State Space

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Recap: Linear versus nonlinear

Process dynamics: uncontrolled and controlled

Example on stabilizing state feedback

Linear versus nonlinear dynamics

Fixed points and stability ideas outside control

A common problem across science and engineering:

Find the solutions or roots of a nonlinear equation $f(x) = 0$

Issue: may have multiple roots

Issue: usually no known analytical solutions

Question: **how to design algorithms** to calculate the roots using a computer?

Fixed point recursion: an algorithm to numerically solve $f(x) = 0$

Idea: rewrite the original equation $f(x) = 0$ in the form $x = g(x)$

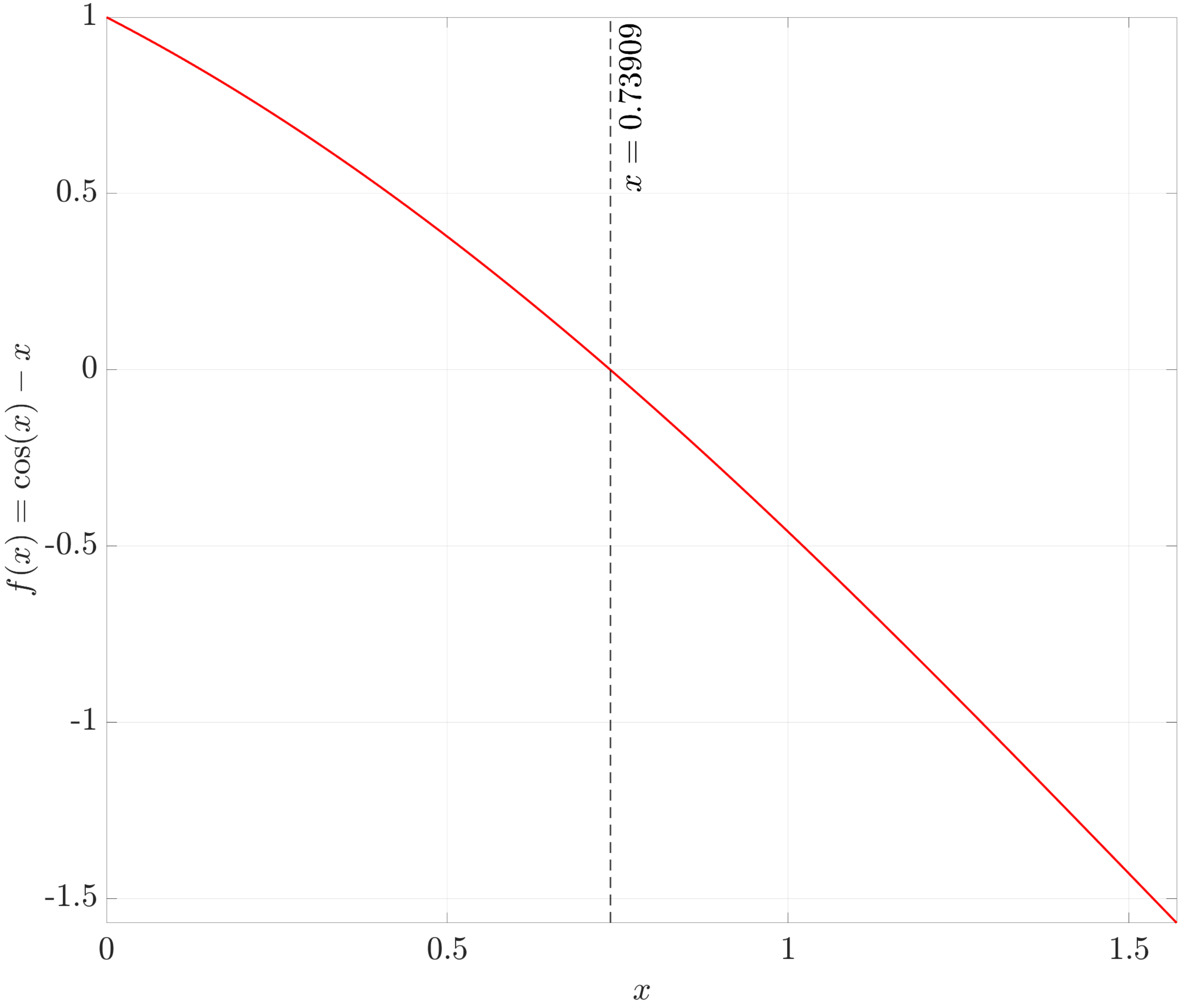
Then iterate $x(t+1) = g(x(t))$ in a computer

