Aero 320: Numerical Methods

Homework 6

Name: ....

Due: November 18, 2013

NOTE: All problems, unless explicitly asked to write a code, are to be done by hand (with the help of a calculator) but **you need to show all the steps**. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Submit your HW by Monday midnight at Room 201, Reed McDonald Building. Late submissions or failure to submit in the required format will receive no credit.

Problem 1

Least squares approximation of a continuous function (15 + 10 + 5 = 30 **points)**

In class and previous homework, you learned how to approximate *discrete* datapoints using least square. In this exercise, you will see that instead of approximating discrete data, we can also approximate a continuous function in least squares sense. This is useful, for example, when the true function, although known, is complicated to evaluate numerically. If we can approximate it using a simpler and "numerically friendly" function, then we can use that approximate function for computational purposes.

Consider any continuous function $f(x)$ in the interval $[-\pi, \pi]$. We want to approximate this function as

$$\widehat{f}(x) = a_0 + \sum_{k=1}^{n} \left(a_k \cos kx + b_k \sin kx\right).$$

(a) Show that the coefficients a_0, a_k, b_k, that minimize the total square error $\int_{-\pi}^{\pi} \left(f(x) - \widehat{f}(x)\right)^2 dx$, are given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx, \quad a_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos kx\,dx, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin kx\,dx, \quad k = 1,\ldots,n.$$

(b) Using your answer in part (a), for $f(x) = e^x$, find the coefficients a_0, a_k, b_k, as functions of k

only. Your final answers should not have any integral.

(c) Use your answer in part (b) to plot the functions $f(x) = e^x$ together with $\widehat{f}(x)$ for $n = 1, 2, 5$, in the interval $[-\pi, \pi]$. In other words, you will be plotting four functions: one true function, and three approximations. Submit a hard copy of your plot. What do you conclude from this plot?

Solution

(a) We need the following identities. (I am writing these identities for clarity of understanding, students don't need to write these in their Homework submission.)

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos kx \, dx &= \frac{1}{k}\left[\sin kx\right]_{x=-\pi}^{x=\pi} = \frac{1}{k}\left(\sin(k\pi) - \sin(-k\pi)\right) = \frac{1}{k}\left(\sin(k\pi) + \sin(k\pi)\right) = \frac{1}{k}(0+0) = 0, \\
\int_{-\pi}^{\pi} \sin kx \, dx &= -\frac{1}{k}\left[\cos kx\right]_{x=-\pi}^{x=\pi} = -\frac{1}{k}\left(\cos(k\pi) - cos(-k\pi)\right) = -\frac{1}{k}\left(\cos(k\pi) - cos(k\pi)\right) = 0.
\end{aligned}$$

In addition, recall that

$$\begin{aligned}
\sin A \cos B &= \frac{1}{2}(2\sin A \cos B) = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right), \\
\sin A \sin B &= \frac{1}{2}(2\sin A \sin B) = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right), \\
\cos A \cos B &= \frac{1}{2}(2\cos A \cos B) = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right).
\end{aligned}$$

As a result, if we consider any two integers m and n, then for $m \neq n$, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin mx \cos nx \, dx &= \frac{1}{2}\int_{-\pi}^{\pi} \left(\sin(m+n)x + \sin(m-n)x\right) dx = \frac{1}{2}\left(-\left[\frac{\cos(m+n)x}{m+n}\right]_{x=-\pi}^{x=+\pi} - \left[\frac{\cos(m-n)x}{m-n}\right]_{x=-\pi}^{x=+\pi}\right) = 0, \\
\int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \frac{1}{2}\int_{-\pi}^{\pi} \left(\cos(m-n)x - \cos(m+n)x\right) dx = \frac{1}{2}\left(\left[\frac{\sin(m-n)x}{m-n}\right]_{x=-\pi}^{x=+\pi} - \left[\frac{\sin(m+n)x}{m+n}\right]_{x=-\pi}^{x=+\pi}\right) = 0, \\
\int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \frac{1}{2}\int_{-\pi}^{\pi} \left(\cos(m-n)x + \cos(m+n)x\right) dx = \frac{1}{2}\left(\left[\frac{\sin(m-n)x}{m-n}\right]_{x=-\pi}^{x=+\pi} + \left[\frac{\sin(m+n)x}{m+n}\right]_{x=-\pi}^{x=+\pi}\right) = 0.
\end{aligned}$$

