Fixed Points and Stability

Abhishek Halder

Dept. of Applied Mathematics
University of California, Santa Cruz

ahalder@ucsc.edu

Recap: Stability and stabilization

Different notions: stable (S), asymptotically stable (AS), globally asymptotically stable (GAS)

Examples from engineering and biology

In engineering, we often need to design the process to be unstable but stabilizable

Example 1: Fixed point in discrete time dynamics

$$\begin{pmatrix} x_1(t+1) & = & -2x_1(t) - 5x_2(t) \\ x_2(t+1) & = & x_1(t) + 3x_2(t) \end{pmatrix}$$

Equilibrium / fixed point ⇔ From where process states do not change over time

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Equilibrium / fixed point ⇔ From where process states do not change over time

∴ (x_1, x_2) is an equilibrium / fixed point if and only if

$x_1(t+1) = x_1(t)$ and $x_2(t+1) = x_2(t)$

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$\therefore (x_1, x_2)$ is an equilibrium / fixed point if and only if

$x_1(t+1) = x_1(t)$ and $x_2(t+1) = x_2(t)$

Unique fixed point

Substituting this in our given dynamics, we get $(x_1, x_2) = (0{,}0)$

Verify this calculation yourself!

Example 2: Fixed point in discrete time dynamics

$$\begin{pmatrix} x_1(t+1) & = & -0.2x_1(t) - 0.5x_2(t) \\ x_2(t+1) & = & x_1(t) + 0.3x_2(t) \end{pmatrix}$$

Find the equilibrium / fixed point(s)

Example 2: Fixed point in discrete time dynamics

$$\begin{pmatrix} x_1(t+1) & = & -0.2x_1(t) - 0.5x_2(t) \\ x_2(t+1) & = & x_1(t) + 0.3x_2(t) \end{pmatrix}$$

Find the equilibrium / fixed point(s)

Unique fixed point

Again, we get $(x_1, x_2) = (0,0)$

Example 3: Fixed point in discrete time dynamics

$$x(t+1) = 2x(t)\big(1 - x(t)\big)$$

Find the equilibrium/fixed point(s)

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$$x(t+1) = 2x(t)\big(1 - x(t)\big)$$

Find the equilibrium/fixed point(s)

We get $x = 0$, and 0.5.

Two fixed points

But are these fixed points stable or unstable?

Example 1

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Example 2

$$\begin{pmatrix} x_1(t+1) & = & -0.2x_1(t) - 0.5x_2(t) \\ x_2(t+1) & = & x_1(t) + 0.3x_2(t) \end{pmatrix}$$

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$\rightsquigarrow (x_1, x_2) = (0,0)$ is unstable