Hope: **if the iterations converge, then it must converge to a root of $f(x) = 0$**

Worry: the iterations may diverge or settle to oscillation

Example: Fixed point recursion

Solve for $x \in \left(0, \frac{\pi}{2}\right)$ **such that** $\underbrace{\cos(x) - x}_{f(x)} = 0$



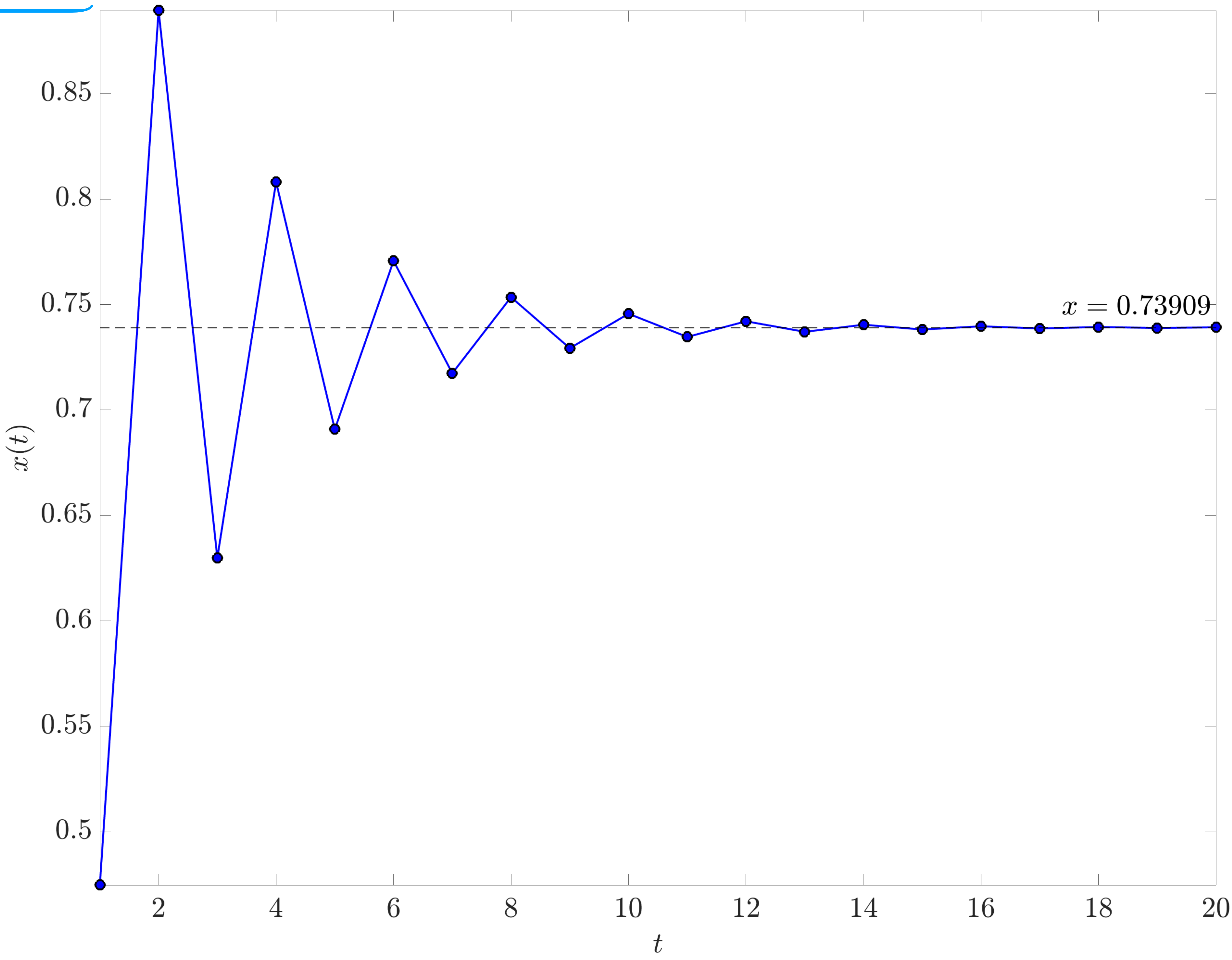
Example: Fixed point recursion

