Aero 320: Numerical Methods

Lab Assignment 15

Fall 2013

Problem 1

Least squares approximation of a continuous function

Find the least squares approximation of the form $\widehat{f}(x) = a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$, for the function

$$f(x) = \begin{cases} -1, & -\pi \le x < 0, \\ +1, & 0 \le x \le \pi. \end{cases}$$

Solution

(a) In the Homework 6, you will show that

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx, \quad a_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos kx\,dx, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin kx\,dx, \quad k = 1,\ldots,n.$$

Then, for our function $f(x)$, as defined in the question, we get

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -1\,dx \;+\; \int_{0}^{\pi} 1\,dx\right) = \frac{1}{2\pi}\left(-x\Big|_{x=-\pi}^{x=0} \;+\; x\Big|_{x=0}^{x=\pi}\right) = \frac{1}{2\pi}\left(-\pi \;+\; \pi\right) = 0,$$

$$\begin{aligned} a_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos kx\,dx \;+\; \int_{0}^{\pi} \cos kx\,dx\right) &= -\frac{1}{\pi}\frac{\sin kx}{k}\Big|_{x=-\pi}^{x=0} \;+\; \frac{1}{\pi}\frac{\sin kx}{k}\Big|_{x=0}^{x=\pi} \\ &= \frac{1}{\pi}\left(0 \;+\; 0\right) = 0, \end{aligned}$$

$$\begin{aligned} b_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin kx\,dx \;+\; \int_{0}^{\pi} \sin kx\,dx\right) &= \frac{1}{\pi}\frac{\cos kx}{k}\Big|_{x=-\pi}^{x=0} \;-\; \frac{1}{\pi}\frac{\cos kx}{k}\Big|_{x=0}^{x=\pi} \\ &= \frac{1}{\pi}\left(\frac{1-(-1)^k}{k}\right) \;-\; \frac{1}{\pi}\left(\frac{(-1)^k - 1}{k}\right) \\ &= \frac{2}{\pi}\left(\frac{1-(-1)^k}{k}\right). \end{aligned}$$

Thus, $\widehat{f}(x) = \frac{2}{\pi}\sum_{k=1}^{n}\left(\frac{1-(-1)^k}{k}\right)\sin kx$. Notice that b_k is zero for any even number k. Try plotting $\widehat{f}$ together with f, for different values of n.