

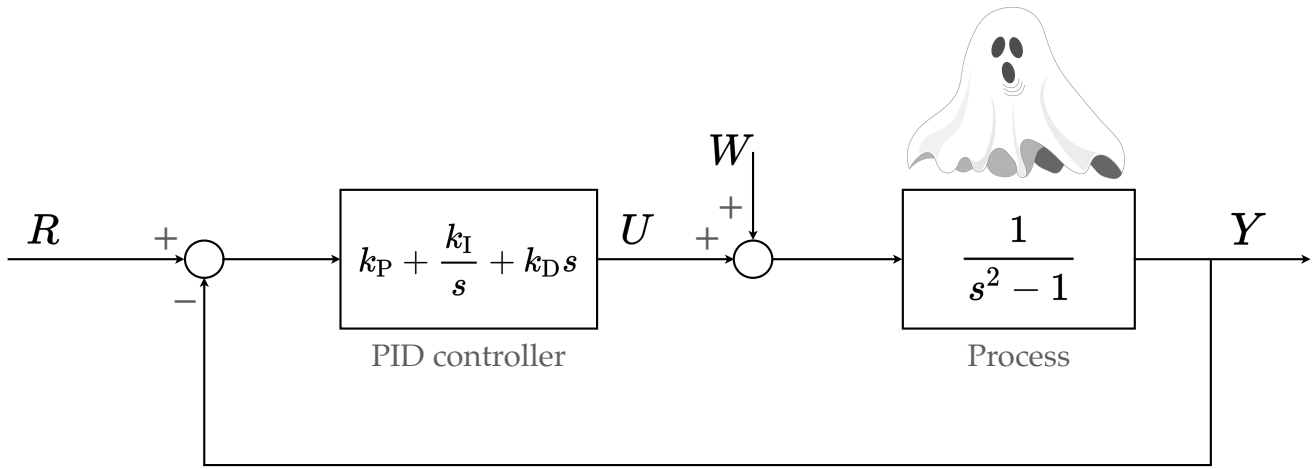
F26 AerE 3310: Flight Control Systems I

Practice Problem Set 6: PID Control Design

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Problem 1



This Halloween, a ghost plans to hover over Ames. To keep constant reference altitude as well as to reject wind disturbance, the ghost would like to use PID control law as shown.

(a) Is the Ghost's uncontrolled altitude dynamics/process transfer function $\frac{1}{s^2 - 1}$ stable? Explain why/why not in one sentence.

Solution for Problem 1(a)

The ghost's process/altitude dynamics transfer function $\frac{1}{s^2 - 1}$ is NOT stable because its denominator polynomial fails the necessary condition for stability (linear term coefficient is zero).

(b) Consider the special case of proportional controller design, i.e., when $k_I = k_D = 0$ and k_P is the only design variable. Also assume $W = 0$ (no disturbance). Derive the transfer function $\frac{Y}{R}$ for this case, and argue why it is impossible to stabilize the Ghost's altitude dynamics in part(a) using proportional control design!

Solution for Problem 1(b)

In this case, the closed loop transfer function comprises of a forward gain $\frac{k_P}{s^2 - 1}$ in unity negative feedback interconnection, i.e.,

$$\frac{Y}{R} = \frac{\frac{k_P}{s^2 - 1}}{1 + \frac{k_P}{s^2 - 1}} = \frac{k_P}{s^2 + k_P - 1}.$$

For the closed loop stability, we need the poles of $\frac{Y}{R}$ to be in the open LHP. However, its denominator polynomial has a linear term with coefficient zero, i.e., fails to satisfy the necessary conditions for stability. Therefore, $\frac{Y}{R}$ is not stable for any real k_P , i.e., proportional controllers cannot stabilize $G(s)$.

(c) Now consider designing PD controller, i.e., fix $k_I = 0$ and think of the pair (k_P, k_D) as design variable. Still continue assuming $W = 0$ (no disturbance). Derive the condition on (k_P, k_D) such that $\frac{Y}{R}$ is stable, i.e., the Ghost's altitude dynamics in part(a) is stabilizable. With such stabilizing (k_P, k_D) , is it possible to eliminate steady state error when tracking unit step reference?

