Aero 320: Numerical Methods

Homework 4

Name: ....

Due: October 21, 2013

NOTE: All problems, unless explicitly asked to write a code, are to be done by hand (with the help of a calculator) but **you need to show all the steps**. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Submit your HW by Monday midnight at Room 201, Reed McDonald Building. Late submissions or failure to submit in the required format will receive no credit.

Problem 1

LU decomposition $(5+3+2+5+2+6+2=25$ **points)**

Let $A = \begin{pmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix}.$

(a) By hand, perform LU decomposition for matrix A. Show all the calculations in *exact arithmetic* (i.e. use fractions throughout).

(b) Use your answer in part (a) to compute $\det(A)$.

(c) From your answer to part (b), what can you say about the solution for the system of linear equations given by $Ax = b$?

(d) Use the LU decomposition from part (a), to solve for vector x such that $Ax = b$.

(e) A matrix is invertible if it has non-zero determinant. From part (b), does A^{-1} exist?

(f) If exists, then A^{-1} can be computed by solving the matrix equation $AX = I$ for square matrix X, where I is the identity matrix of size same as A. Use the LU decomposition from part (a), to compute A^{-1}.

(g) From your answer in part (f), find $x = A^{-1}b$. Compare your result with that found in part (d). Why (d) is a better algorithm to solve $Ax = b$ than directly computing $x = A^{-1}b$, even though $\det(A) \neq 0$?

Problem 2

Vector and matrix norms **(3+3+4+4+6 = 20 points)**

(a) By hand, compute the 1-norm, 2-norm and ∞-norm of the vector $x = \{-\sqrt{3} \quad -6 \quad 4 \quad 2\}^\top$.

(b) For any $n \times 1$ vector x, the following holds:

$$\| x \|_\infty \quad \leq \quad \| x \|_2 \quad \leq \quad \| x \|_1 \quad \leq \quad \sqrt{n} \, \| x \|_2 \quad \leq \quad n \, \| x \|_\infty \, .$$

Verify this relation for the vector in part (a).

(c) By hand, compute the 1-norm, 2-norm, ∞-norm, and Frobenius norm of the matrix

$$M = \begin{pmatrix} 3 & 5 & 7 \\ 2 & -6 & 4 \\ -1 & 2 & 8 \end{pmatrix} .$$

(d) Consider any $n \times n$ orthogonal matrix Q. Compute $\| Q \|_2$, $\| Q \|_F$.

(e) Give examples of matrix A such that (i) $\| A \|_1 < \| A \|_\infty$, (ii) $\| A \|_1 = \| A \|_\infty$, and (iii) $\| A \|_1 > \| A \|_\infty$.

Problem 3

Condition number and ill-conditioned problems **($(11 + 5 + (5 + 3 + 1) = 25$ points)**

The *condition number* $\kappa_* (A)$ of a matrix A, is defined as $\kappa_* (A) = \| A \|_* \| A^{-1} \|_*$, where $*$ is any matrix norm. For example, if we use 2-norm of matrix, then we get $\kappa_2 (A) = \| A \|_2 \| A^{-1} \|_2$, etc. A system of linear equations of the form $Ax = b$, is said to be *ill-conditioned* if small changes in A or b produce large changes in the solution x.

(a) Consider solving $Ax = b$, where $A = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix}$. You may verify that $\det(A) = 1$, and hence there is unique solution. Using your favorite method (Gauss elimination or LU) solve for x when (i) $b = \{23 \quad 32 \quad 33 \quad 31\}^\top$, (ii) $b = \{22.9 \quad 32.1 \quad 32.9 \quad 31.1\}^\top$, (iii) $b =$

$\{22.99 \quad 32.01 \quad 32.99 \quad 31.01\}^{\top}$. Try to compute your answers as accurately as you can. What do you think is happening?

(b) Compute $\kappa_{\infty}(A)$ for the matrix in part (a).

(c) A Hilbert matrix H of size $n \times n$ has entries: $H_{ij} = \dfrac{1}{i+j-1}$. Write a C++ code to compute $\kappa_2(H)$ for $n = 2, 4, 8, 16, 32$. Submit a hard copy of your code, and a plot of $\kappa_2(H)$ versus n. What is your conclusion from this plot?

Problem 4

Jacobi and Gauss-Seidel iteration $(10 + 4 + 4 + 4 + 8 = 30$ **points)**

In this exercise, you will see that for some problems, Jacobi method may converge with any initial guess, but the Gauss-Seidel method may fail.

(a) First, write C++ codes to iteratively solve the system of linear equations $Ax = b$ using *Jacobi method* and *Gauss-Seidel method*. If the k^{th} iterate is the vector x_k, then the convergence condition is that the *2-norm relative error* becomes less than the tolerance $\epsilon = 10^{-4}$, that is:

$$\frac{\| Ax_k - b \|_2}{\| b \|_2} < \epsilon.$$

Define the maximum number of iterations to be 500 to stop the code in case it diverges. Submit the hard copies for your codes.

(b) Test your codes in part (a), for $A = \begin{pmatrix} 3 & -5 & 2 \\ 5 & 4 & 3 \\ 2 & 5 & 3 \end{pmatrix}$, and $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Start with initial guess $x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Report, if possible, the number of iterations needed for each method to converge.

(c) How does your answer to part (b) change if we modify the convergence criterion to be $\dfrac{\| Ax_k - b \|_{\infty}}{\| b \|_{\infty}} < \epsilon$? Explain the change, if any, you observe, compared to part (b).

(d) Repeat part (c) with the convergence criterion $\dfrac{\| Ax_k - b \|_1}{\| b \|_1} < \epsilon$.

(e) For part (b), (c), (d) above, plot the corresponding relative error versus iteration number k on the same figure.