A Short Review of Laplace Transform

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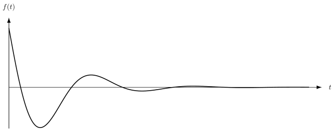
Notations

- $\mathbb{R}$ denotes the set of real numbers
- $\mathbb{C}$ denotes the set of complex numbers
- Re denotes the real part of a complex number, Im denotes its imaginary part
- $j := \sqrt{-1}$
- $F(s) = \mathcal{L}\left[f(t)\right]$ reads "$F(s)$ is the Laplace transform of $f(t)$"

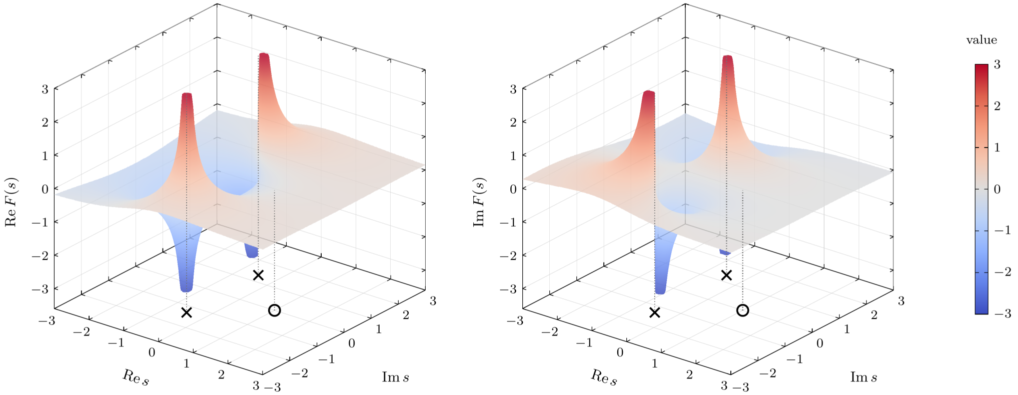
Definition of Laplace Transform

- Laplace transform $\mathcal{L}[\cdot]$ is a machine that takes a real function of time $t \in [0, \infty)$ and returns a complex function of the complex variable $s \in \mathbb{C}$
- $F(s) = \mathcal{L}[f(t)] := \int_0^\infty e^{-st} f(t) \mathrm{d}t$
- Inverse Laplace transform $\mathcal{L}^{-1}[\cdot]$ undoes this operation, i.e., $f(t) = \mathcal{L}^{-1}[F(s)]$
- Domain of the function $F(\cdot)$ depends on the choice of the function $f(\cdot)$



Example

$$F(s) = \frac{s-1}{s^2+s+2}$$



zero ○ at $s = 1$; poles × at $s = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$; surfaces clipped at ± 3

Powers

- $\mathcal{L}\left[k\right] = \dfrac{k}{s}$ for constant $k \in \mathbb{R}$, check by direct integration, domain: $\Re s > 0$

- $\mathcal{L}\left[t^n\right] = \dfrac{n!}{s^{n+1}}$ for whole $n = 0, 1, 2, \ldots$, check by induction, domain: $\Re s > 0$

Exponential, Sine, Cosine

- $\mathcal{L}\left[e^{zt}\right] = \dfrac{1}{s-z}$ for constant $z \in \mathbb{C}$, check by direct integration, domain: $\Re s > \Re z$
- Specialize the above for pure imaginary z, i.e., $z = j\omega$ with constant $\omega \in \mathbb{R}$
- ... to get two results in one shot:

$$\mathcal{L}\left[\cos\left(\omega t\right)\right] = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}\left[\sin\left(\omega t\right)\right] = \frac{\omega}{s^2+\omega^2}, \quad \text{domain: } \Re s > 0$$

Examples

$$\mathcal{L}\left[e^{-5t}\right] = \frac{1}{s+5}, \quad \mathcal{L}\left[2\cos(3t) - 4\sin(5t)\right] = 2 \cdot \frac{s^3 - 10s^2 + 25s - 90}{(s^2+9)(s^2+25)}$$

How to compute $\mathcal{L}^{-1}[\cdot]$

- Pattern recognition

Examples

$$\mathcal{L}^{-1}\left[\frac{7}{s+1}\right] = 7e^{-t}, \quad \mathcal{L}^{-1}\left[\frac{1}{s^2+36}\right] = \frac{1}{6}\sin(6t)$$

- If $F(s)$ is rational, then first do partial fraction expansion, then pattern recognition

Example

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{4s+5}{s^3(s+2)(s-1)^2}\right] &= \mathcal{L}^{-1}\left[\frac{69/8}{s} + \frac{23/4}{s^2} + \frac{5/2}{s^3} + \frac{1/24}{s+2} + \frac{-26/3}{s-1} + \frac{3}{(s-1)^2}\right] \\ &= \frac{69}{8} + \frac{23}{4}t + \frac{5}{4}t^2 + \frac{1}{24}e^{-2t} - \frac{26}{3}e^t + 3te^t\end{aligned}$$

Properties of Laplace Transform: Arithmetic

- Linearity

  $\mathcal{L}\left[af(t) + bg(t)\right] = aF(s) + bG(s)$ for $a, b \in \mathbb{C}$