Solution for Problem 1(c)

In this case, we get the transfer function

$$\frac{Y}{R} = \frac{\frac{k_P + k_D s}{s^2 - 1}}{1 + \frac{k_P + k_D s}{s^2 - 1}} = \frac{k_P + k_D s}{s^2 + k_D s + (k_P - 1)},$$

for which necessary conditions of stability gives $k_D > 0$, $k_P > 1$. Since the denominator polynomial of $\frac{Y}{R}$ is a quadratic, the necessary condition is also sufficient (by Routh-Hurwitz criterion). Hence, $G(s)$ is stabilizable by any PD controller in

$$\{(k_P, k_D) \in \mathbb{R}^2 \mid k_P > 1, k_D > 0\}.$$

Let $y_u(t)$ (resp. $Y_u(s)$) denote the time domain output (resp. s domain output) for the unit step input. Under the stabilizability conditions $k_D > 0$, $k_P > 1$, we find the DC gain

$$\lim_{t \rightarrow \infty} y_u \stackrel{\text{FVT}}{=} \lim_{s \rightarrow 0} s Y_u = \lim_{s \rightarrow 0} \left(s \cdot \frac{1}{s} \cdot \frac{k_P + k_D s}{s^2 + k_D s + (k_P - 1)} \right) = \frac{k_P}{k_P - 1},$$

which is $\neq 1$ for any real k_P . So it is impossible to achieve asymptotic tracking of unit step reference using a stabilizing PD controller.

(d) Next, consider PID control design, i.e., think of the triple (k_P, k_I, k_D) as design variable. Also consider nonzero disturbance, i.e., $W \neq 0$. Since Y then depends on both the reference input R and the disturbance input W , we use input-output linearity to write

$$Y = G_{R \rightarrow Y} R + G_{W \rightarrow Y} W.$$

Derive the “reference to output transfer function” $G_{R \rightarrow Y}$ and the “disturbance to output transfer function” $G_{W \rightarrow Y}$. Show all calculations.

Use your answer to derive the condition on (k_P, k_I, k_D) such that the Ghost’s altitude dynamics is stabilizable by a PID controller.

Hint: To determine $G_{R \rightarrow Y}$, simply zero out the disturbance W in the diagram, and apply the (forward gain)/(1 + loop gain) to the unity negative feedback interconnection.

To determine $G_{W \rightarrow Y}$, zero out the reference R in the diagram, reshape the diagram such that W appears as the left most input, and the process $\frac{1}{s^2 - 1}$ as the forward gain, and the PID controller $k_P + \frac{k_I}{s} + k_D s$ in the feedback path. Then apply the (forward gain)/(1 + loop gain) to this negative feedback interconnection.

Solution for Problem 1(d)

Denote the controller transfer function as $K(s) = k_P + \frac{k_I}{s} + k_D s$. Following the hints, we find the “reference to output transfer function”

$$G_{R \rightarrow Y} = \frac{KG}{1 + KG} = \frac{k_D s^2 + k_P s + k_I}{s^3 + k_D s^2 + (k_P - 1)s + k_I}.$$

Likewise, we find the “disturbance to output transfer function”

$$G_{W \rightarrow Y} = \frac{G}{1 + KG} = \frac{s}{s^3 + k_D s^2 + (k_P - 1)s + k_I}.$$

To ensure stabilizability, we need all roots of the denominator cubic polynomial

$$s^3 + k_D s^2 + (k_P - 1)s + k_I = 0$$

to be in the strict LHP. We find the necessary conditions (all coefficients positive) as

$$1 > 0, \quad k_D > 0, \quad k_P > 1, \quad k_I > 0.$$

For sufficient condition, we apply the Routh-Hurwitz criterion from Lec. 13 p. 7, to get

$$k_D(k_P - 1) > k_I.$$

So for the PID controller to stabilize the altitude of the Ghost, the (k_P, k_I, k_D) must be in the three dimensional region

$$\left\{ (k_P, k_I, k_D) \in \mathbb{R}^3 \mid k_P > 1, \quad k_I > 0, \quad k_D > 0, \quad k_D(k_P - 1) > k_I \right\}.$$

(e) Choose the triple (k_P, k_I, k_D) satisfying the conditions you derived in part (d). Now we wonder whether such an arbitrary stabilizing PID controller can also guarantee performance in terms of reference tracking and disturbance rejection.

Use your $G_{R \rightarrow Y}$ from part(d) to explain why asymptotic tracking of unit step reference is guaranteed for this system using PID control design. This fixes the limitation of PD control observed in part(c).

Use your $G_{W \rightarrow Y}$ from part(d) to explain why asymptotic disturbance rejection of unit step disturbance is guaranteed for this system using PID control design.

Solution for Problem 1(e)

To check unit step reference tracking, we use our answer in part 1(d), to compute the corresponding output limit:

$$\lim_{t \rightarrow \infty} y_u \stackrel{\text{FVT}}{=} \lim_{s \rightarrow 0} s Y_u = \lim_{s \rightarrow 0} \left(s \cdot \frac{1}{s} \cdot \frac{k_D s^2 + k_P s + k_I}{s^3 + k_D s^2 + (k_P - 1)s + k_I} \right) = \frac{k_I}{k_I} = 1,$$

for any stabilizing PID controller in

$$\left\{ (k_P, k_I, k_D) \in \mathbb{R}^3 \mid k_P > 1, k_I > 0, k_D > 0, k_D(k_P - 1) > k_I \right\}.$$

Thus, any stabilizing PID controller derived in part 1(d) has guaranteed asymptotic tracking of unit step reference.

