Aero 320: Numerical Methods

Homework 3

Name: ....

Due: October 7, 2013

NOTE: All problems, except the last, are to be done by hand (with the help of a calculator) but you need to show all the steps. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Submit your HW by Monday midnight at Room 201, Reed McDonald Building. Late submissions or failure to submit in the required format will receive no credit.

Problem 1

Basics on matrix algebra **($3 \times 7 = 21$ points)**

Find whether the following statements are *true* or *false*. If you think a statement is *true*, then prove that statement. If *false*, then provide a counterexample.

(a) Any matrix can be written as the sum of a symmetric and a skew-symmetric matrix.

(b) Let the square matrices M and N are of same size. If $\det(M) = 0$, and $\det(N) = 0$, then $\det(M+N) = 0$.

(c) Given square matrices X and Y of same size, $\operatorname{tr}(XY - YX)$ need not be zero.

(d) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. A and B have same eigenvalues.

(e) Consider the matrices in part (d). Eigenvalues of AB are the *product* of eigenvalues of A and eigenvalues of B. Eigenvalues of $A+B$ are the *sum* of eigenvalues of A and eigenvalues of B.

(f) Determinant of any orthogonal matrix is ± 1.

(g) Let $A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then A^{2013} is $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$.

Solution

(a) **True**. $A = \dfrac{A + A^\top}{2} + \dfrac{A - A^\top}{2}$. Let $P = \dfrac{A + A^\top}{2}$, and $Q = \dfrac{A - A^\top}{2}$. Then $A = P + Q$. On the other hand, $P^\top = \dfrac{1}{2}\left(A + A^\top\right)^\top = \dfrac{1}{2}\left(A + A^\top\right) = P$. Hence, P is a symmetric matrix. Similarly, $Q^\top = \dfrac{1}{2}\left(A - A^\top\right)^\top = \dfrac{1}{2}\left(A^\top - A\right) = -Q$. Hence, Q is a skew-symmetric matrix.

(b) **False**. For example, let $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\det(M) = 0$, $\det(N) = 0$, but $\det(M + N) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$.

(c) **False**. Because $\operatorname{tr}(XY - YX) = \operatorname{tr}(XY) - \operatorname{tr}(YX) = 0$. So $\operatorname{tr}(XY - YX)$ must always be zero.

(d) **True**. Because, $\operatorname{eig}(A) = \operatorname{eig}(B) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(e) **False**. We get $AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\operatorname{eig}(AB) = \begin{pmatrix} +i \\ -i \end{pmatrix}$, where $i = \sqrt{-1}$. Again, $A + B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\operatorname{eig}(A + B) = \begin{pmatrix} +\sqrt{2} \\ -\sqrt{2} \end{pmatrix}$.

(f) **True**. For any orthogonal matrix Q, we have $QQ^\top = I$. Hence $\det\left(QQ^\top\right) = \det(I) = 1$. However, $\det\left(QQ^\top\right) = \det(Q)\det\left(Q^\top\right) = (\det(Q))^2$. Hence, $(\det(Q))^2 = 1 \Rightarrow \det(Q) = \pm 1$.

(g) **False**. Notice that $A^2 = A\,A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Next, $A^3 = A\,A\,A = A^2\,A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. As a result, any power of A, greater than 3, will be a zero matrix. Thus, $A^{2013} = A\,A\,A \ldots 2013 \text{ times} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Problem 2

Uniqueness of the solution for a system of linear equations $\qquad$ ($3 \times 3 = 9$ points)

Show that the following system of equations:

$$\begin{array}{ccccccc} x_1 & + & 4x_2 & + & \alpha x_3 & = & 6, \\ 2x_1 & - & x_2 & + & 2\alpha x_3 & = & 3, \\ \alpha x_1 & + & 3x_2 & + & x_3 & = & 5, \end{array}$$

has (a) unique solution if $\alpha = 0$, (b) no solution when $\alpha = -1$, and (c) infinitely many solutions when $\alpha = 1$.

Solution

(a) A system of linear equations of the form $Ax = b$ has unique solution if and only if $\det(A) \neq 0$. In our case, $A = \begin{pmatrix} 1 & 4 & \alpha \\ 2 & -1 & 2\alpha \\ \alpha & 3 & 1 \end{pmatrix}$, and $b = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}$. Now, $\det(A) = \alpha^2 - 1 = 0$. For $\alpha = 0$, we have $\det(A) \neq 0$, hence there is a unique solution.