- Time scaling

  $\mathcal{L}\left[f(at)\right] = \frac{1}{|a|}F\left(\frac{s}{a}\right)$ for $a \in \mathbb{R} \setminus \{0\}$

Properties of Laplace Transform: t Domain Calculus

- t domain derivative

$$\mathcal{L}\left[f^n(t)\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} \dot{f}(0) - \ldots - f^{(n-1)}(0) \text{ for } n = 0, 1, 2, \ldots$$

Examples

$$\mathcal{L}\left[\dot{f}\right] = sF(s) - f(0), \quad \mathcal{L}\left[\ddot{f}\right] = s^2 F(s) - sf(0) - \dot{f}(0)$$

- t domain integral and convolution

$$\mathcal{L}\left[f * g\right] := \mathcal{L}\left[\int_0^t f(\tau) g(t-\tau) \mathrm{d}\tau\right] = F(s)G(s)$$

Examples

For $g(t) = 1$ we get $\mathcal{L}\left[\int_0^t f(\tau)\mathrm{d}\tau\right] = \frac{F(s)}{s}$. Notice that $\mathcal{L}\left[f(t)g(t)\right] \neq F(s)G(s)$

Properties of Laplace Transform: s Domain Calculus

- s domain derivative

  $\mathcal{L}\left[t^n f(t)\right] = (-1)^n F^{(n)}(s)$ for $n = 0, 1, 2, \ldots$

Examples

$$\mathcal{L}\left[t f(t)\right] = -F'(s), \quad \mathcal{L}\left[t^2 f(t)\right] = F''(s)$$

- s domain integral

  $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\sigma)\mathrm{d}\sigma$. The special case $s = 0$ gives $\displaystyle\int_0^\infty \frac{f(t)}{t}\mathrm{d}t = \int_0^\infty F(\sigma)\mathrm{d}\sigma$

Example

$$\mathcal{L}\left[\frac{\sin t}{t}\right] = \arctan\frac{1}{s}, \quad \int_0^\infty \frac{\sin t}{t}\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}\sigma}{\sigma^2+1} = \left[\arctan\sigma\right]_{\sigma=0}^{\sigma=\infty} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Properties of Laplace Transform: Shift

- t domain shift

  $\mathcal{L}\left[f(t-\tau)\right] = e^{-s\tau}F(s)$ for $t > \tau$

- s domain shift

  $\mathcal{L}\left[e^{at}f(t)\right] = F(s-a)$ for $a \in \mathbb{C}$

Examples

$$\mathcal{L}^{-1}\left[\frac{2}{(s-3)^3}\right] = e^{3t}t^2, \quad \mathcal{L}^{-1}\left[\frac{1}{(s+5)^2+4}\right] = \frac{1}{2}\,e^{-5t}\sin(2t)$$

Initial Value Theorem (IVT)

- When applicable, IVT says: $\lim_{t \downarrow 0} f(t) = \lim_{s \uparrow \infty} sF(s)$
- Applicable when $\lim_{t \downarrow 0} f(t)$ exists and $f(t) = \mathcal{O}(e^{ct})$ for some $c \in \mathbb{R}$
- For rational $F(s)$, applicable if $F(s)$ is strictly proper
- Rational $F(s)$ is strictly proper if degree(denominator poly)<degree(numerator poly)

Examples

IVT does not apply to $F(s) = \dfrac{s-1}{s+2}$ but applies to $F(s) = \dfrac{s-1}{s^2+s+2}$

Final Value Theorem (FVT)

- When applicable, FVT says: $\lim_{t\uparrow\infty} f(t) = \lim_{s\to 0} sF(s)$
- Applicable when $F(s)$ has no poles in closed right half plane (RHP) except at most one (i.e., non-repeated) pole at origin

Examples

FVT does not apply to $F(s) = \dfrac{2s+5}{s^2(s+3)}$ with poles $0, 0, -3$ (repeated poles at origin)

FVT does not apply to $F(s) = \dfrac{s}{s^2+1}$ with poles $j, -j$ (both poles in closed RHP)

FVT does not apply to $F(s) = \dfrac{1}{s(s-2)}$ with poles $0, 2$ (even though 0 is non-repeated, 2 is in closed RHP)

FVT applies to $F(s) = \dfrac{1}{s(s+2)}$ with poles $0, -2$ (0 is non-repeated, -2 is NOT in closed RHP), so $\lim_{t\uparrow\infty} f(t) = \lim_{s\to 0} \dfrac{1}{s+2} = \dfrac{1}{2}$. Check via $f(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s+2)}\right] = \frac{1}{2}(1-e^{-2t})$

Exercise

- Compute $\mathcal{L}\left[7t^5 - 4\cos(3t) + 9e^{-t}\right]$, $\mathcal{L}\left[(t-2)^5\right]$ for $t > 2$, $\mathcal{L}\left[\int_0^t \frac{\sin\theta}{\theta}\mathrm{d}\theta\right]$

- Compute $\mathcal{L}^{-1}\left[1\right]$, $\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]$, $\mathcal{L}^{-1}\left[\log\left(\frac{s-10}{s-20}\right)\right]$

- For $F(s) = \frac{s}{(s+5)(s^2+s+1)}$, use FVT if possible, to compute $\lim_{t\uparrow\infty} f(t)$.
  Compute the same without FVT