Solve for $x \in \left(0, \frac{\pi}{2}\right)$ **such that** $\underbrace{\cos(x) - x}_{f(x)} = 0$

Rewrite: $x = \underbrace{\cos(x)}_{g(x)}$

Algorithm: iterate $x(t+1) = \cos(x(t))$

Here t denotes iteration index



Example: Process dynamics in state space

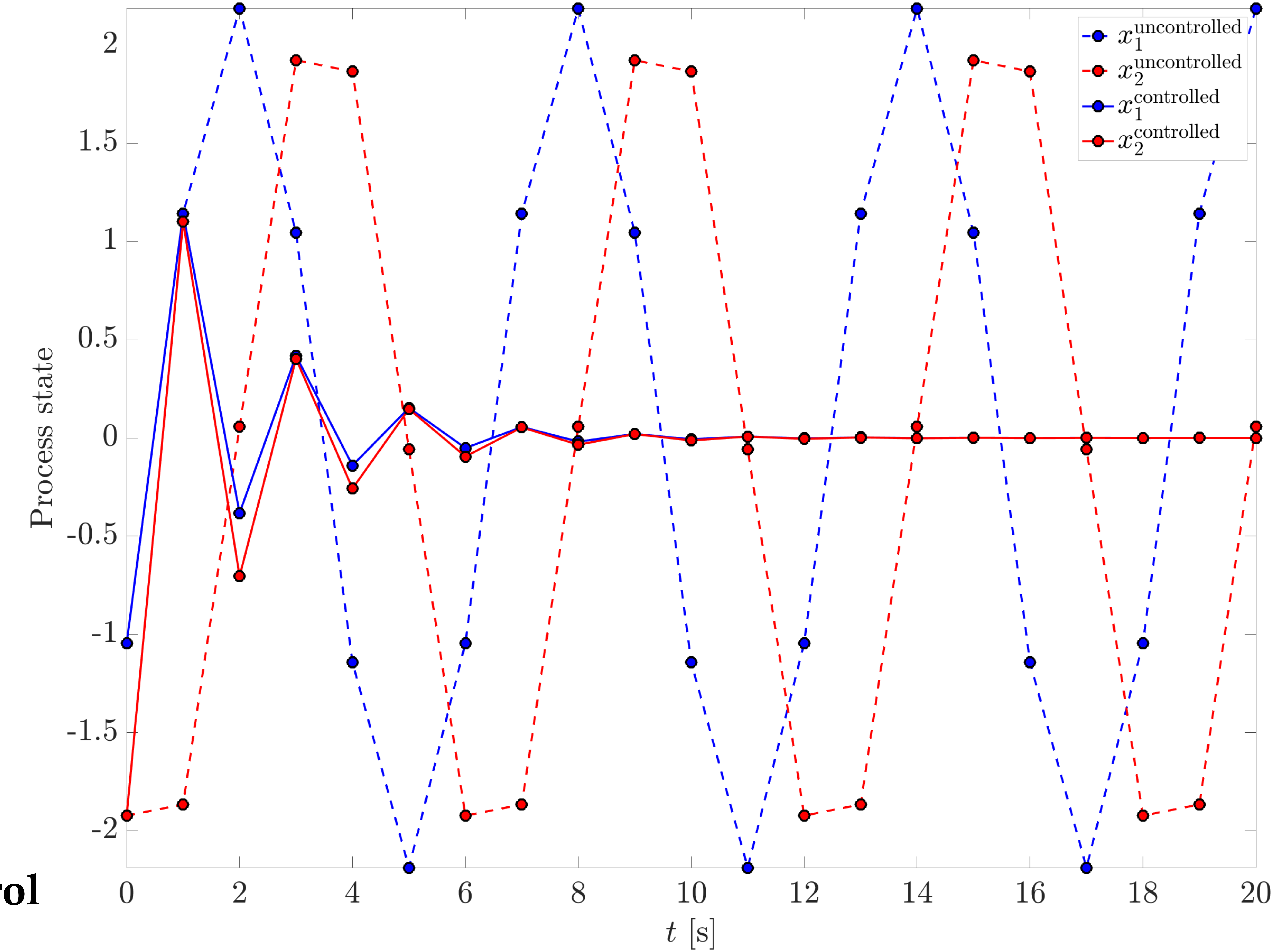
$$x_1(t+1) = \frac{1}{2}x_1(t) - \frac{\sqrt{3}}{2}x_2(t)$$

