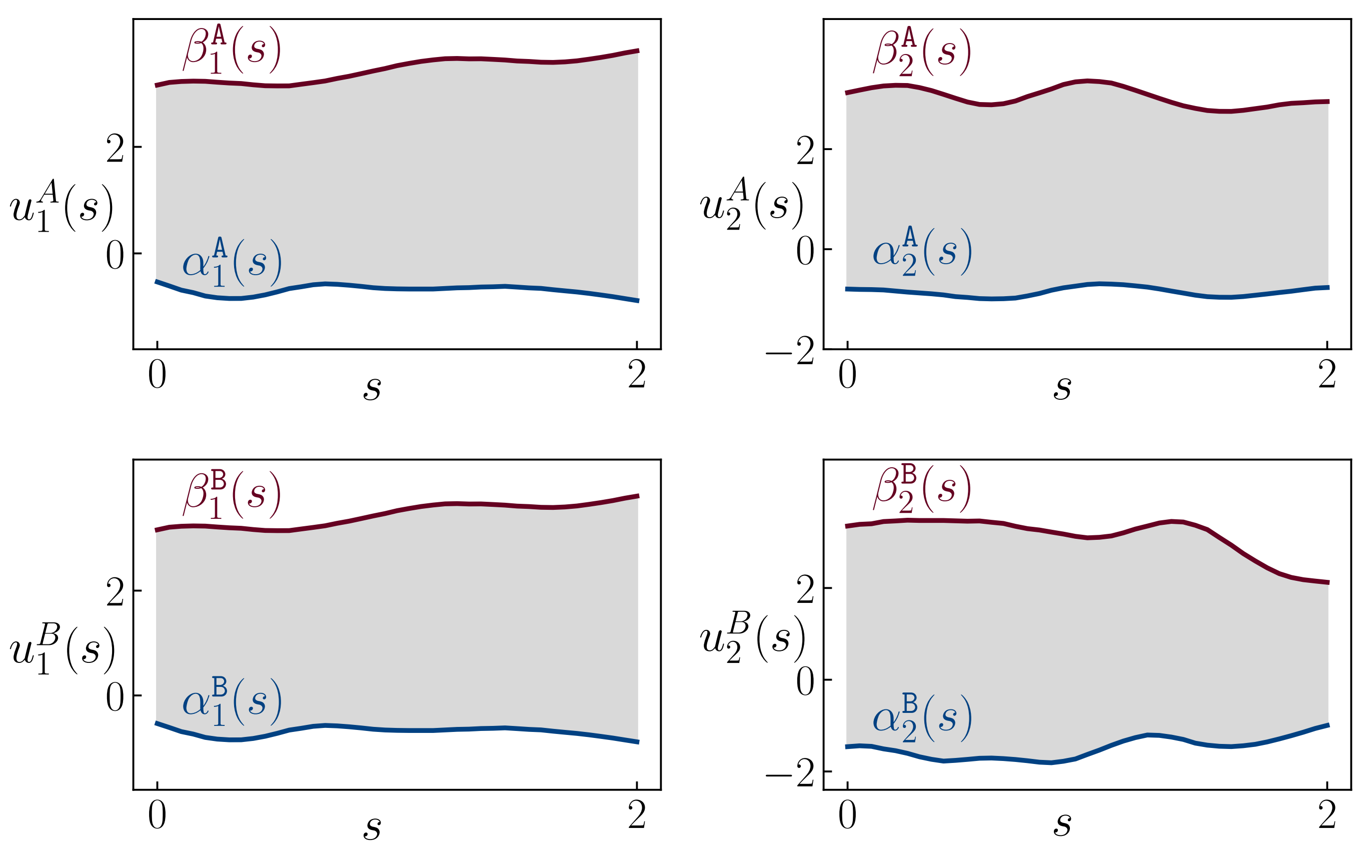
$$\dot{\boldsymbol{x}}^i = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \boldsymbol{x}^i + \left(\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right) \boldsymbol{u}^i, \;\; i \in \{\mathrm{A}, \mathrm{B}\}, \;\; \boldsymbol{r} = (3, 2)^\top$$

Decoupled dynamics for each block

∞-norm ball

$\mathcal{U}(s) := [\alpha_1(s), \beta_1(s)] \times [\alpha_2(s), \beta_2(s)]$ for all $0 \leq s \leq t$

