

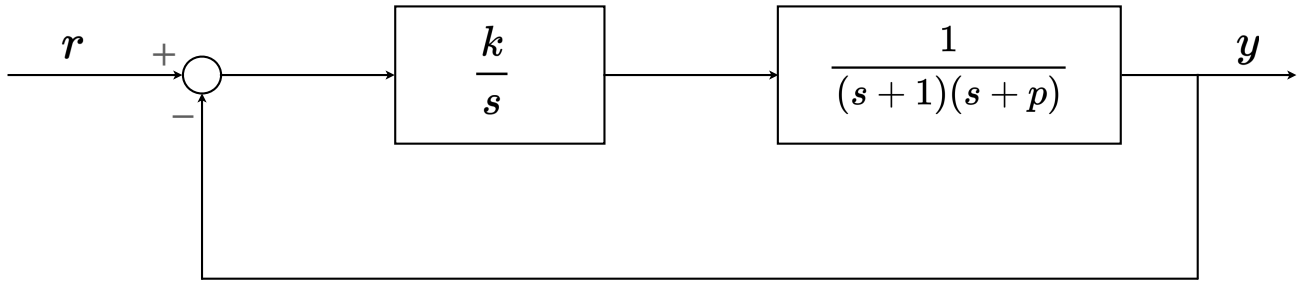
F26 AerE 3310: Flight Control Systems I

Practice Problem Set 5: Stability

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Problem 1



Consider the closed loop control system for a robot where the block $\frac{k}{s}$ denotes the controller transfer function, and the block $\frac{1}{(s+1)(s+p)}$ denotes to the process transfer function.

The constants p, k are both parameters that can be designed. We want to figure out what are good design values for the pair (p, k) .

Determine a set of inequalities to be satisfied by the pair (p, k) for guaranteeing overall system to be stable. Show all your calculations. Then graphically plot the stability region in (p, k) parameter coordinate.

Solution for Problem 1

The overall transfer function is a negative feedback interconnection of $\frac{k}{s} \cdot \frac{1}{(s+1)(s+p)}$ and 1. Therefore, the overall transfer function is

$$\frac{\frac{k}{s(s+1)(s+p)}}{1 + \frac{k}{s(s+1)(s+p)}} = \frac{k}{s(s+1)(s+p) + k} = \frac{k}{s^3 + (p+1)s^2 + ps + k}.$$

For stability, the necessary condition is that all coefficients of the denominator polynomial $\text{den}(s) := s^3 + (p+1)s^2 + ps + k$ are positive, which requires

$$1 > 0, \quad p+1 > 0, \quad p > 0, \quad k > 0,$$

i.e., (p, k) must lie in the strict positive orthant $\mathbb{R}_{>0}^2$ satisfying $p > 0, k > 0$. However, this is not sufficient.

Applying the Routh-Hurwitz condition from Lec. 13 p. 7, we get an extra condition: $p^2 + p > k$. Therefore, we conclude that the admissible region for stability is

$$\{p > 0, k > 0, p^2 + p > k\} = \{(p, k) \in \mathbb{R}_{>0}^2 \mid p^2 + p > k\}.$$

Therefore, the stability region is the bottom exterior of the parabola $p^2 + p = k$, intersected with the positive quadrant $\mathbb{R}_{>0}^2$, shown as shaded region in Fig. 1.

