The Ground Cost for Optimal Transport of Angular Velocity

Presenter:

Yongxin Chen, Georgia Institute of Technology

Authors:

Karthik Elamvazhuthi, Los Alamos National Lab
Abhishek Halder, Iowa State University

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Problem: Minimum Effort Steering of Spin PDF

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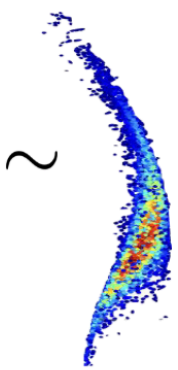
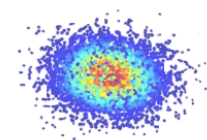
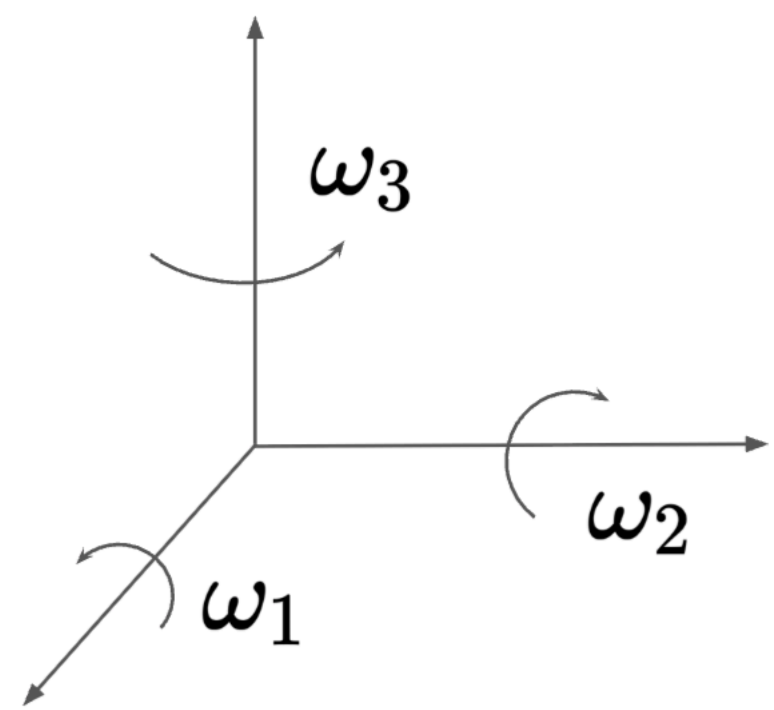
$\boldsymbol{\omega}(t=0) \sim$  ρ_0 $\quad$ $\boldsymbol{\omega}(t=t_{\mathrm{f}}) \sim$  ρ_{f}

Motivation: stochastic spin stabilization over a given deadline

Initial and terminal PDF: estimation errors, desired statistical accuracy



Controlled Spin Dynamics: Euler Equations

Define (scaled) state $\boldsymbol{x} := \boldsymbol{J} \odot \boldsymbol{\omega}$, control $\boldsymbol{u} := \boldsymbol{\tau}$

positive vector of principal moments of inertia

vector of control torques

and parameters $\alpha := \dfrac{1}{J_3} - \dfrac{1}{J_2}, \quad \beta := \dfrac{1}{J_1} - \dfrac{1}{J_3}, \quad \gamma := \dfrac{1}{J_2} - \dfrac{1}{J_1}$

Note that $\alpha + \beta + \gamma = 0$

Controlled ODE:

$$\dot{x}_1 = \alpha x_2 x_3 + u_1,$$
$$\dot{x}_2 = \beta x_3 x_1 + u_2,$$
$$\dot{x}_3 = \gamma x_1 x_2 + u_3,$$

known to be controllable

Stochastic Optimal Control Problem

$$\underset{(\xi,\boldsymbol{u})}{\arg\inf} \int_0^{t_{\mathrm{f}}} \int_{\mathbb{R}^3} \frac{1}{2}\boldsymbol{u}^\top \boldsymbol{u}\,\underbrace{\xi(t,\boldsymbol{x})}_{\text{controlled state PDF}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

