F26 AerE 3310: Flight Control Systems I

Practice Problem Set 1: Modeling and Linearization

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Problem 1

Denote the nth order derivative of $x(t)$ with respect to t as $x^{(n)}(t) := \dfrac{\mathrm{d}^n x}{\mathrm{d}t^n}$.

(a) By defining appropriate state vector $\boldsymbol{x}$, write the following control system in state space form.

$$x^{(2026)}(t) + 2026\, x^{(2025)}(t) + 2025\, x^{(2024)}(t) + \ldots + 2\, x^{(1)}(t) + x(t) = u, \quad y(t) = x(t).$$

Solution. As usual, let $x_1 := x, x_2 := x^{(1)}, \ldots, x_{2026} := x^{(2025)}$. This gives the first order ODEs:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \ldots, \quad \dot{x}_{2025} = x_{2026}, \quad \dot{x}_{2026} = -x_1 - 2\, x_2 \ldots - 2025\, x_{2025} - 2026\, x_{2026} + u.$$

Defining the state vector $\boldsymbol{x} := (x_1, x_2, \ldots, x_{2026})^\top$, we obtain

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{2025} \\ \dot{x}_{2026} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -1 & -2 & -3 & \ldots & -2026 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2025} \\ x_{2026} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u, \quad y = (1\ 0\ 0\ \ldots\ 0) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2025} \\ x_{2026} \end{pmatrix} + 0 \cdot u,$$

which is in the standard state space form $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$, $y = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}u$.

(b) From your answer in part (a), is this a linear or nonlinear control system? Is this an LTI or LTV system or neither? Explain why.

Solution. This is a linear, in fact, LTI control system since all coefficient matrices are constants.

(c) How will your answers to parts (a)-(b) change for the following control system?

$$x^{(2026)}(t) + \sin(2026t) x^{(2025)}(t) + \sin(2025t) x^{(2024)}(t) + \ldots + \sin(2t) x^{(1)}(t) + \sin(t) x(t) = u, \quad y(t) = x(t).$$

Solution. Using the same state vector $\boldsymbol{x}$ as in part (a), we now get the state space form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{2025} \\ \dot{x}_{2026} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\sin(t) & -\sin(2t) & -\sin(3t) & \dots & -\sin(2026t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2025} \\ x_{2026} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u,$$

$$y = (1\ 0\ 0\ \dots\ 0) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2025} \\ x_{2026} \end{pmatrix} + 0 \cdot u.$$

This is a linear, in fact LTV control system since the state matrix has explicit dependence on time t.

Problem 2

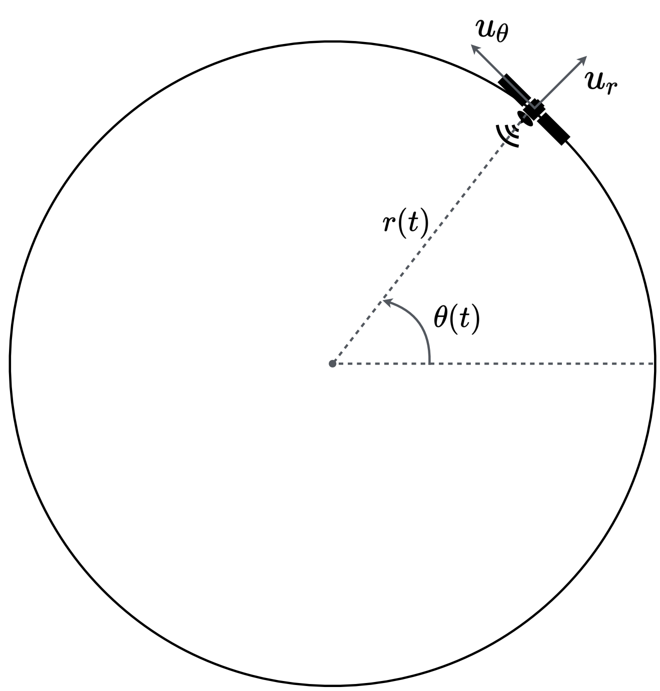

Figure 1: Satellite in circular orbit.

Suppose we want to put an Earth-orbiting satellite of constant mass m in a circular orbit in the equatorial plane. As shown in Fig. 1, let $(r(t), \theta(t))$ denote the satellite's polar coordinate at time t. The satellite's onboard thrusters can apply radial thrust u_r, and tangential thrust u_θ.