To check unit step disturbance rejection, we use our answer in part 1(d), to compute the corresponding output limit:

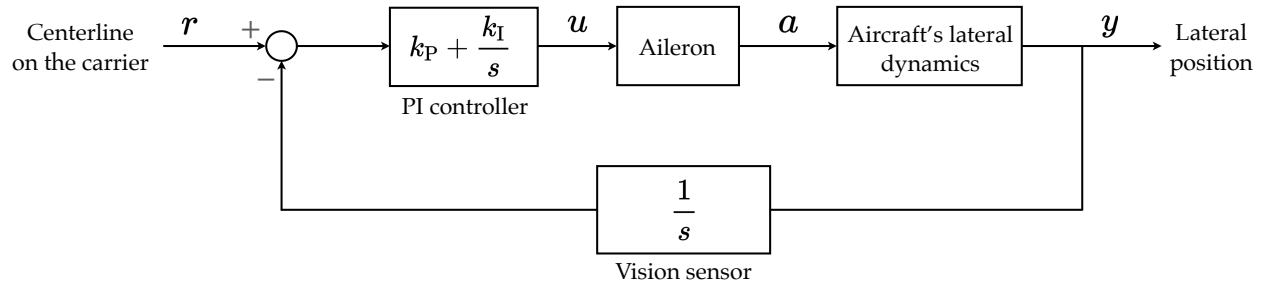
$$\lim_{t \rightarrow \infty} y_u \stackrel{\text{FVT}}{=} \lim_{s \rightarrow 0} s Y_u = \lim_{s \rightarrow 0} \left(s \cdot \frac{1}{s} \cdot \frac{s}{s^3 + k_D s^2 + (k_P - 1)s + k_I} \right) = \frac{0}{k_I} = 0,$$

for any stabilizing PID controller in

$$\left\{ (k_P, k_I, k_D) \in \mathbb{R}^3 \mid k_P > 1, k_I > 0, k_D > 0, k_D(k_P - 1) > k_I \right\}.$$

Thus, any stabilizing PID controller derived in part 1(d) has guaranteed rejection of unit step disturbance.

Problem 2



Consider a navy pilot landing on an aircraft carrier as shown in the top figure. The pilot engages an automatic PI controller to land along the centerline of the runway on the carrier's deck. The block diagram shows the lateral position control loop for this problem. Assume zero initial conditions.

(a) The “Aileron” block gets control input $u(t)$ and outputs actuation signal $a(t)$. They are related through a differential equation

$$\dot{a} - a = u.$$

The “Aircraft's lateral dynamics” block gets actuation input $a(t)$ and outputs lateral position $y(t)$. They are related through a differential equation

$$\ddot{y} + 5y = a.$$

Derive the closed loop transfer function $\frac{Y}{R}$ where $Y(s)$ is the Laplace transform of $y(t)$, and $R(s)$ is the Laplace transform of $r(t)$. Show all your calculations.

Solution for Problem 1(a)

Taking Laplace transforms of the given ODEs, we get

$$sA - A = U \Rightarrow \frac{A}{U} = \frac{1}{s-1},$$

$$s^2Y + 5Y = A \Rightarrow \frac{Y}{A} = \frac{1}{s^2 + 5}.$$

So the forward gain = $\left(k_P + \frac{k_I}{s}\right) \frac{1}{(s-1)(s^2+5)}$. Similarly, the loop gain = $\left(k_P + \frac{k_I}{s}\right) \frac{1}{s(s-1)(s^2+5)}$.
Therefore, the closed loop transfer function

$$\begin{aligned} \frac{Y}{R} &= \frac{\text{forward gain}}{1 + \text{loop gain}} \\ &= \frac{\left(k_P + \frac{k_I}{s}\right) \frac{1}{(s-1)(s^2+5)}}{1 + \left(k_P + \frac{k_I}{s}\right) \frac{1}{s(s-1)(s^2+5)}} \\ &= \frac{k_P s^2 + k_I s}{s^2(s-1)(s^2+5) + k_P s + k_I} \quad (\text{clearing the denominators}) \\ &= \frac{k_P s^2 + k_I s}{s^5 - s^4 + 5s^3 - 5s^2 + k_P s + k_I}. \end{aligned}$$

(b) Use your derived closed loop transfer function in part (a) to answer: can this PI controller guarantee closed loop stability for controlling lateral position of the aircraft?

If your answer is “yes”, then determine the range of such k_P, k_I . If your answer is “no”, the argue why not in 1 sentence.

Solution for Problem 1(b)

No it cannot (meaning there does not exist such real numbers k_P, k_I). This is because the denominator polynomial of $\frac{Y}{R}$ in part (a) has at least one negative coefficient (for s^4 , also for s^2), thus failing the necessary condition for stability (so no need to check Routh-Hurwitz criterion).