subject to

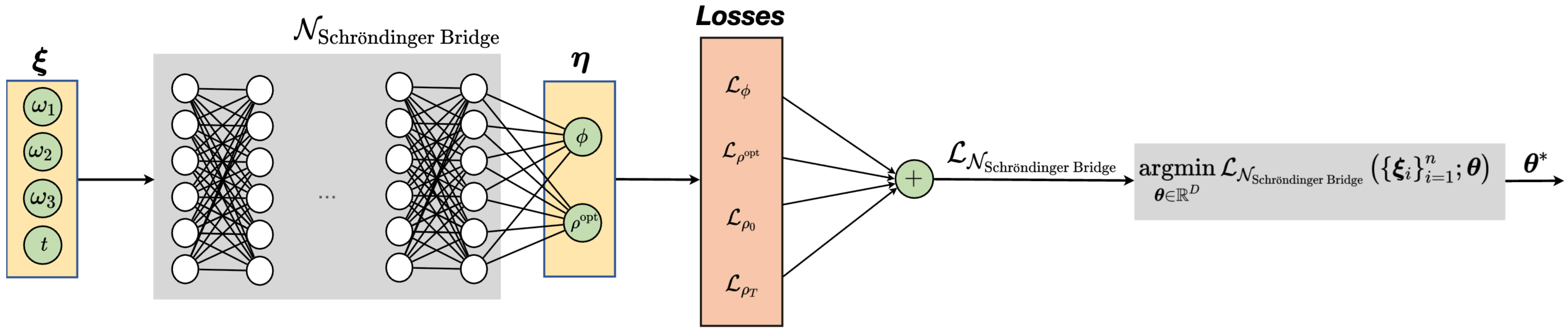
$$\underbrace{\begin{aligned} \dot{x}_1 &= \alpha x_2 x_3 + u_1, \\ \dot{x}_2 &= \beta x_3 x_1 + u_2, \\ \dot{x}_3 &= \gamma x_1 x_2 + u_3, \end{aligned}}_{\text{controlled dynamical constraints}}$$

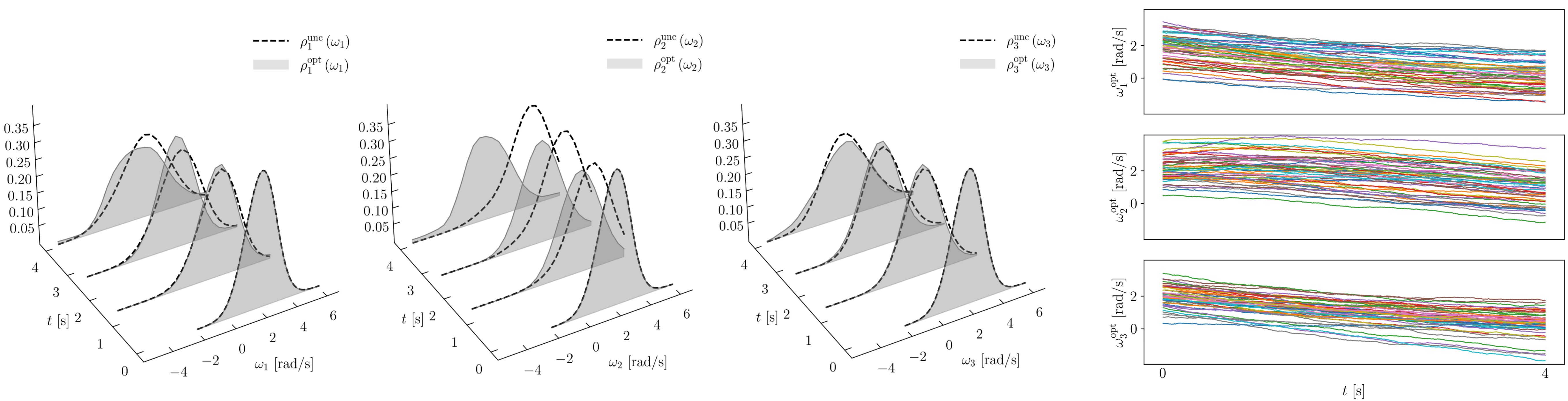
$$\underbrace{\boldsymbol{x}_0 \sim \xi_0 := \frac{\rho_0(\boldsymbol{x} \oslash \boldsymbol{J})}{J_1 J_2 J_3}, \quad \boldsymbol{x}_{\mathrm{f}} \sim \xi_{\mathrm{f}} := \frac{\rho_{\mathrm{f}}(\boldsymbol{x} \oslash \boldsymbol{J})}{J_1 J_2 J_3}}_{\text{initial and terminal statistics constraints}}$$