(b) For $\alpha = \pm 1$, $\det(A) = 0$, and it is not clear, just by looking at the determinant, when we have *no solution*, and when we have *infinitely many solutions*. To see what happens for $\alpha = -1$, we start solving the system $Ax = b$, as follows. For $\alpha = -1$, we perform: (first equation) $-\frac{1}{2}\times$ (second equation). This yields $x_2 = 1$. Substituting $x_2 = 1$ in third equation results $-x_1 + x_3 = 2$. Similarly, substituting $x_2 = 1$ in second equation results $x_1 - x_3 = 2$. Adding them together, we get:

$$(-x_1 + x_3) + (x_1 - x_3) = 2 + 2, \Rightarrow\ 0 = 4,$$

which is impossible. Hence, for $\alpha = -1$, the system has no solution.

(c) For $\alpha = 1$, we get

$$\begin{aligned} x_1 + 4x_2 + x_3 &= 6, \\ 2x_1 - x_2 + 2x_3 &= 3, \\ x_1 + 3x_2 + x_3 &= 5. \end{aligned}$$

Eliminating x_2 from first and second equation yields $x_1 + x_3 = 2$, which in turn results $x_2 = 1$. We get exactly the same results if we eliminate x_2 from the second and third equations. Thus, choosing x_3 as the “free variable”, the solution set is given by:

$$x_1 = 2 - x_3, \quad x_2 = 1, \quad x_3 = x_3.$$

In other words, there are infinitely many solutions, depending on what value we choose for x_3.

Problem 3

Gauss elimination for an ill-conditioned system $\qquad$ ($5 + 5 + 5 = 15$ points)

Consider a system of linear equations that is “ill-conditioned”, meaning it may be difficult to obtain the numerical solution accurately (see Section 2.4 in textbook). One such system, written in augmented matrix form, is

$$\left(\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 8 & -6 & -8 & -6 \\ -1 & 2 & 3 & 4 \end{array}\right).$$

(a) By doing Gauss elimination *using exact arithmetic* (use fractions throughout), show that the exact solution is $x = \{1,\ 1,\ 1\}^\top$.

(b) Now compute the solution using Gauss elimination *with only 3 significant digits* in each operation. How does your answer compare with part (a)?

(c) Compute the solution to the system by changing the coefficient in *row 1, column 1 position* to 3.1 instead of 3. Use your calculator's full precision. How does this answer compare to the previous solutions in part (a) and (b)?

Solution

(a) We perform the following forward elimination:

$$\left(\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 8 & -6 & -8 & -6 \\ -1 & 2 & 3 & 4 \end{array}\right) \xrightarrow[\text{Row 3}-\left(-\frac{1}{3}\right)\text{Row 1}]{\text{Row 2}-\frac{8}{3}\text{Row 1}} \left(\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 0 & -\frac{34}{3} & -\frac{56}{3} & -30 \\ 0 & \frac{8}{3} & \frac{13}{3} & 7 \end{array}\right) \xrightarrow{\text{Row 3}-\left(-\frac{8}{34}\right)\text{Row 2}} \left(\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 0 & -\frac{34}{3} & -\frac{56}{3} & -30 \\ 0 & 0 & -\frac{1}{17} & -\frac{1}{17} \end{array}\right).$$

Then, back substitution yields: $-\frac{1}{17}x_3 = -\frac{1}{17} \Rightarrow x_3 = 1, \Rightarrow x_2 = 1, \Rightarrow x_1 = 1.$

(b) Gauss elimination with 3 significant digits results:

$$\left(\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 8 & -6 & -8 & -6 \\ -1 & 2 & 3 & 4 \end{array}\right) \xrightarrow[\text{Row 3+0.333Row 1}]{\text{Row 2}-\text{2.67Row 1}} \left(\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ -0.01 & -11.3 & -18.7 & -30.0 \\ -0.001 & 2.67 & 4.33 & 7.00 \end{array}\right) \xrightarrow{\text{Row 3+0.236Row 2}}$$

$$\left(\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ -0.01 & -11.34 & -18.68 & -30.03 \\ -0.003 & 0.003 & -0.0832 & -0.087 \end{array}\right).$$

Then, back substitution results $x_3 \approx 1.05$, $x_2 \approx 0.919$, $x_1 \approx 0.987$. Your answer may slightly vary depending on how you do the row operations. But your answer should be close to the answer in part (a), meaning *round-off error* during row-operations, does not have significant impact on the solution.