On the other hand, for $m = n$, we get

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin nx \cos nx \, dx &= \frac{1}{2}\int_{-\pi}^{\pi} \sin(2nx) \, dx = -\frac{1}{4n}\left[\cos(2nx)\right]_{x=-\pi}^{x=+\pi} = -\frac{1}{4n}\left[\cos(2n\pi) - \cos(-2n\pi)\right] = 0, \\
\int_{-\pi}^{\pi} \sin^2 nx \, dx &= \frac{1}{2}\int_{-\pi}^{\pi} \left(1 - \cos(2nx)\right) dx = \frac{1}{2}2\pi - \frac{1}{2}\left[\frac{\sin(2nx)}{2n}\right]_{x=-\pi}^{x=+\pi} = \pi - \frac{1}{4n}(0+0) = \pi, \\
\int_{-\pi}^{\pi} \cos^2 nx \, dx &= \frac{1}{2}\int_{-\pi}^{\pi} \left(1 + \cos(2nx)\right) dx = \frac{1}{2}2\pi + \frac{1}{2}\left[\frac{\sin(2nx)}{2n}\right]_{x=-\pi}^{x=+\pi} = \pi + \frac{1}{4n}(0+0) = \pi.
\end{aligned}$$

Now, if we denote the *total square error* as $E = \int_{-\pi}^{\pi} \left(f(x) - \widehat{f}(x)\right)^2 \, dx$, then to make E minimum (or "least"), we need