$$x_2(t+1) = \frac{\sqrt{3}}{2}x_1(t) + \frac{1}{2}x_2(t) + u$$

$u = 0 \rightsquigarrow$ Oscillation

$u = -x_1 - x_2 \rightsquigarrow$ GAS

Stabilizing state feedback control



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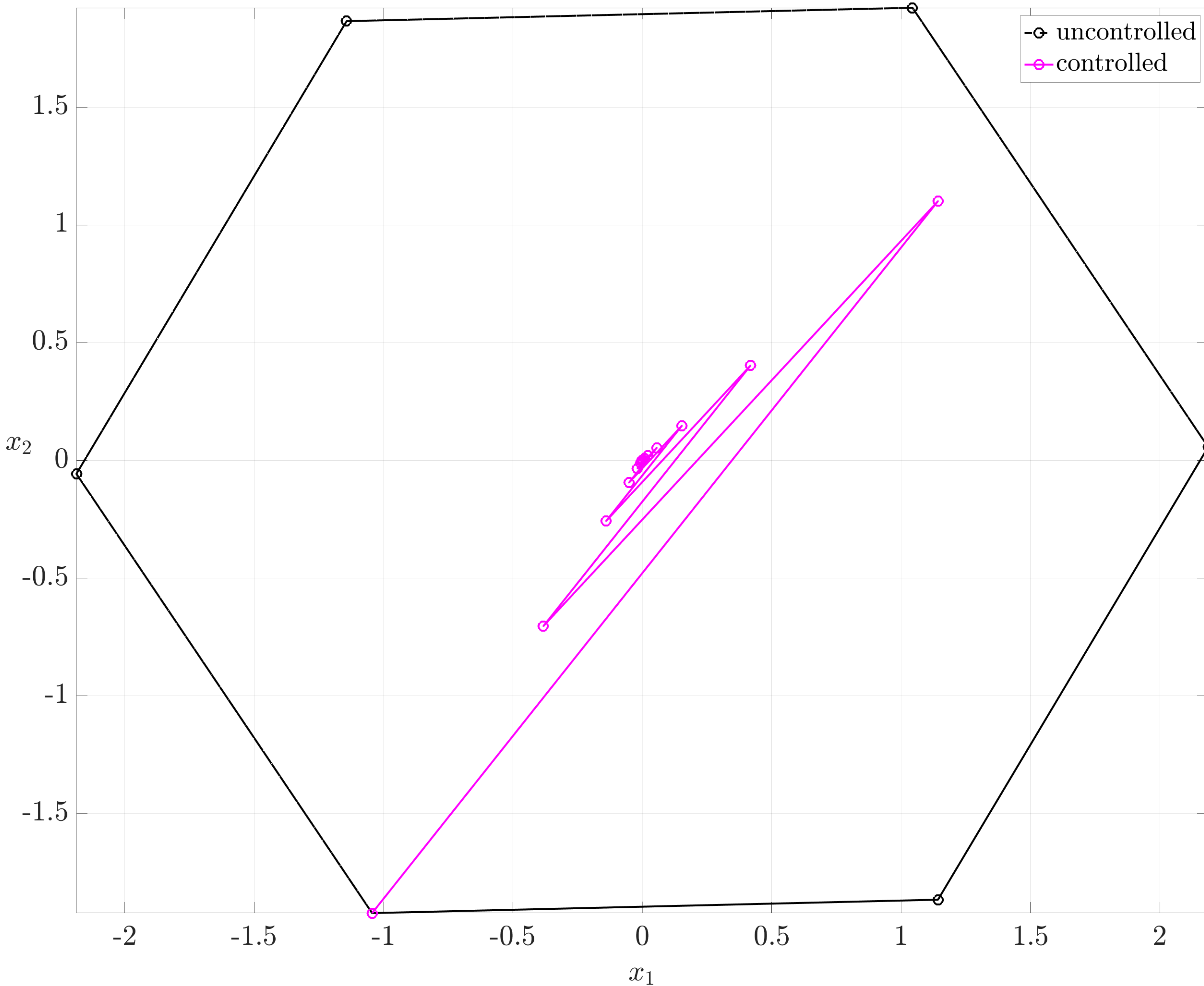


