Probabilistic Uncertainties and State Estimation

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Recap: uncertainties in control systems

Sources of uncertainties: initial conditions, parameters, process and measurement noise

Types of uncertainties: epistemic versus aleatoric, deterministic versus probabilistic

Deterministic uncertainties $\equiv$ set valued uncertainties

Exercise: deterministic uncertainties in control systems

A robot moving in 2D is experiencing process noise $(w_x(t), w_y(t))$ such that for all time t:

$$w_x^2 + 3w_y^2 \leq 10 \quad \text{OR} \quad 5w_x^2 + w_y^2 \leq 20$$

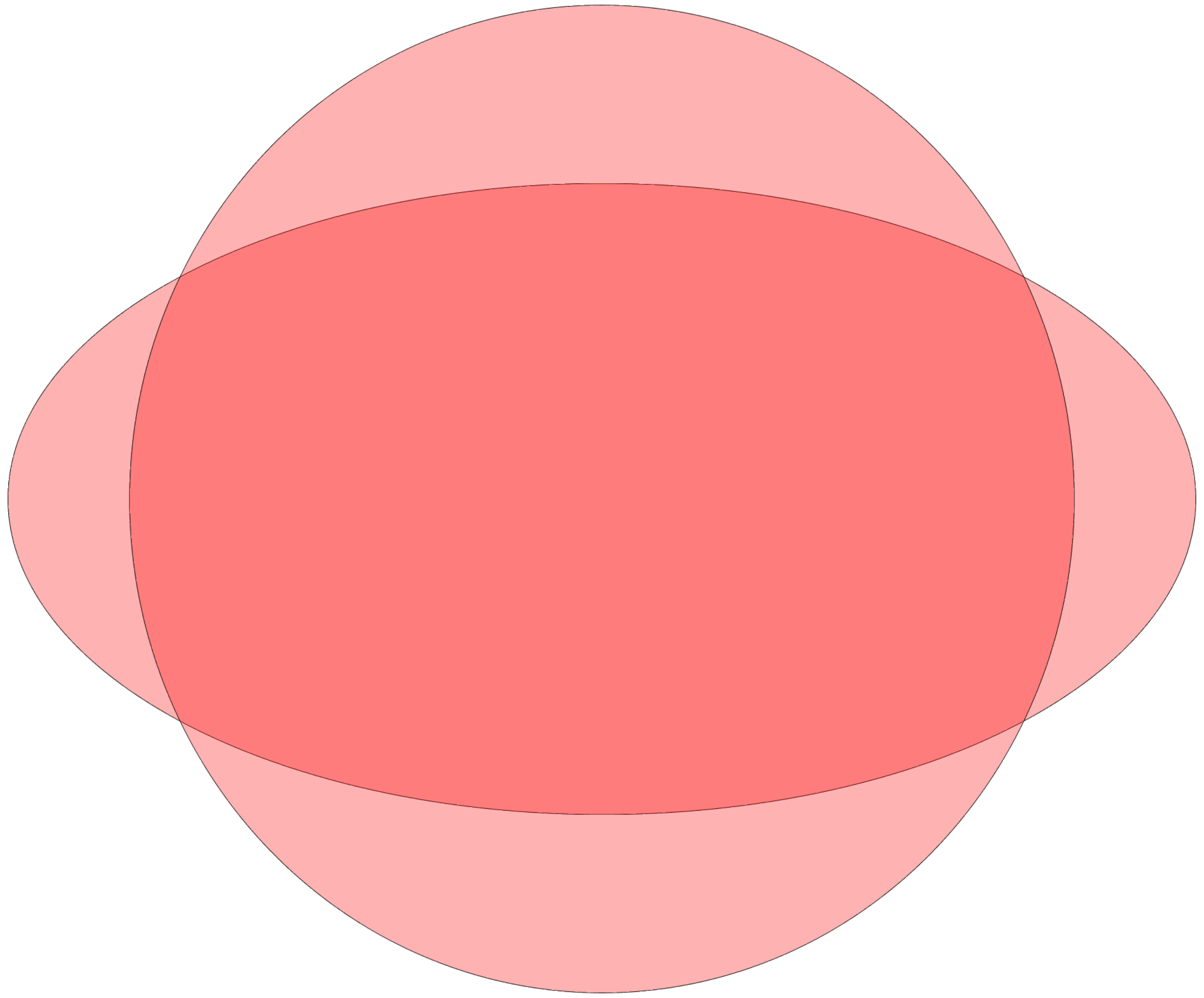
So the disturbance set is

Exercise: deterministic uncertainties in control systems

A robot moving in 2D is experiencing process noise $(w_x(t), w_y(t))$ such that for all time t:

$$w_x^2 + 3w_y^2 \leq 10 \quad \text{OR} \quad 5w_x^2 + w_y^2 \leq 20$$

So the disturbance set is **union of elliptical discs**



Probabilistic/stochastic uncertainties

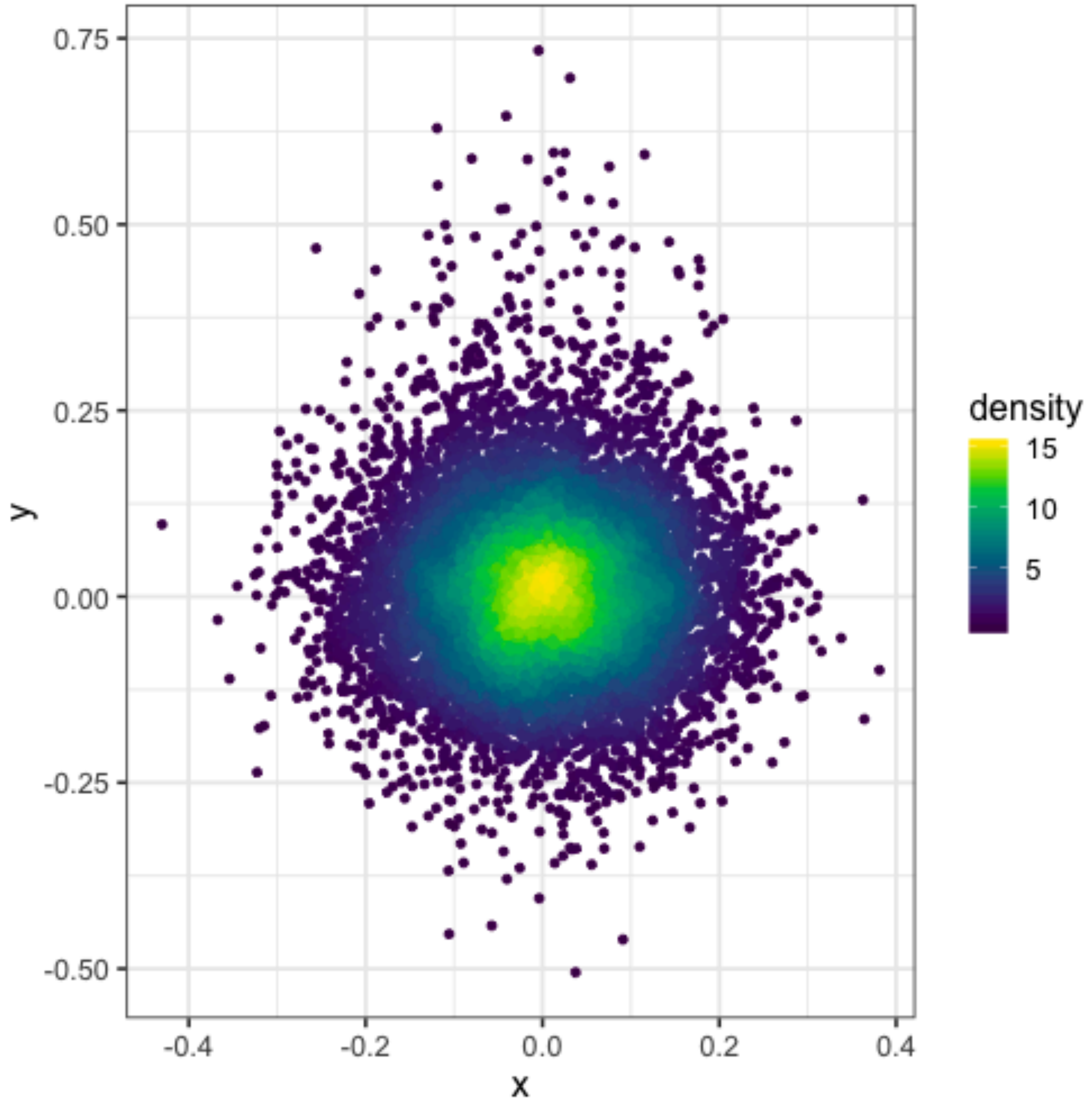
Initial conditions: $x_1(0), \ldots, x_n(0)$ may not be exactly known [but we may know that some values are "more likely" than the other]