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Example 3

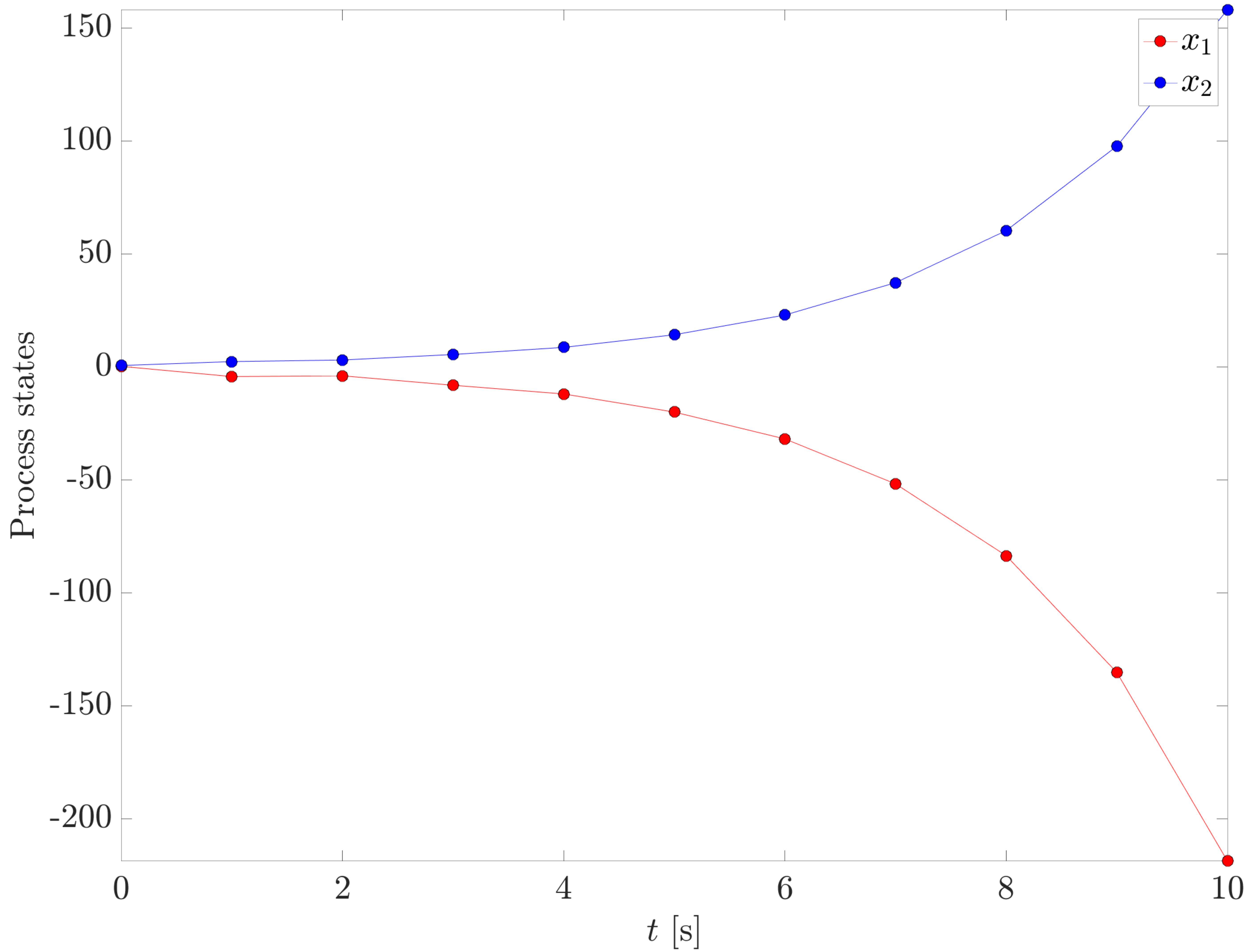
$$x(t+1) = 2x(t)\big(1 - x(t)\big)$$

$\rightsquigarrow x = 0$ is unstable, $x = 0.5$ is AS but not GAS

But are these fixed points stable or unstable?

Example 1

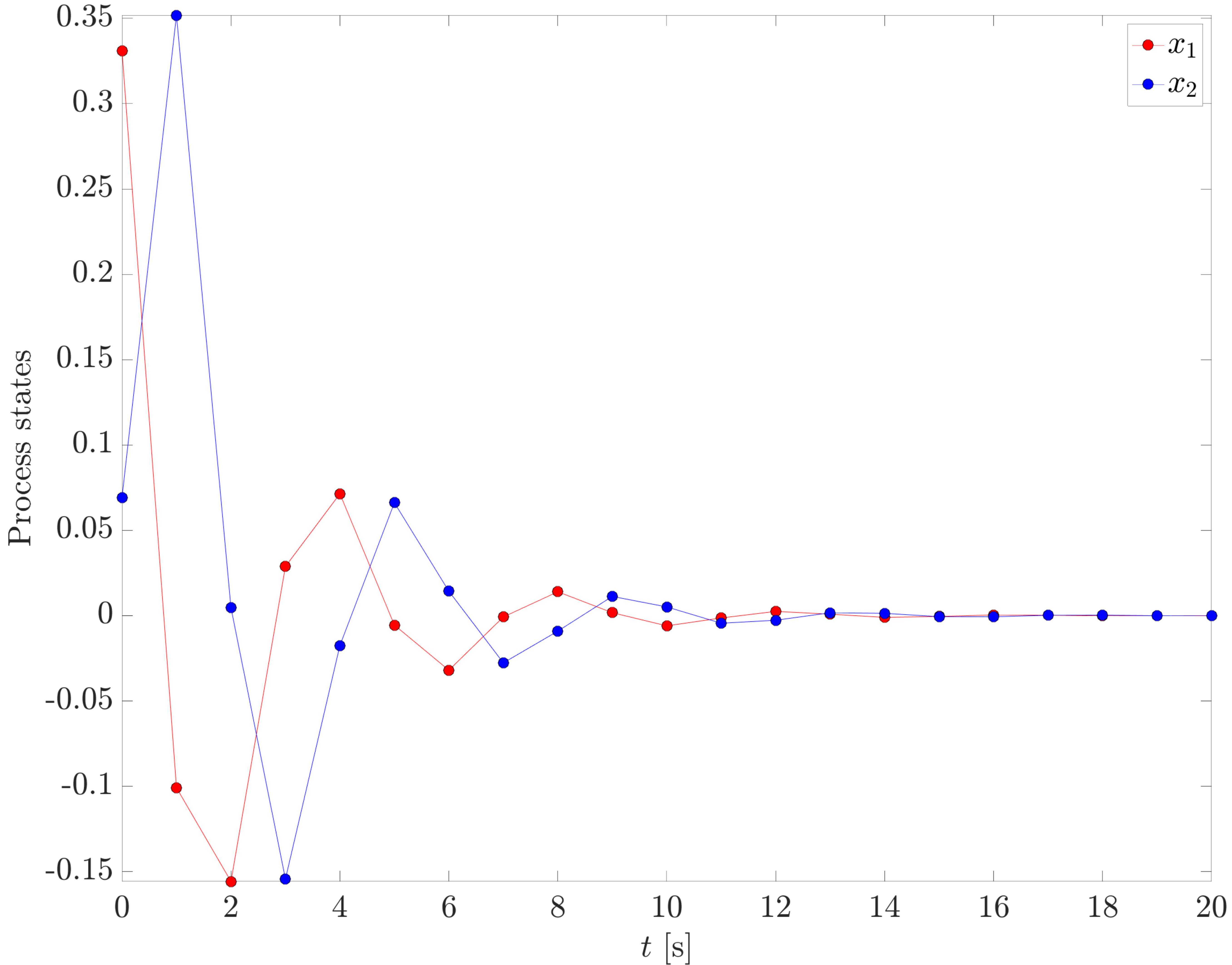
⇝ $(x_1, x_2) = (0,0)$ is unstable



But are these fixed points stable or unstable?

Example 2

⇝ $(x_1, x_2) = (0,0)$ is GAS



But are these fixed points stable or unstable?

Example 3

$\rightsquigarrow$ $x = 0$ is unstable, $x = 0.5$ is AS but not GAS