$$\begin{aligned}
&\frac{\partial E}{\partial a_0} = 0 \\
\Rightarrow \quad & \frac{\partial}{\partial a_0} \int_{-\pi}^{\pi} \left(f(x) - \widehat{f}(x)\right)^2 \, dx = \int_{-\pi}^{\pi} \frac{\partial}{\partial a_0} \left(f(x) - \widehat{f}(x)\right)^2 \, dx = 2 \int_{-\pi}^{\pi} \left(f - \widehat{f}\right) \left(- \underbrace{\frac{\partial \widehat{f}}{\partial a_0}}_{1} \right) \, dx = 0 \\
\Rightarrow \quad & \int_{-\pi}^{\pi} \widehat{f}(x) \, dx = \int_{-\pi}^{\pi} f(x) \, dx \\
\Rightarrow \quad & a_0 \int_{-\pi}^{\pi} dx + \sum_{k=1}^{n} \left(a_k \underbrace{\int_{-\pi}^{\pi} \cos kx \, dx}_{0} + b_k \underbrace{\int_{-\pi}^{\pi} \sin kx \, dx}_{0} \right) = \int_{-\pi}^{\pi} f(x) \, dx \\
\Rightarrow \quad & a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, .
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{\partial E}{\partial a_k} = 0 \\
\Rightarrow \quad & \frac{\partial}{\partial a_k} \int_{-\pi}^{\pi} \left(f(x) - \widehat{f}(x)\right)^2 \, dx = \int_{-\pi}^{\pi} \frac{\partial}{\partial a_k} \left(f(x) - \widehat{f}(x)\right)^2 \, dx = 2 \int_{-\pi}^{\pi} \left(f - \widehat{f}\right) \left(- \underbrace{\frac{\partial \widehat{f}}{\partial a_k}}_{\cos kx} \right) \, dx = 0 \\
\Rightarrow \quad & \int_{-\pi}^{\pi} \widehat{f}(x) \cos kx \, dx = \int_{-\pi}^{\pi} f(x) \cos kx \, dx \\
\Rightarrow \quad & a_0 \underbrace{\int_{-\pi}^{\pi} \cos kx \, dx}_{0} + \sum_{j=1}^{n} \left(a_j \underbrace{\int_{-\pi}^{\pi} \cos jx \cos kx \, dx}_{=\pi \text{ for } j=k, \text{ else } =0} + b_j \underbrace{\int_{-\pi}^{\pi} \sin jx \cos kx \, dx}_{0} \right) = \int_{-\pi}^{\pi} f(x) \cos kx \, dx \\
\Rightarrow \quad & a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial E}{\partial b_k} = 0 \\
\Rightarrow \quad & \frac{\partial}{\partial b_k} \int_{-\pi}^{\pi} \left(f(x) - \widehat{f}(x)\right)^2 \, dx = \int_{-\pi}^{\pi} \frac{\partial}{\partial b_k} \left(f(x) - \widehat{f}(x)\right)^2 \, dx = 2 \int_{-\pi}^{\pi} \left(f - \widehat{f}\right) \left(- \underbrace{\frac{\partial \widehat{f}}{\partial b_k}}_{\sin kx} \right) \, dx = 0 \\
\Rightarrow \quad & \int_{-\pi}^{\pi} \widehat{f}(x) \sin kx \, dx = \int_{-\pi}^{\pi} f(x) \sin kx \, dx \\
\Rightarrow \quad & a_0 \underbrace{\int_{-\pi}^{\pi} \sin kx \, dx}_{0} + \sum_{j=1}^{n} \left(a_j \underbrace{\int_{-\pi}^{\pi} \cos jx \sin kx \, dx}_{0} + b_j \underbrace{\int_{-\pi}^{\pi} \sin jx \sin kx \, dx}_{=\pi \text{ for } j=k, \text{ else } =0} \right) = \int_{-\pi}^{\pi} f(x) \sin kx \, dx \\
\Rightarrow \quad & b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx.
\end{aligned}$$

(b) For $f(x) = e^x$, from part (a), we get

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x \, dx = \frac{1}{2\pi} \left[e^x\right]_{x=-\pi}^{x=+\pi} = \frac{1}{2\pi} \left(e^{\pi} - e^{-\pi}\right).$$

Next, from part (a), we have

$$
\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos kx\, dx \\
&= \frac{1}{\pi}\left(\left[e^x \frac{\sin kx}{k}\right]_{x=-\pi}^{x=+\pi} - \int_{-\pi}^{\pi} e^x \frac{\sin kx}{k}\, dx\right) && \text{(by doing integration by parts)} \\
&= \frac{1}{\pi}\left(\left[e^x \frac{\sin kx}{k} + e^x \frac{\cos kx}{k^2}\right]_{x=-\pi}^{x=+\pi} - \frac{1}{k^2}\int_{-\pi}^{\pi} e^x \cos kx\, dx\right) && \text{(by doing integration by parts again)} \\
&= \frac{1}{\pi k^2}\left[e^x\left(k\sin kx + \cos kx\right)\right]_{x=-\pi}^{x=+\pi} - \frac{1}{k^2}\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos kx\, dx}_{a_k} \\
\Rightarrow \left(1+\frac{1}{k^2}\right) a_k &= \frac{1}{\pi k^2}\left(e^{\pi}\left(\underbrace{k\sin k\pi}_{0} + \cos k\pi\right) - e^{-\pi}\left(\underbrace{-k\sin k\pi}_{0} + \cos k\pi\right)\right) \\
\Rightarrow a_k &= \frac{\cos k\pi}{\pi\left(k^2+1\right)}\left(e^{\pi} - e^{-\pi}\right) = \frac{(-1)^k}{\pi\left(k^2+1\right)}\left(e^{\pi} - e^{-\pi}\right).
\end{aligned}
$$

