Aero 320: Numerical Methods

Homework 6

Name: ....

Due: November 18, 2013

NOTE: All problems, unless explicitly asked to write a code, are to be done by hand (with the help of a calculator) but **you need to show all the steps**. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Submit your HW by Monday midnight at Room 201, Reed McDonald Building. Late submissions or failure to submit in the required format will receive no credit.

Problem 1

Least squares approximation of a continuous function $(15+10+5 = 30$ **points)**

In class and previous homework, you learned how to approximate *discrete* datapoints using least square. In this exercise, you will see that instead of approximating discrete data, we can also approximate a continuous function in least squares sense. This is useful, for example, when the true function, although known, is complicated to evaluate numerically. If we can approximate it using a simpler and "numerically friendly" function, then we can use that approximate function for computational purposes.

Consider any continuous function $f(x)$ in the interval $[-\pi, \pi]$. We want to approximate this function as

$$\widehat{f}(x) = a_0 + \sum_{k=1}^{n} \left(a_k \cos kx + b_k \sin kx\right).$$

(a) Show that the coefficients a_0, a_k, b_k, that minimize the total square error $\int_{-\pi}^{\pi} \left(f(x) - \widehat{f}(x)\right)^2 dx$, are given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\, dx, \quad a_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos kx\, dx, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin kx\, dx, \quad k = 1,\ldots,n.$$

(b) Using your answer in part (a), for $f(x) = e^x$, find the coefficients a_0, a_k, b_k, as functions of k

only. Your final answers should not have any integral.

(c) Use your answer in part (b) to plot the functions $f(x) = e^x$ together with $\widehat{f}(x)$ for $n = 1, 2, 5$, in the interval $[-\pi, \pi]$. In other words, you will be plotting four functions: one true function, and three approximations. Submit a hard copy of your plot. What do you conclude from this plot?

Problem 2

Spline interpolation **($5 + 10 + 30 = 45$ points)**

(a) Determine if the following function $S_1(x)$ is a linear spline or not. Why/why not?

$$S_1(x) = \begin{cases} x, & -1 \le x \le 0.5, \\ 0.5 + 2\,(x - 0.5)\,, & 0.5 \le x \le 2, \\ x + 1.5, & 2 \le x \le 4. \end{cases}$$

(b) Determine the values of a, b, c, if possible, such that the following becomes a cubic spline. Show all the steps in your calculation.

$$S_3(x) = \begin{cases} 4 + ax + 2x^2 - \frac{1}{6}x^3, & 0 \le x \le 1, \\ 1 - \frac{4}{3}(x-1) + b\,(x-1)^2 - \frac{1}{6}\,(x-1)^3\,, & 1 \le x \le 2, \\ 1 + c\,(x-2) + (x-2)^2 - \frac{1}{6}\,(x-2)^3\,, & 2 \le x \le 3. \end{cases}$$

(c) Write a C++ code for spline interpolation. Your code should have a function called `InterpSpline` that will take the datapoints (x_i, y_i) and the degree of spline (1 for linear spline, 3 for cubic spline etc.) as input, and would return the vector of coefficients of spline polynomials. Write another function `InterpPoly` that only takes the datapoints (x_i, y_i) as input and returns a single interpolating polynomial over these data.

Use your code to do the *linear spline*, *cubic spline*, and *polynomial* interpolations of the monthly average high temperature data, measured at an airport, from January through December, as shown below.

Month	Average high Temp.
1	54.6
2	54.4
3	67.1
4	78.3
5	85.3
6	88.7
7	96.9
8	97.6
9	84.1
10	80.1
11	68.8
12	61.1

Submit a plot for the linear spline, cubic spline, and the polynomial interpolants computed from your code, with the datapoints clearly marked. Also submit a hard copy of your code.

Problem 3

Bezier curve and B-spline **(**15+10 = 25 **points)**

A B-spline curve segment is given by the following control points:

$$P_0 = (-1, -1)\,, \quad P_1 = (1, -1)\,, \quad P_2 = (1, 1)\,, \quad P_3 = (-1, 1)\,.$$

(a) Find the Bezier control points that will produce the same curve segment. Show all your calculations.

(b) Submit a plot of the curve segment.