(c) We perform the following forward elimination:

$$\left(\begin{array}{ccc|c} 3.1 & 2 & 4 & 9 \\ 8 & -6 & -8 & -6 \\ -1 & 2 & 3 & 4 \end{array}\right) \xrightarrow[\text{Row } 3-(-\frac{1}{3.1})\text{Row } 1]{\text{Row } 2-\frac{8}{3.1}\text{Row } 1} \left(\begin{array}{ccc|c} 3.1 & 2 & 4 & 9 \\ 0 & -11.16129032 & -18.33258064 & -29.22580645 \\ 0 & 2.64516129 & 4.290322581 & 6.903225807 \end{array}\right)$$

$$\xrightarrow{\text{Row } 3-(\frac{2.645}{-11.161})\text{Row } 2} \left(\begin{array}{ccc|c} 3.1 & 2 & 4 & 9 \\ 0 & -11.16129032 & -18.33258064 & -29.22580645 \\ 0 & 0 & -0.0520231207 & -0.0231213877 \end{array}\right).$$

Then, back substitution yields: $x_3 = 0.4444444587, \Rightarrow x_2 = 1.888888866, \Rightarrow x_1 = 1.111111107$. Compared to part (a) and (b), this answer is drastically different. This shows that a small change in problem, can cause a large change in solution, meaning the problem is *ill-conditioned*.

Problem 4

Gauss elimination algorithm: standard versus partial pivoting (5+2+3+5 = 15 **points)**

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}.$$

(a) Show that the system of linear equations $Ax = b$ has infinitely many solutions.

(b) Describe the set of all possible solutions.

(c) Use Gauss elimination to solve $Ax = b$ *using exact arithmetic*. Because there are infinitely many solutions, it is unreasonable to expect one particular solution to be computed. What does happen?

(d) Repeat part (c) with partial pivoting. What are your conclusions?

Solution

(a) We solve the following system:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1, \\ 4x_1 + 5x_2 + 6x_3 &= 3, \\ 7x_1 + 8x_2 + 9x_3 &= 5. \end{aligned}$$

Eliminating x_3 from the first and second equation, we get $2x_1 + x_2 = 1$. If we eliminate x_3 from the second and third equation, we again get $2x_1 + x_2 = 1$. Substituting $x_2 = 1 - 2x_1$ in any of the three equations, we get $x_3 = x_1 - \frac{1}{3}$. Thus, there are infinitely many solutions, depending on what numerical value we fix for any one variable, say x_1.

(b) Let us choose x_1 as the "free variable". Then the solution set is:

$$x_1 = x_1, \quad x_2 = 1 - 2x_1, \quad x_3 = x_1 - \frac{1}{3}.$$

(c) We get

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 5 \end{array}\right) \xrightarrow[\text{Row } 3 - 7\text{Row } 1]{\text{Row } 2 - 4\text{Row } 1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & -6 & -12 & -2 \end{array}\right) \xrightarrow{\text{Row } 3 - 2\text{Row } 2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Since the last row becomes zero, there are infinite solutions, as we checked manually in part (a) and (b).

(d) With partial pivoting:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 5 \end{array}\right) \xrightarrow{\text{Row } 3 \leftrightarrow \text{Row } 1} \left(\begin{array}{ccc|c} 7 & 8 & 9 & 5 \\ 4 & 5 & 6 & 3 \\ 1 & 2 & 3 & 1 \end{array}\right) \xrightarrow[\text{Row } 2 = \text{Row } 2 - 4\text{Row } 1, \text{Row } 3 = \text{Row } 3 - \text{Row } 1]{\text{Row } 1 = \frac{1}{7}\text{Row } 1} \left(\begin{array}{ccc|c} 1 & \frac{8}{7} & \frac{9}{7} & \frac{5}{7} \\ 0 & \frac{3}{7} & \frac{6}{7} & \frac{1}{7} \\ 0 & \frac{6}{7} & \frac{12}{7} & \frac{2}{7} \end{array}\right)$$

$$\xrightarrow[\text{Row } 3 = \text{Row } 3 - \frac{6}{7}\text{Row } 2]{\text{Row } 2 = \frac{7}{2}\text{Row } 2} \left(\begin{array}{ccc|c} 1 & \frac{8}{7} & \frac{9}{7} & \frac{5}{7} \\ 0 & 1 & 2 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Again, we see that the last row becomes zero, thus arriving at the same conclusion as in part (a), (b) and (c).

Problem 5

Forces along the links of a truss $(10 + 30 = 40$ **points)**

The following is a schematic of a mechanical structure (such as a bridge), modeled by the truss as follows.

As shown in the diagram, the truss has 10 joints, numbered in circles. It has 17 rigid links, also numbered in the diagram along the line segments. Given the loads F_1, F_2, F_3, F_4, our objective is to compute the forces $f_1, f_2, \ldots, f_{17}$ along the rigid links of the structure.