Similarly, we get $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin kx\, dx = \frac{1}{\pi\left(k^2+1\right)}\left[-ke^x\cos kx + e^x \sin kx\right]_{x=-\pi}^{x=+\pi} = -\frac{k\,(-1)^k}{\pi\left(k^2+1\right)}\left(e^{\pi} - e^{-\pi}\right).$

(c) Combining the expressions for a_0, a_k, and b_k from part (b), we arrive at

$$
\widehat{f}\,(x) = \frac{e^{\pi} - e^{-\pi}}{\pi}\left[\frac{1}{2} + \sum_{k=1}^{n} \frac{(-1)^k}{\left(k^2+1\right)}\left(\cos kx - k\sin kx\right)\right].
$$

The plots, generated from the MATLAB file `HW6Problem1c.m`, are shown below. The figure shows that keeping more terms (large n) in the least square approximate function $\widehat{f}(x)$ introduces unwanted oscillations, while the true function $f(x)$ is monotone increasing.

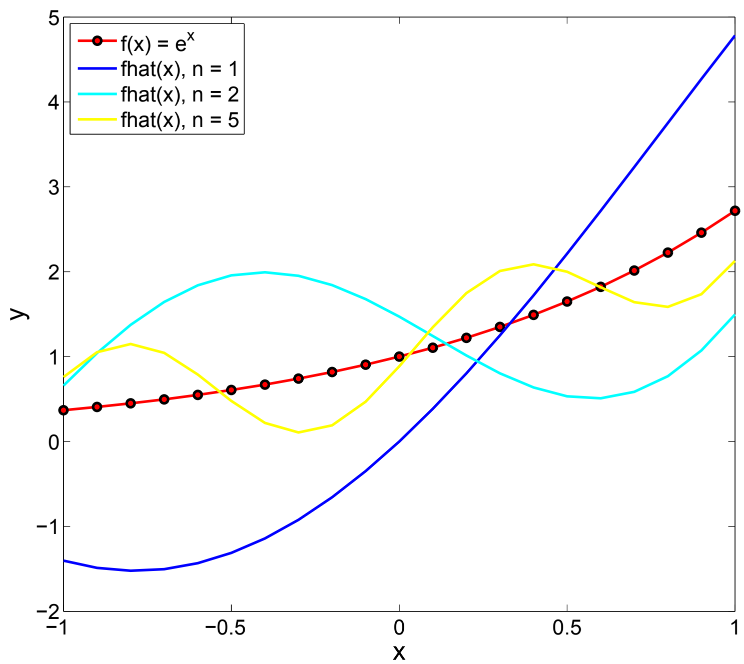

Figure 1: The true function $f(x) = e^x$, and its three least square approximations $\widehat{f}(x)$, for $n = 1, 2, 5$.

Problem 2

Spline interpolation **($5 + 10 + 30 = 45$ points)**

(a) Determine if the following function $S_1(x)$ is a linear spline or not. Why/why not?

$$S_1(x) = \begin{cases} x, & -1 \leq x \leq 0.5, \\ 0.5 + 2\left(x - 0.5\right), & 0.5 \leq x \leq 2, \\ x + 1.5, & 2 \leq x \leq 4. \end{cases}$$

(b) Determine the values of a, b, c, if possible, such that the following becomes a cubic spline. Show all the steps in your calculation.

$$S_3(x) = \begin{cases} 4 + ax + 2x^2 - \frac{1}{6}x^3, & 0 \leq x \leq 1, \\ 1 - \frac{4}{3}(x-1) + b\left(x-1\right)^2 - \frac{1}{6}\left(x-1\right)^3, & 1 \leq x \leq 2, \\ 1 + c\left(x-2\right) + \left(x-2\right)^2 - \frac{1}{6}\left(x-2\right)^3, & 2 \leq x \leq 3. \end{cases}$$

(c) Write a C++ code for spline interpolation. Your code should have a function called `InterpSpline` that will take the datapoints (x_i, y_i) and the degree of spline (1 for linear spline, 3 for cubic spline etc.) as input, and would return the vector of coefficients of spline polynomials. Write another function `InterpPoly` that only takes the datapoints (x_i, y_i) as input and returns a single interpolating polynomial over these data.