Due to inverse squared gravitational pull toward the center of Earth, the satellite's controlled dynamics is

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{mk}{r^2} + u_r, \tag{1a}$$

$$mr\ddot{\theta} = -2m\dot{r}\dot{\theta} + u_\theta, \tag{1b}$$

where the gravitational constant $k > 0$.

(a) Let the input $\boldsymbol{u} := (u_1, u_2)^\top = (u_r/m, u_\theta/m)^\top$, and the output $\boldsymbol{y} := (y_1, y_2)^\top = (r, \theta)^\top$. By defining appropriate state vector $\boldsymbol{x}$, express the satellite's controlled dynamics in standard state space form (Lec. 3, 4). Show all details of your derivations.

Solution for Problem 1(a)

Let the state vector $\boldsymbol{x} := (x_1, x_2, x_3, x_4)^\top = \left(r, \dot{r}, \theta, \dot{\theta}\right)^\top$. Then the state equation (1) becomes

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix}}_{\dot{\boldsymbol{x}}} = \underbrace{\begin{pmatrix} x_2 \\ x_1 x_4^2 - \dfrac{k}{x_1^2} + u_1 \\ x_4 \\ -\dfrac{2x_2x_4}{x_1} + \dfrac{u_2}{x_1} \end{pmatrix}}_{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})}, \tag{2}$$

and from the given definition of $\boldsymbol{y}$, the output equation becomes

$$\boldsymbol{y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{u}. \tag{3}$$

Equations (2) and (3) are in the standard state space form.

(b) We would like the satellite to follow a nominal trajectory such that in the absence of control, starting from any initial condition $\theta_0 := \theta(t = 0)$, it stays in a circular orbit of some desired constant radius r_0 with a desired constant angular velocity ω_0. In other words, the desired nominal trajectory is $(\boldsymbol{x}_0(t), \boldsymbol{u}_0(t)) = \left(\begin{pmatrix} r_0 \\ 0 \\ \theta_0 + \omega_0 t \\ \omega_0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$. Intuitively, for arbitrary constants r_0, ω_0, k, such a circular orbit may not be possible. Show that such nominal trajectory is possible if and only if the constants r_0, ω_0, k satisfy a condition that you need to derive.

Solution for Problem 1(b)

Since the nominal trajectory must satisfy the state equation, we have $\dot{\boldsymbol{x}}_0 = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0)$. This gives

$$\begin{pmatrix} 0 \\ 0 \\ \omega_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ r_0\omega_0^2 - \dfrac{k}{r_0^2} \\ \omega_0 \\ 0 \end{pmatrix},$$

which holds if and only if $r_0^3\omega_0^2 = k$, and the desired circular orbit is possible.

(c) Assuming the condition you derived in part (b) relating r_0, ω_0, k holds, linearize your state space model from part (a) about the nominal trajectory given in part (b). Show all your calculation. Is this linearized state space model an LTI or LTV system? Explain in 1 sentence.

Solution for Problem 1(c)

Following Lec. 4-5, we compute the linearized state space model:

$$\begin{aligned}
\Delta\dot{\boldsymbol{x}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ x_4^2 + \dfrac{2k}{x_1^3} & 0 & 0 & 2x_1x_4 \\ 0 & 0 & 0 & 1 \\ \dfrac{2x_2x_4}{x_1^2} & -\dfrac{2x_4}{x_1} & 0 & -\dfrac{2x_2}{x_1} \end{bmatrix}_{(\boldsymbol{x}_0(t),\boldsymbol{u}_0(t))} \Delta\boldsymbol{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{x_1} \end{bmatrix}_{(\boldsymbol{x}_0(t),\boldsymbol{u}_0(t))} \Delta\boldsymbol{u} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{2\omega_0}{r_0} & 0 & 0 \end{bmatrix} \Delta\boldsymbol{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{r_0} \end{bmatrix} \Delta\boldsymbol{u}, \\
\Delta\boldsymbol{y} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Delta\boldsymbol{x} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Delta\boldsymbol{u},
\end{aligned}$$

where we used the condition derived in part (b) in the $(2,1)$th entry of the state matrix.

The linearized state space model above is an LTI system since all the coefficient matrices turned out to be time invariant even though the nominal trajectory was time-varying. This is exactly what we explained in Lec. 4 p. 8 second bullet point.