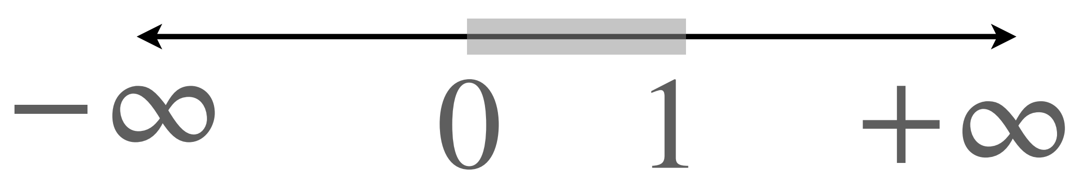
State Space

What does it mean: It is the space of process state variables

Mathematically: It is a **set** in which the collection of state variables belong to

Examples:

$x(t+1) = r\,x(t)\big(1 - x(t)\big), \quad 0 < r \leq 4,$ State space: [0,1]



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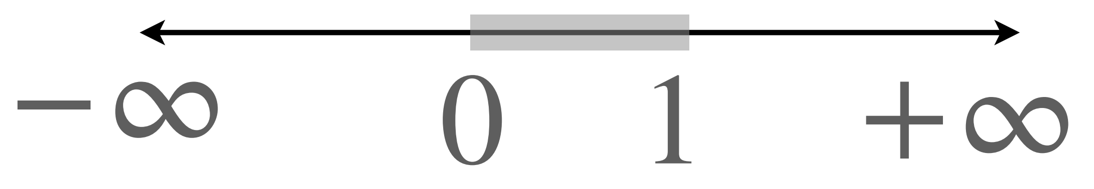
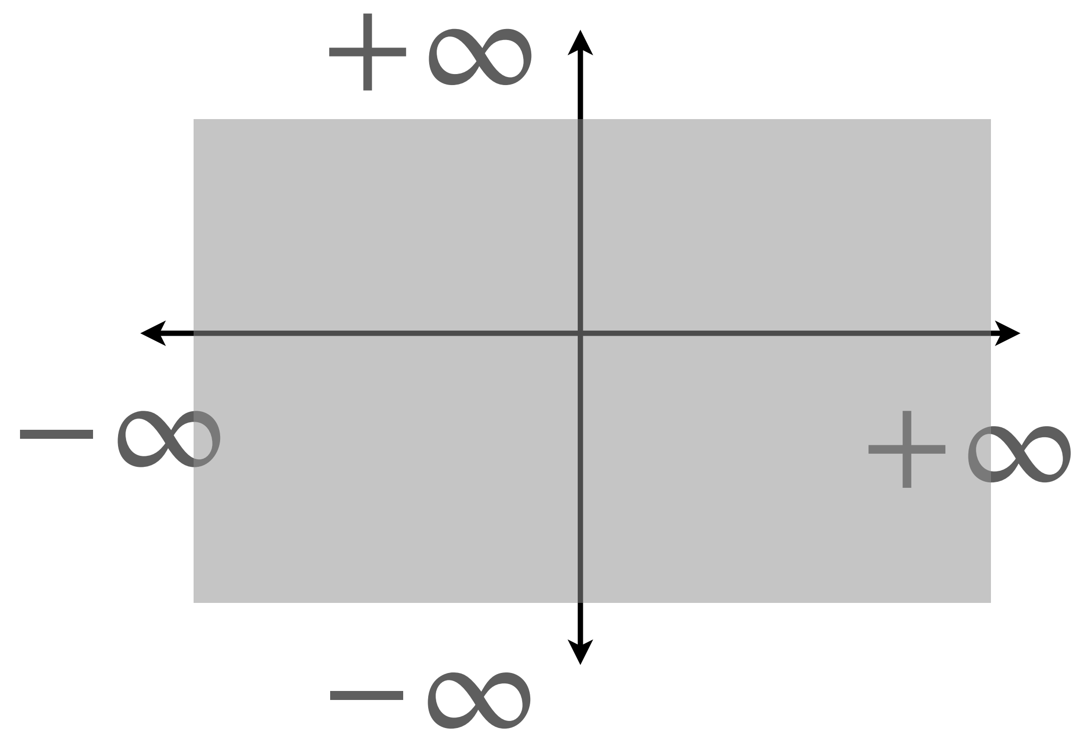


Line interval

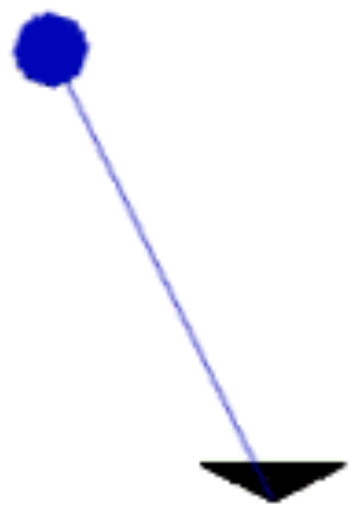
$$x_1(t+1) = \frac{1}{2}x_1(t) - \frac{\sqrt{3}}{2}x_2(t)$$
$$x_2(t+1) = \frac{\sqrt{3}}{2}x_1(t) + \frac{1}{2}x_2(t)$$