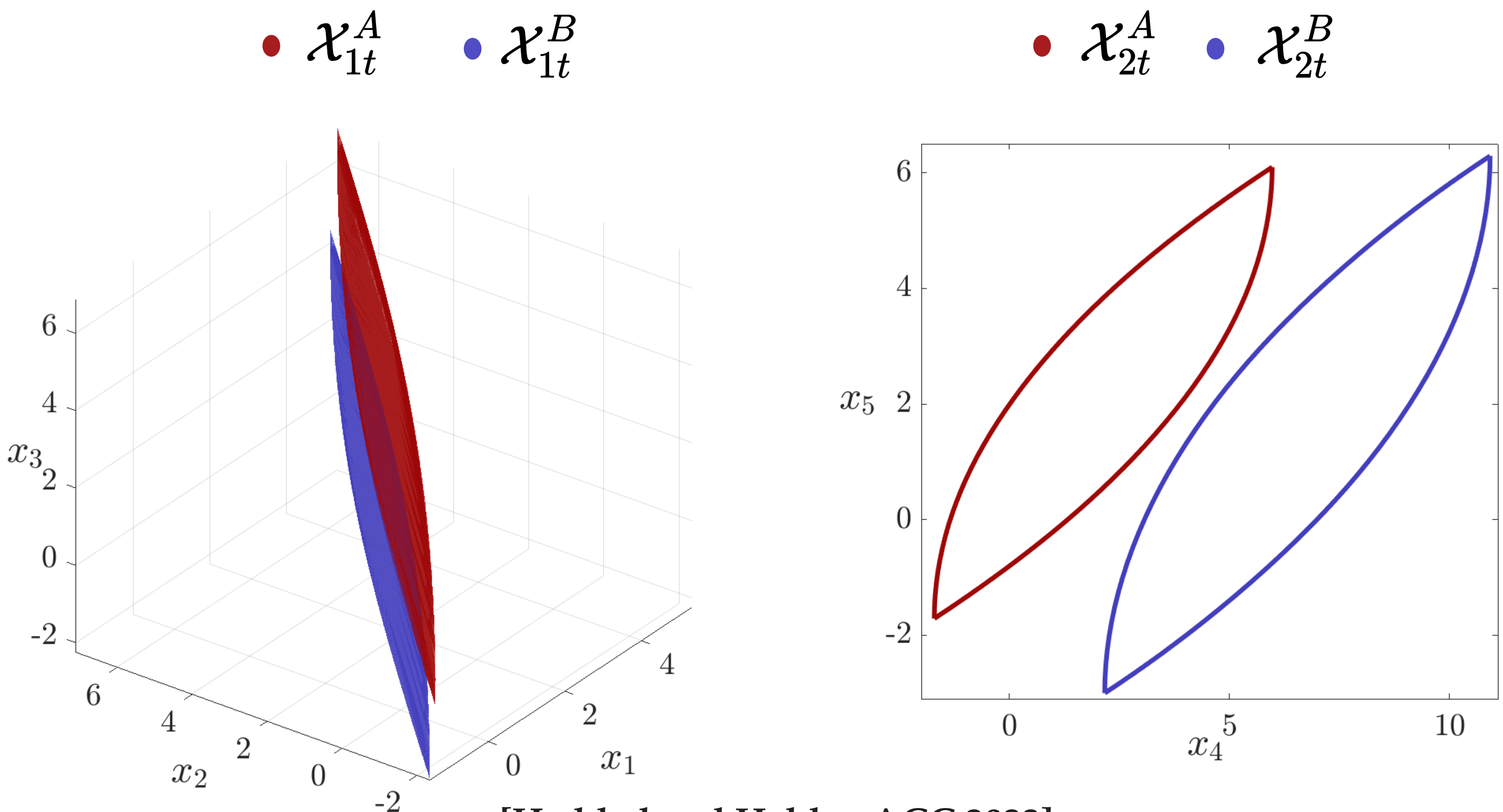
Input Uncertainties



$$\mathcal{X}_t^i = \mathcal{X}_{1t}^i \times \mathcal{X}_{2t}^i = \mathcal{X}_{1t}^i \dotplus \mathcal{X}_{2t}^i$$

$$h_{\mathcal{X}}^i = h_{\mathcal{X}_{1t}}^i + h_{\mathcal{X}_{2t}}^i \quad i \in \{\mathrm{A}, \mathrm{B}\}.$$