Use your code to do the *linear spline*, *cubic spline*, and *polynomial* interpolations of the monthly average high temperature data, measured at an airport, from January through December, as shown below.

Month	Average high Temp.
1	54.6
2	54.4
3	67.1
4	78.3
5	85.3
6	88.7
7	96.9
8	97.6
9	84.1
10	80.1
11	68.8
12	61.1

Submit a plot for the linear spline, cubic spline, and the polynomial interpolants computed from your code, with the datapoints clearly marked. Also submit a hard copy of your code.

Solution

(a) Yes, $S_1(x)$ is a linear spline.

We have three intervals. Let us denote the line segments as $S_{11}(x)$, $S_{12}(x)$ and $S_{13}(x)$ over the first, second and third interval, respectively. $S_1(x)$ is a linear spline because it satisfies continuity at the *knot points* (where two different line segments meet). In other words, $S_{11}(0.5) = 0.5 = 0.5 + 2(0.5 - 0.5) = S_{12}(0.5)$, and $S_{12}(2) = 0.5 + 2(2 - 0.5) = 2 + 1.5 = S_{13}(2)$.

(b) Let $S_{31}(x)$, $S_{32}(x)$ and $S_{33}(x)$ be the cubic polynomials over the first, second and third interval, respectively. For $S_3(x)$ to be a cubic spline, we must have continuity: $S_{31}(1) = S_{32}(1)$, and $S_{32}(2) = S_{33}(2)$. In addition, we must match slopes: $S'_{31}(1) = S'_{32}(1)$, and $S'_{32}(2) = S'_{33}(2)$. Furthermore, we must match curvatures (second derivatives) $S''_{31}(1) = S''_{32}(1)$, and $S''_{32}(2) = S''_{33}(2)$.

From $S_{31}(1) = S_{32}(1)$, we get $a = -\frac{29}{6}$. From $S_{32}(2) = S_{33}(2)$, we get $b = \frac{3}{2}$. From $S'_{31}(1) = S'_{32}(1)$, we get $a = -\frac{29}{6}$ again. From $S'_{32}(2) = S'_{33}(2)$, we get $c = \frac{7}{6}$. From $S''_{31}(1) = S''_{32}(1)$, we get $b = \frac{3}{2}$ again. From $S''_{32}(2) = S''_{33}(2)$, we get $b = \frac{3}{2}$ yet again. So for $a = -\frac{29}{6}, b = \frac{3}{2}$, and $c = \frac{7}{6}$, all necessary conditions are met for $S_3(x)$ to be a cubic spline.

(c) The plot is shown below. As expected, the (11^{th} degree) polynomial interpolation shows oscillations, but the splines don't. This plot is generated using the C++ code `HW6problem2c.cpp` (attached) that takes the file input `HW6problem2cData.dat` (attached). When you execute the C++ code (with `n` $= 3$, and `nInterp` $= 1000$), it outputs three data files, which are then called by the MATLAB file `HW6problem2c_PlotCplusplusData.m` (attached) for plotting.