Prior Results in Yan *et al.*, CDC23

Thm. Existence-uniqueness of minimizer $\left(\xi^{\rm opt}(t,\boldsymbol{x}),\boldsymbol{u}^{\rm opt}(t,\boldsymbol{x})\right)$ is guaranteed if the endpoint PDFs have finite second moments

Numerics. Solved conditions of optimality (coupled PDEs) using a custom Physics Informed Neural Network (PINN)





Goal of this Work

Better understand connections with optimal mass transport (OMT)

Our Stochastic Optimal Control Problem is an Instance of Generalized OMT (GOMT)

	OMT	Our problem
Dynamic version	$\underset{(\xi,\boldsymbol{u})}{\arg\inf} \int_0^{t_{\mathrm{f}}} \int_{\mathbb{R}^3} \frac{1}{2}\boldsymbol{u}^\top \boldsymbol{u}\xi(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$ $\dot{\boldsymbol{x}} = \boldsymbol{u},$ $\boldsymbol{x}_0 \sim \xi_0\,, \quad \boldsymbol{x}_{\mathrm{f}} \sim \xi_{\mathrm{f}}$	$\underset{(\xi,\boldsymbol{u})}{\arg\inf} \int_0^{t_{\mathrm{f}}} \int_{\mathbb{R}^3} \frac{1}{2}\boldsymbol{u}^\top \boldsymbol{u}\xi(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$ $\dot{x}_1 = \alpha x_2 x_3 + u_1,$ $\dot{x}_2 = \beta x_3 x_1 + u_2,$ $\dot{x}_3 = \gamma x_1 x_2 + u_3,$ $\boldsymbol{x}_0 \sim \xi_0\,, \quad \boldsymbol{x}_{\mathrm{f}} \sim \xi_{\mathrm{f}}$
Static version	$\underset{\pi\in\Pi_2(\xi_0,\xi_{\mathrm{f}})}{\arg\inf} \int_{\mathbb{R}^3\times\mathbb{R}^3} \boxed{\frac{1}{2t_{\mathrm{f}}}\lVert\boldsymbol{x}_0 - \boldsymbol{x}_{\mathrm{f}}\rVert_2^2}\,\mathrm{d}\pi\left(\boldsymbol{x}_0,\boldsymbol{x}_{\mathrm{f}}\right)$ Here **ground cost** is scaled Euclidean distance squared, straight lines are minimal geodesics	$\underset{\pi\in\Pi_2(\xi_0,\xi_{\mathrm{f}})}{\arg\inf} \int_{\mathbb{R}^3\times\mathbb{R}^3} \boxed{c(\boldsymbol{x}_0,\boldsymbol{x}_{\mathrm{f}})}\mathrm{d}\pi(\boldsymbol{x}_0,\boldsymbol{x}_{\mathrm{f}})$ What can be said about this **ground cost**?