Parameters: $p_1, \ldots, p_s$ may not be exactly known [ditto]

Noises: $w_1, \ldots, w_q$ and $v_1, \ldots, v_r$ if present, are unknown and unmeasured [ditto]

This is more information than simply specifying the set where the values must belong to

Example: probabilistic/stochastic uncertainties by density



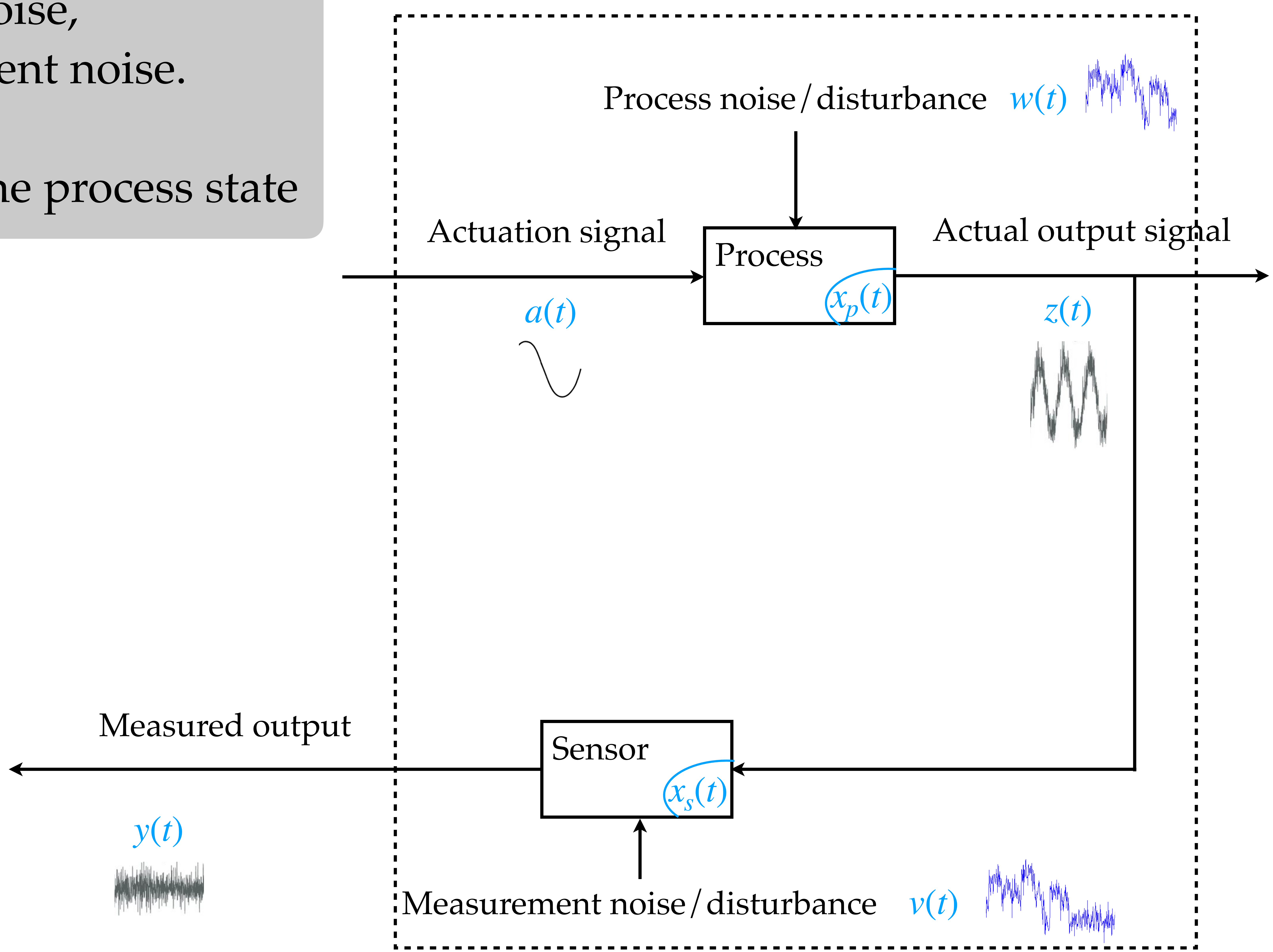
With high probability $(x, y) = (0,0)$

State estimation problem in control

State estimation problem in control

Given process model + process noise,
measurement model + measurement noise.