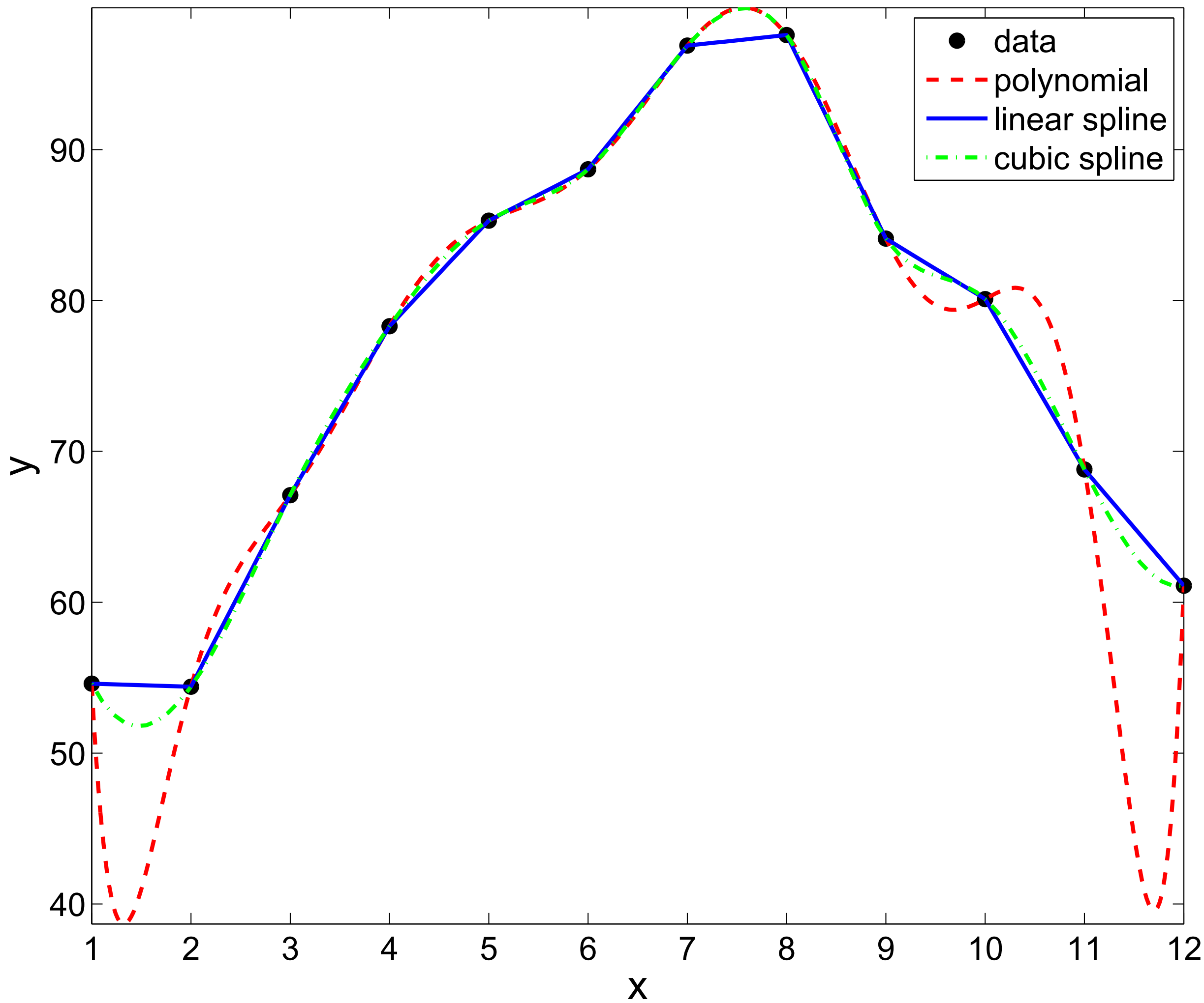

Figure 2: The datapoints (*black circles*) plotted with polynomial (*red dashed*), linear spline (*blue solid*) and cubic spline (*green dash dotted*) interpolants.

It is possible (but you were not asked to) to solve this entire problem in MATLAB, by simply running the file `HW6problem2c.m` (attached).

Problem 3

Bezier curve and B-spline (15+10 = 25 **points**)

A B-spline curve segment is given by the following control points:

$$P_0 = (-1, -1)\,, \quad P_1 = (1, -1)\,, \quad P_2 = (1, 1)\,, \quad P_3 = (-1, 1)\,.$$

(a) Find the Bezier control points that will produce the same curve segment. Show all your calculations.