Static Version is Appealing because It

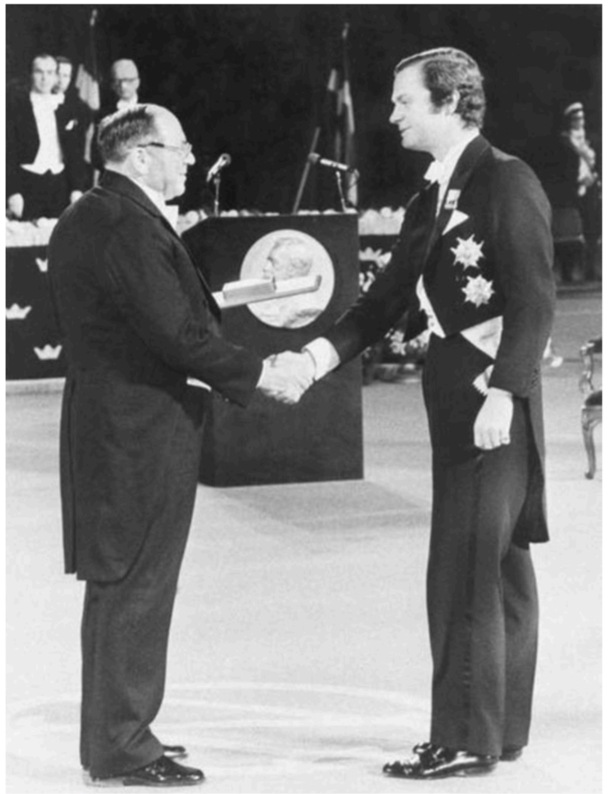
Gives optimal coupling between the initial and terminal stochastic states

Is a linear program

All these are possible if we know the ground cost c

History. OT static version is due to Kantorovich (1941-42)

1975 Nobel Prize in Economics for this work



Finding Our Ground Cost c

Need to compute length of the minimal sub-Riemannian geodesic

Difficult to apply Pontryagin's min principle to analytically compute:

$$c(\boldsymbol{x}_0, \boldsymbol{x}_\mathrm{f}) = \underset{\boldsymbol{u}}{\text{minimum}} \int_0^{t_\mathrm{f}} \frac{1}{2} \|\boldsymbol{u}\|_2^2 \, \mathrm{d}t$$

subject to

$$\begin{aligned} \dot{x}_1 &= \alpha x_2 x_3 + u_1, \\ \dot{x}_2 &= \beta x_3 x_1 + u_2, \\ \dot{x}_3 &= \gamma x_1 x_2 + u_3, \end{aligned}$$

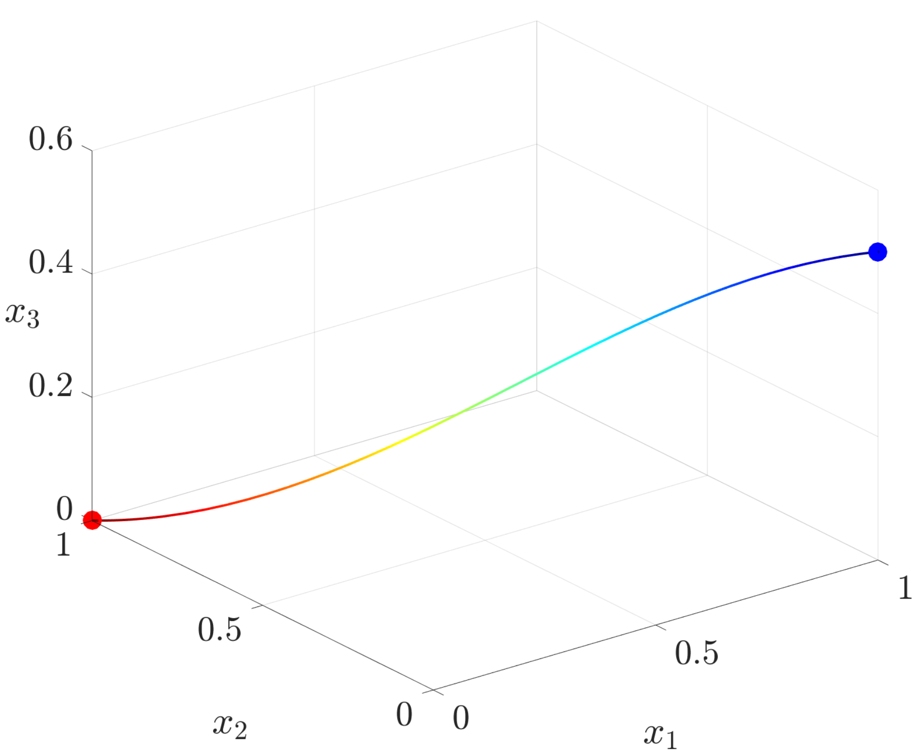
$$\boldsymbol{x}(t=0) = \boldsymbol{x}_0 \text{ (given)}, \quad \boldsymbol{x}(t=t_\mathrm{f}) = \boldsymbol{x}_\mathrm{f} \text{ (given)}$$