Compute the "best" estimate of the process state

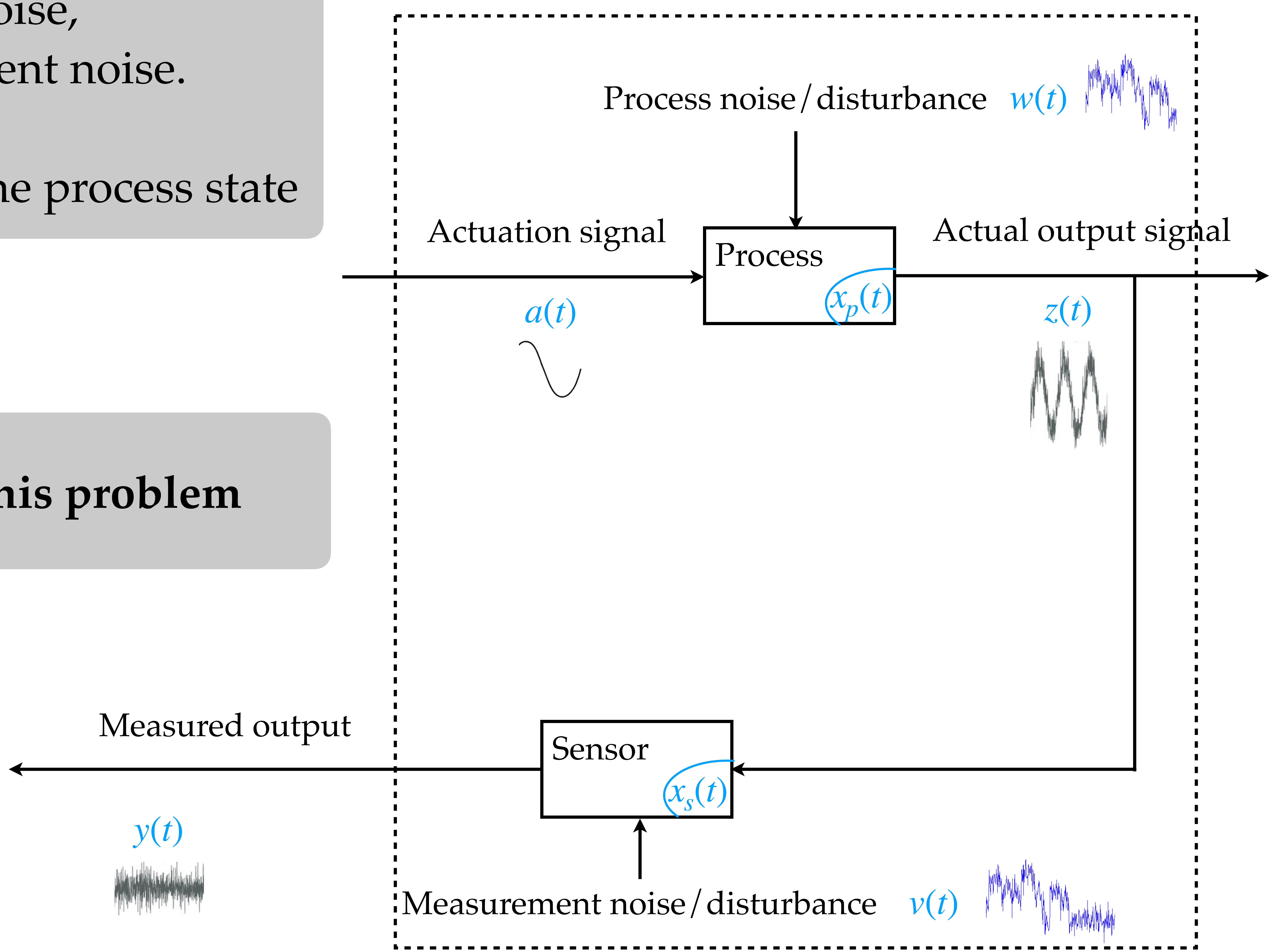


State estimation problem in control

Given process model + process noise,
measurement model + measurement noise.

Compute the "best" estimate of the process state

Filter = an algorithm to solve this problem



From Lec. 14, slide 6:

Process model:

$$x_1(t+1) = f_1\left(x_1(t),\ldots,x_n(t),u_1(t),\ldots,u_m(t),w_1(t),\ldots,w_q(t)\right)$$

$$x_2(t+1) = f_2\left(x_1(t),\ldots,x_n(t),u_1(t),\ldots,u_m(t),w_1(t),\ldots,w_q(t)\right)$$

$$\vdots = \vdots$$

$$x_n(t+1) = f_n\left(x_1(t),\ldots,x_n(t),u_1(t),\ldots,u_m(t),w_1(t),\ldots,w_q(t)\right)$$

Measurement model:

$$y_1(t) = g_1\left(x_1(t),\ldots,x_n(t),u_1(t),\ldots,u_m(t),v_1(t),\ldots,v_r(t)\right)$$

$$y_2(t) = g_2\left(x_1(t),\ldots,x_n(t),u_1(t),\ldots,u_m(t),v_1(t),\ldots,v_r(t)\right)$$

$$\vdots = \vdots$$

$$y_p(t) = g_p\left(x_1(t),\ldots,x_n(t),u_1(t),\ldots,u_m(t),v_1(t),\ldots,v_r(t)\right)$$

Given $f_1,\ldots,f_n, g_1,\ldots,g_p$,

and feedback $u_1,\ldots,u_m$,

as well as probabilistic descriptions

of $w_1,\ldots,w_q, v_1,\ldots,v_r$,

estimate $x_1,\ldots,x_n$

We also have the history of noisy measurements (raw data) up until the time t

What do we mean by "best estimate"

True process states $x_1(t), \ldots, x_n(t)$ are probabilistic

Example (nonlinear control system): three process states (x_1, x_2, θ) and two controls (V, ω)

$$x_1(t+1) = x_1(t) + V(t)\Delta t \times \cos\theta(t)$$
$$x_2(t+1) = x_2(t) + V(t)\Delta t \times \sin\theta(t)$$
$$\theta(t+1) = \theta(t) + \Delta t \times \omega(t) + w(t)$$

We know what the statistics of the process noise $w(t)$ is

So our estimates $\hat{x}_1(t), \hat{x}_2(t), \hat{\theta}(t)$ will be **probabilistic**

What do we mean by "best estimate"

Common way to measure "best": minimum mean squared error (MMSE)

Find $\hat{x}(t)$ that minimizes

Expected value of $\left[(x(t) - \hat{x}(t))^2 \mid \text{history of measurements up until time } t\right]$

It turns out that the minimizer is what is called "conditional expectation of the process state"

Many algorithms: Kalman filter, particle filters