State space: $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$



Two dimensional coordinate plane

Example: state space for simple pendulum



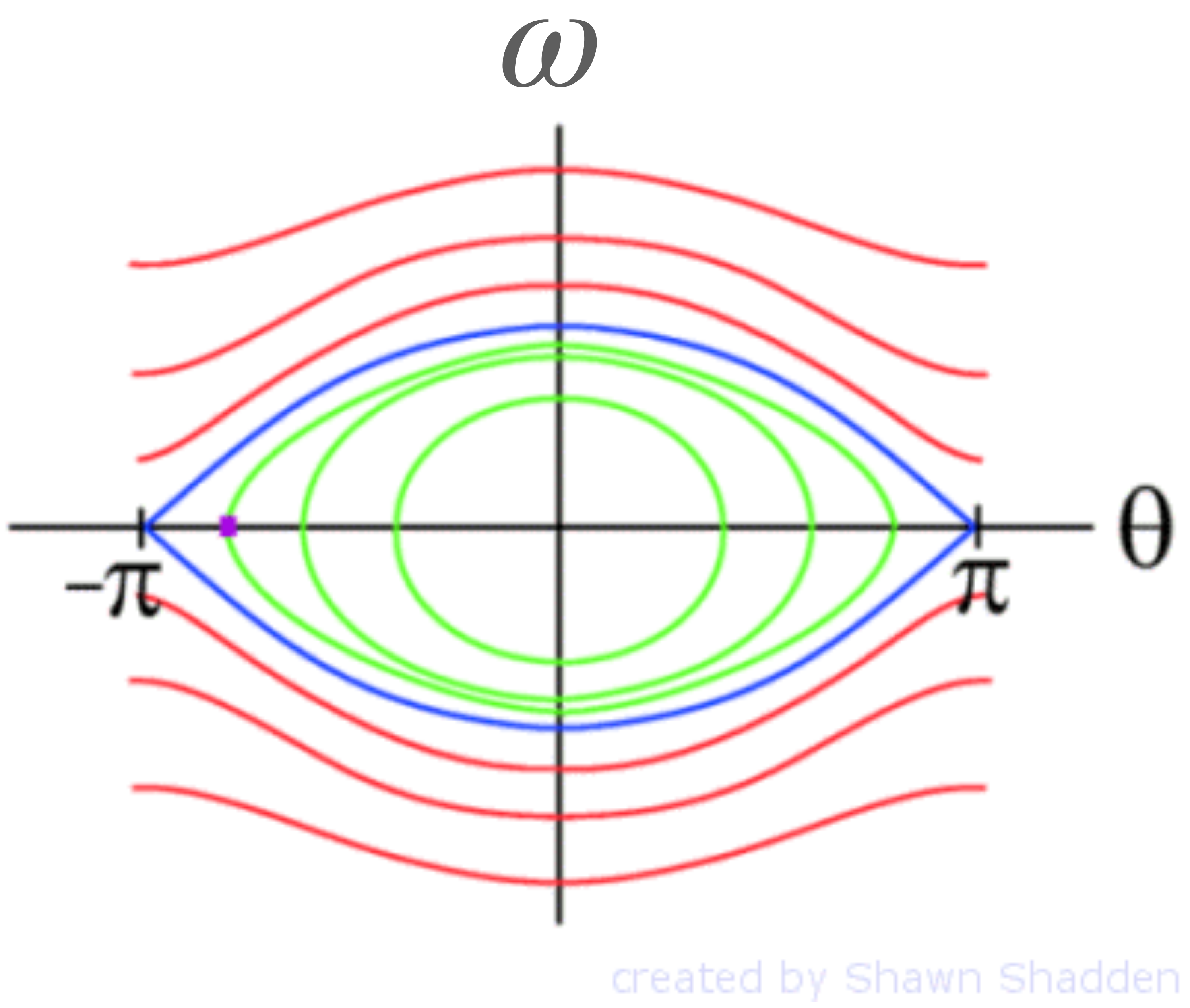


Image credit: Shawn Shadden

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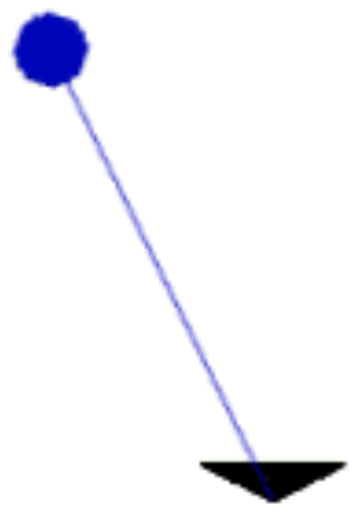