To do this, let us introduce $\alpha = \sin(45°) = \cos(45°)$. Now, for each joint, we write the force balance equations in horizontal and vertical directions:

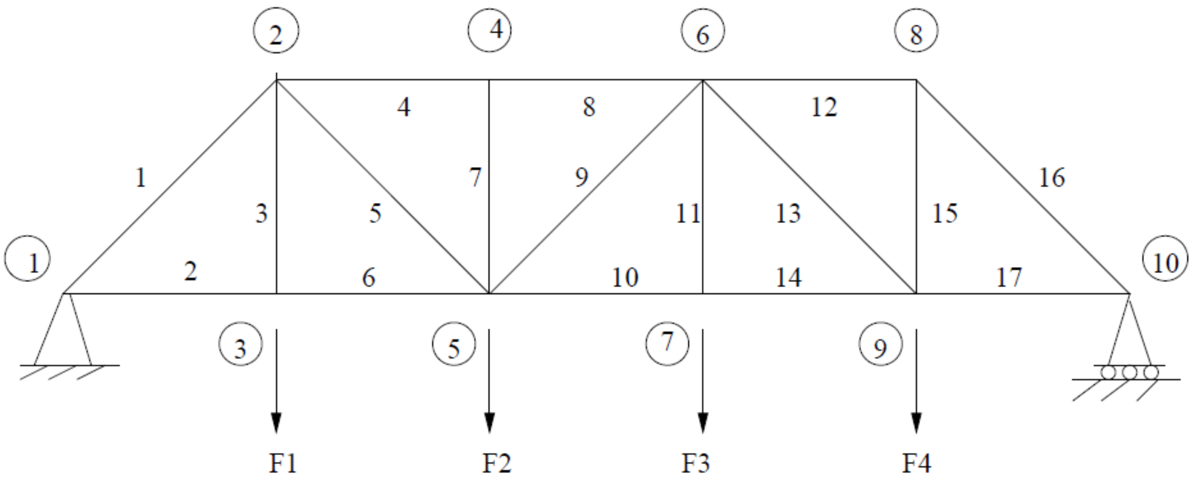

Figure 1: Schematic of the truss structure.

$$\text{Joint 2}\begin{cases}-\alpha f_1 + f_4 + \alpha f_5 = 0,\\ -\alpha f_1 + f_3 + \alpha f_5 = 0,\end{cases}$$

$$\text{Joint 3}\begin{cases}-f_2 + f_6 = 0,\\ -f_3 + F_1 = 0,\end{cases}$$

$$\text{Joint 4}\begin{cases}-f_4 + f_8 = 0,\\ f_7 = 0,\end{cases}$$

$$\text{Joint 5}\begin{cases}-\alpha f_5 - f_6 + \alpha f_9 + f_{10} = 0,\\ -\alpha f_5 - f_7 + \alpha f_9 + F_2 = 0,\end{cases}$$

$$\text{Joint 6}\begin{cases}-f_8 - \alpha f_9 + f_{12} + \alpha f_{13} = 0,\\ -\alpha f_9 + f_{11} + \alpha f_{13} = 0,\end{cases}$$

$$\text{Joint 7}\begin{cases}-f_{10} + f_{14} = 0,\\ -f_{11} + F_3 = 0,\end{cases}$$

$$\text{Joint 8}\begin{cases}-f_{12} + \alpha f_{16} = 0,\\ f_{15} - \alpha f_{16} = 0,\end{cases}$$

$$
\text{Joint 9}\begin{cases}-\alpha f_{13} - f_{14} + f_{17} = 0,\\ -\alpha f_{13} - f_{15} + F_4 = 0,\end{cases}
$$

$$
\text{Joint 10}\begin{cases}-\alpha f_{16} - f_{17} = 0.\end{cases}
$$

(a) Write the above system in the matrix equation form $Ax = b$, where $x = \{f_1, f_2, \ldots, f_{17}\}^\top$.

(b) Write a computer code to solve for x using Gauss elimination, when the external loads are (i) $F_1 = 10, F_2 = 15, F_3 = 0, F_4 = 10$; and (ii) $F_1 = 10, F_2 = 0, F_3 = 20, F_4 = 0$. Compare the results obtained with and without partial pivoting.

Solution

$$
\text{(a) } A = \begin{pmatrix}
-\alpha & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\alpha & 0 & 1 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\alpha & -1 & 0 & 0 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\alpha & 0 & -1 & 0 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\alpha & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 1 & 0 & \alpha & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \alpha & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\alpha & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & -1 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & -1
\end{pmatrix},\ b = \begin{pmatrix}0\\0\\0\\-F_1\\0\\0\\0\\-F_2\\0\\0\\0\\-F_3\\0\\0\\0\\-F_4\\0\end{pmatrix}.
$$

(b) Please see the code attached.