(b) Submit a plot of the curve segment.

Solution

(a) Let us denote the B-spline curve as $BS(u)$ and the Bezier curve as $BZ(u)$. We would like to find Bezier control points $Q_0 = (x_0, y_0), Q_1 = (x_1, y_1), Q_2 = (x_2, y_2)$, and $Q_3 = (x_3, y_3)$, such that $BS(u) = BZ(u)$.

First, we write $BS(u)$ and $BZ(u)$ in terms of their respective basis functions.

$$\begin{aligned} BS(u) &= b_{-1}(u) P_0 + b_0(u) P_1 + b_1(u) P_2 + b_2(u) P_3, \\ BZ(u) &= B_{3,0}(u) Q_0 + B_{3,1}(u) Q_1 + B_{3,2}(u) Q_2 + B_{3,3}(u) Q_3, \end{aligned}$$

where the B-spline basis functions are (as in slide # 29, ch-3-b)

$$b_{-1}(u) = \frac{(1-u)^3}{6}, \quad b_0 = \frac{u^3}{2} - u^2 + \frac{2}{3}, \quad b_1 = \frac{1}{2}\left(-u^3 + u^2 + u + \frac{1}{3}\right), \quad b_2 = \frac{u^3}{6},$$

and the Bezier basis functions are (Bernstein polynomials $B_{n,k}(u) = \frac{n!}{k!(n-k)!} u^k (1-u)^{n-k}$, see slides # 12 and 14, ch-3-b)

$$B_{3,0}(u) = (1-u)^3, \quad B_{3,1}(u) = 3u(1-u)^2, \quad B_{3,2}(u) = 3u^2(1-u), \quad B_{3,3}(u) = u^3.$$

To get $BS(u) = BZ(u)$, we must have $BS_x(u) = BZ_x(u)$ (x-coordinate match) and $BS_y(u) = BZ_y(u)$ (y-coordinate match). This means

$$\begin{aligned} -b_{-1}(u) + b_0(u) + b_1(u) - b_2(u) &= B_{3,0}(u) x_0 + B_{3,1}(u) x_1 + B_{3,2}(u) x_2 + B_{3,3}(u) x_3, \\ -b_{-1}(u) - b_0(u) + b_1(u) + b_2(u) &= B_{3,0}(u) y_0 + B_{3,1}(u) y_1 + B_{3,2}(u) y_2 + B_{3,3}(u) y_3. \end{aligned}$$

The above equations, after substituting the basis functions in both sides, result

$$\begin{aligned} 0u^3 - u^2 + u + \frac{2}{3} &= (-x_0 + 3x_1 - 3x_2 + x_3) u^3 + (3x_0 - 6x_1 + 3x_2) u^2 + (-3x_0 + 3x_1) u + x_0, \\ -\frac{2}{3}u^3 + u^2 + u - \frac{2}{3} &= (-y_0 + 3y_1 - 3y_2 + y_3) u^3 + (3y_0 - 6y_1 + 3y_2) u^2 + (-3y_0 + 3y_1) u + y_0. \end{aligned}$$

Equating the coefficients of monomials in u from both sides, we get

$$Q_0 = \left(\frac{2}{3}, -\frac{2}{3}\right), Q_1 = \left(1, -\frac{1}{3}\right), Q_2 = \left(1, \frac{1}{3}\right), Q_3 = \left(\frac{2}{3}, \frac{2}{3}\right).$$

(b) The curve segment is plotted below with B-spline and Bezier control points. The plot is made with MATLAB code `HW6Problem3b.m` (attached).