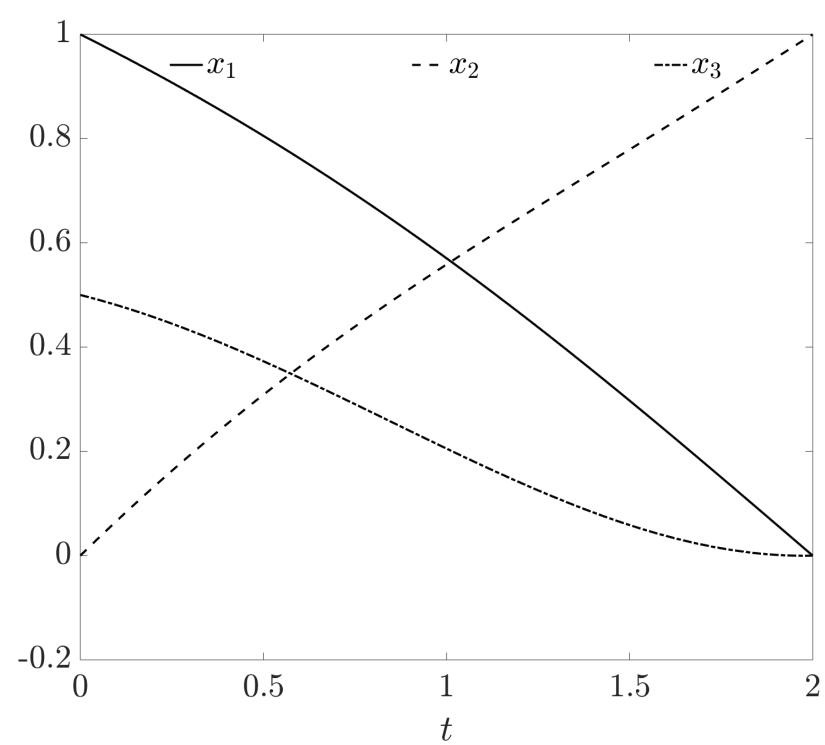
Main Results

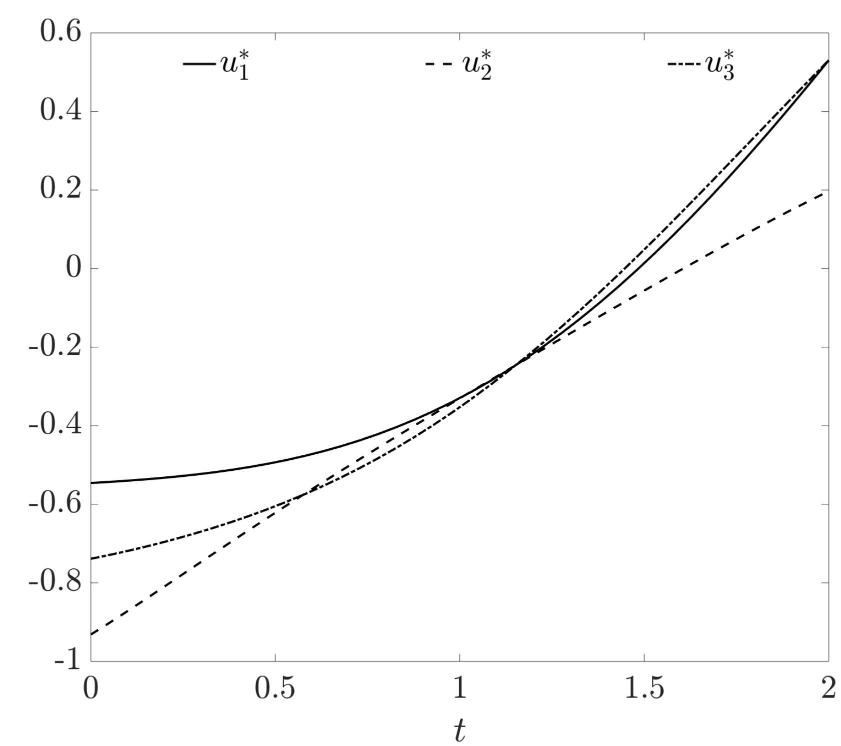
Ground Cost Inequality. $c(\boldsymbol{x}_0, \boldsymbol{x}_{\mathrm{f}}) \leq \dfrac{\|\boldsymbol{x}_0\|_2^2}{t_{\mathrm{f}}} + \dfrac{\|\boldsymbol{x}_{\mathrm{f}}\|_2^2}{t_{\mathrm{f}}}$

A Feasible Controller. $\boldsymbol{u}^*(\boldsymbol{x}) := \left(-\dfrac{\|\boldsymbol{x}_0 - \boldsymbol{x}_{\mathrm{f}}\|_2}{t_{\mathrm{f}}} - \dfrac{\langle \boldsymbol{x} - \boldsymbol{x}_{\mathrm{f}}, \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x}_{\mathrm{f}}) + \boldsymbol{b}\rangle}{\|\boldsymbol{x} - \boldsymbol{x}_{\mathrm{f}}\|_2}\right) \dfrac{\boldsymbol{x} - \boldsymbol{x}_{\mathrm{f}}}{\|\boldsymbol{x} - \boldsymbol{x}_{\mathrm{f}}\|_2}$

where $\boldsymbol{A} := \begin{bmatrix} 0 & \alpha x_{\mathrm{f}3} & \alpha x_{\mathrm{f}2} \\ \beta x_{\mathrm{f}3} & 0 & \beta x_{\mathrm{f}1} \\ \gamma x_{\mathrm{f}2} & \gamma x_{\mathrm{f}1} & 0 \end{bmatrix}$, $\boldsymbol{b} := \begin{pmatrix} \alpha x_{\mathrm{f}2} x_{\mathrm{f}3} \\ \beta x_{\mathrm{f}3} x_{\mathrm{f}1} \\ \gamma x_{\mathrm{f}1} x_{\mathrm{f}2} \end{pmatrix}$







$\boldsymbol{x}_0 = (1, 0, 0.5)^\top$ to $\boldsymbol{x}_{\mathrm{f}} = (0, 1, 0)^\top$ with $J_1 = 1, J_2 = 2, J_3 = 3,$ over $[0, t_{\mathrm{f}}] = [0, 2]$

When Constructed Feasible Controller is Optimal

For any translated norm invariant system, the GOMT ground cost

$$c\left(\boldsymbol{x}_0, \boldsymbol{x}_{\mathrm{f}}\right) = \frac{1}{2t_{\mathrm{f}}}\|\boldsymbol{x}_0 - \boldsymbol{x}_{\mathrm{f}}\|_2^2$$

even though geodesics are not straight lines in general.

Proof idea.

Key technique from Athans et al., TAC 1963

Creatively mix of different variants of Cauchy-Schwarz inequality

Take Home Messages

OMT and its generalizations are now part of a mature discipline

Less structural results for GOMT problem instances with nonlinear drift

State-of-the-art remains diffusion regularization and numerical solution

Future Work.

Stochastic steering of attitude-spin over tangent bundle $\mathcal{T}\mathrm{SO}(3) \simeq \mathrm{SO}(3) \times \mathbb{R}^3$

Thank You