Example



[Hadded and Halder, ACC 2022]

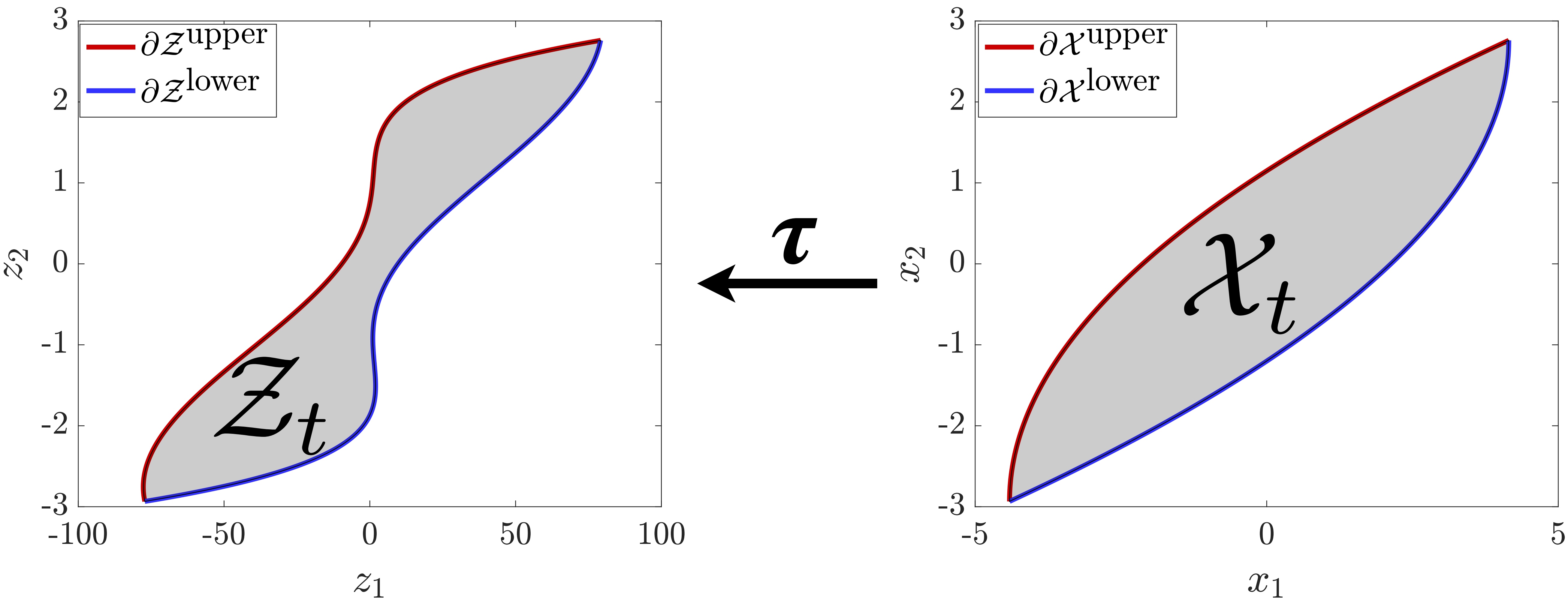
$(\tilde{p}_1^*, \tilde{p}_2^*) = (0, -0.54)$ **Computation time ≈ 1.24 s**

$\mathcal{X}_{1t}^{\mathrm{A}} \cap \mathcal{X}_{1t}^{\mathrm{B}} \neq \varnothing$ **and** $\mathcal{X}_{2t}^{\mathrm{A}} \cap \mathcal{X}_{2t}^{\mathrm{B}} = \varnothing \quad \Leftrightarrow \quad \boxed{\mathcal{X}_{t}^{\mathrm{A}} \cap \mathcal{X}_{t}^{\mathrm{B}} = \varnothing}$

Ongoing Work

Map them back to original coordinate z via known diffeomorphism

Compute the reach set and its functionals in normal coordinate x



$\mathrm{vol}\left(\mathcal{Z}_t\right) = \mathbf{206.7362}$

$\mathrm{vol}\left(\mathcal{X}_t\right) = \mathbf{15.4292}$

Thank You