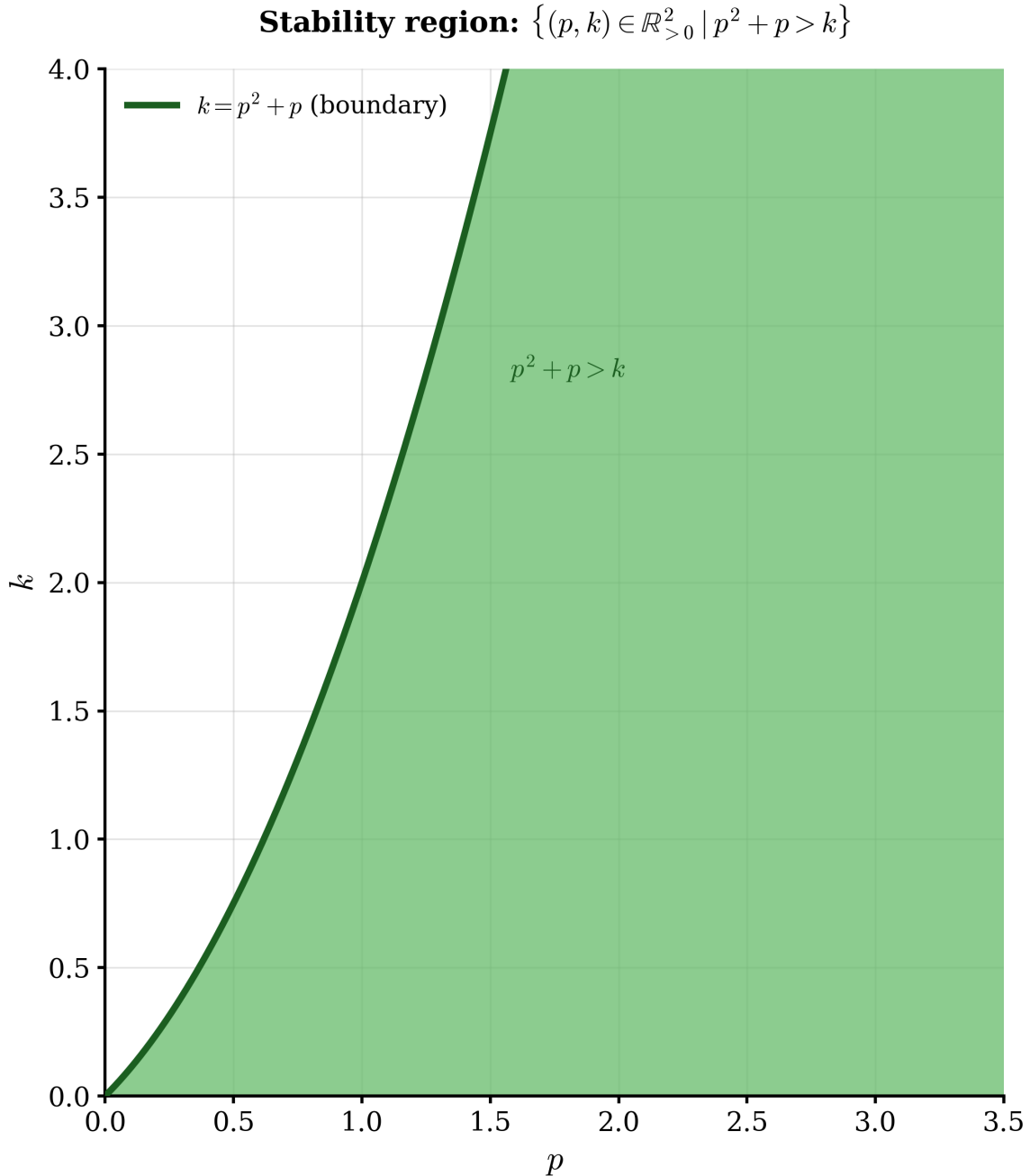


Figure 1: Stability region in (p, k) coordinate.

Problem 2



Figure 2: Figure skater tilting and spinning while keeping skate location on ground fixed.

The figure skater shown in Fig. 2 is tilting and spinning while keeping her left foot/skate location on ground fixed. To do so, she is controlling her tilt angle $\theta(t)$ and spin angle $\phi(t)$ with control torque $u(t)$ while maintaining constant spin-axis angular momentum h created by the stretching of her arms (and aligning right leg with right arm).