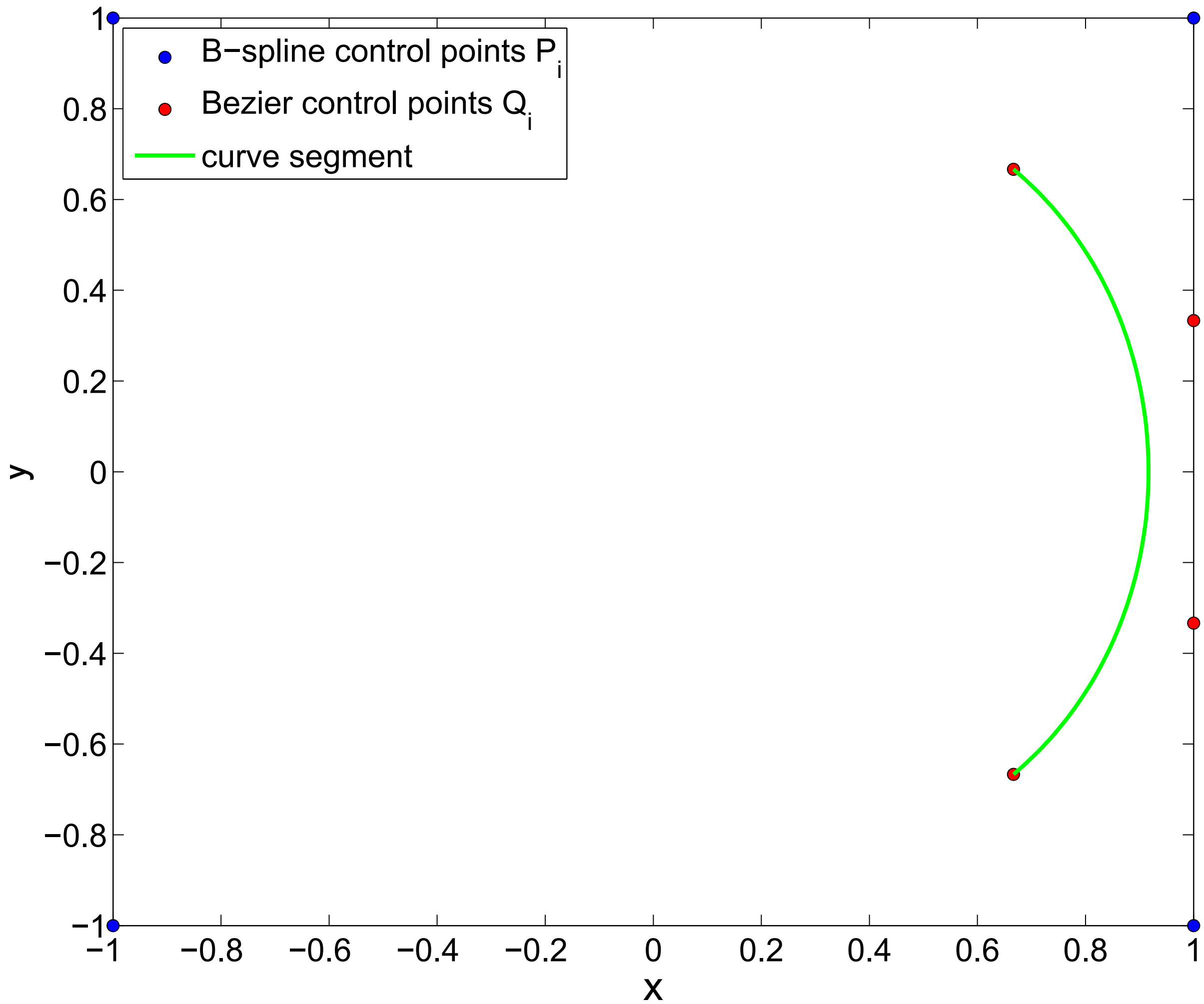

Figure 3: The curve segment (*green line*), the B-spline control points (*blue circles*) P_0, P_1, P_2, P_3, and the Bezier control points (*red circles*) Q_0, Q_1, Q_2, Q_3, computed in part (a).