The following are constant positive parameters: m = mass of the skater, g = acceleration due to gravity, ℓ = distance between the fixed skate location at ground and the skater's center of mass (filled dark circle in Fig. 2), h = magnitude of spin-axis angular momentum, I = moment of inertia of the skater about the fixed skate location on ground, J = moment of inertia of the skater about the axis (solid line) connecting the fixed skate location and the center of mass.

(a) The linearized equations of motion for the figure skater is

$$\begin{aligned} I\ddot{\theta} &= mg\ell\theta - h\dot{\phi}, \\ J\ddot{\phi} &= h\dot{\theta} + u. \end{aligned}$$

Assume zero initial conditions. Derive the transfer function from u to θ , that is, $G_{u \rightarrow \theta}$. Also derive the transfer function from u to ϕ , that is, $G_{u \rightarrow \phi}$.

Solution for Problem 2(a)

Let $\Theta(s) := \mathcal{L}[\theta(t)]$, $\Phi(s) := \mathcal{L}[\phi(t)]$, $U(s) := \mathcal{L}[u(t)]$. Taking the Laplace transform to both sides of the equations of motion with zero initial conditions, we get:

$$Is^2\Theta = mg\ell\Theta - hs\Phi, \quad (1a)$$

$$Js^2\Phi = hs\Theta + U. \quad (1b)$$

Eliminating Φ from (1), we obtain

$$G_{u \rightarrow \theta}(s) = \frac{\Theta(s)}{U(s)} = \frac{-h/(IJ)}{s \left(s^2 + \frac{h^2}{IJ} - \frac{mg\ell}{I} \right)}. \quad (2)$$

To eliminate Θ from (1), we use (2) to substitute for $\Theta(s) = G_{u \rightarrow \theta}(s)U(s)$ in (1b), to obtain

$$G_{u \rightarrow \phi}(s) = \frac{\Phi(s)}{U(s)} = \frac{(1/J)s^2 - \frac{mg\ell}{IJ}}{s^2 \left(s^2 + \frac{h^2}{IJ} - \frac{mg\ell}{I} \right)}. \quad (3)$$

(b) Mathematically argue which of the two transfer functions derived in part (a) is/are stable/not stable for arbitrary positive parameters m, g, ℓ, h, I, J .

Solution for Problem 2(b)

Both transfer functions (2) and (3) are not stable because the quadratic factor polynomial in their denominators fail the necessary conditions of having all coefficients positive. Also, both of them have poles at the origin (which is not in open LHP).

(c) Mathematically argue which of the two transfer functions derived in part (a) is/are stabilizable/not stabilizable by unity feedback (positive or negative, see Lec. 11 p. 4-7) for arbitrary positive parameters m, g, ℓ, h, I, J .

Solution for Problem 2(c)

With unity feedback, the closed loop transfer functions are (Lec. 11, p. 11)

$$\frac{G_{u \rightarrow \theta}(s)}{1 \pm G_{u \rightarrow \theta}(s)} = \frac{-h/(IJ)}{s \left(s^2 + \frac{h^2}{IJ} - \frac{mg\ell}{I} \right) \mp h/(IJ)}, \quad (4)$$

and

$$\frac{G_{u \rightarrow \phi}(s)}{1 \pm G_{u \rightarrow \phi}(s)} = \frac{(1/J)s^2 - \frac{mg\ell}{IJ}}{s^2 \left(s^2 + \frac{h^2}{IJ} - \frac{mg\ell}{I} \right) \pm \left((1/J)s^2 - \frac{mg\ell}{IJ} \right)}. \quad (5)$$

Since the denominator polynomials in both (4) and (5) fail the necessary conditions for stability (all coefficients had to be positive), so both $G_{u \rightarrow \theta}, G_{u \rightarrow \phi}$ are not stabilizable by unity feedback. So the stabilization need to be achieved by more sophisticated control than simple unity feedback.