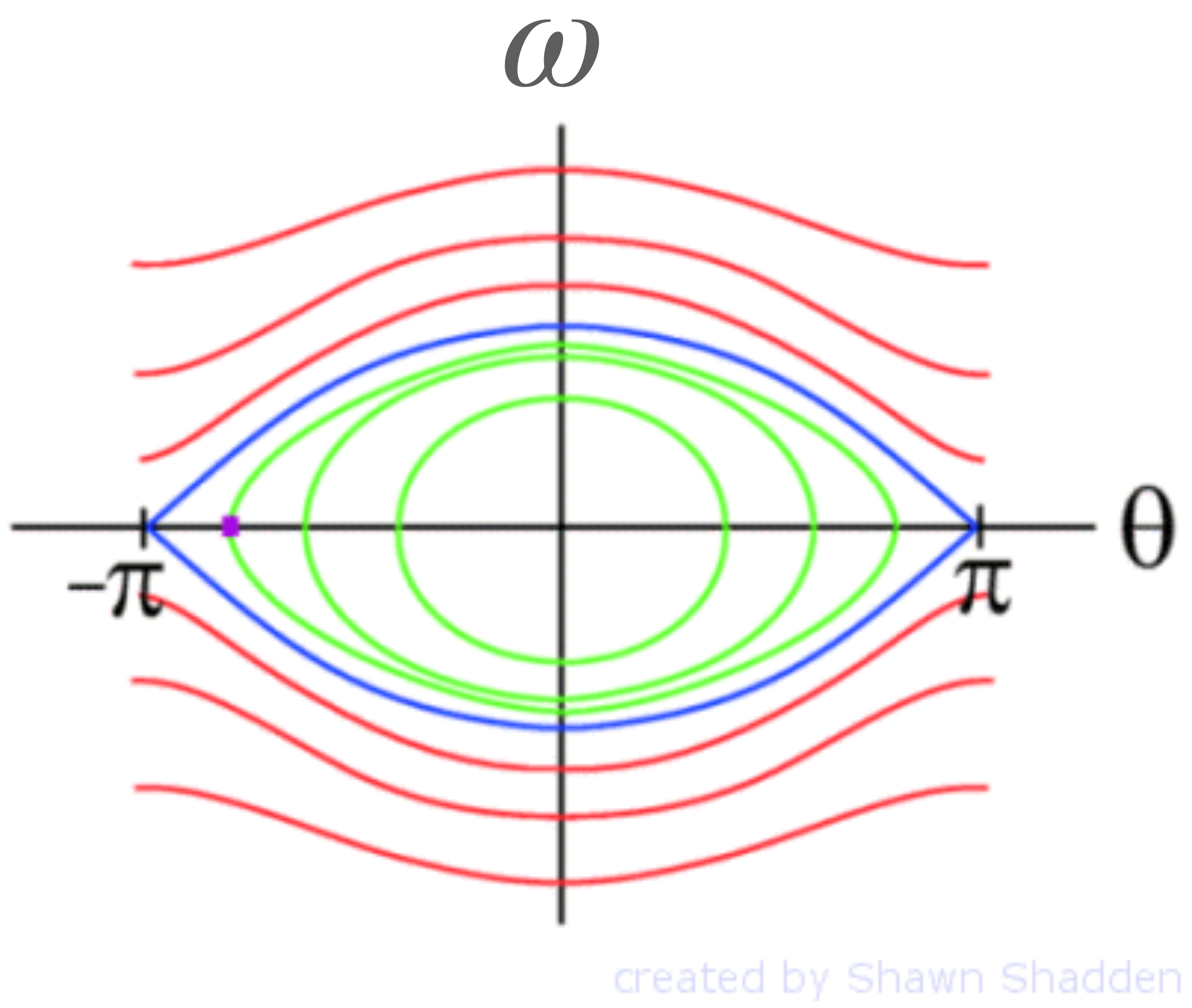

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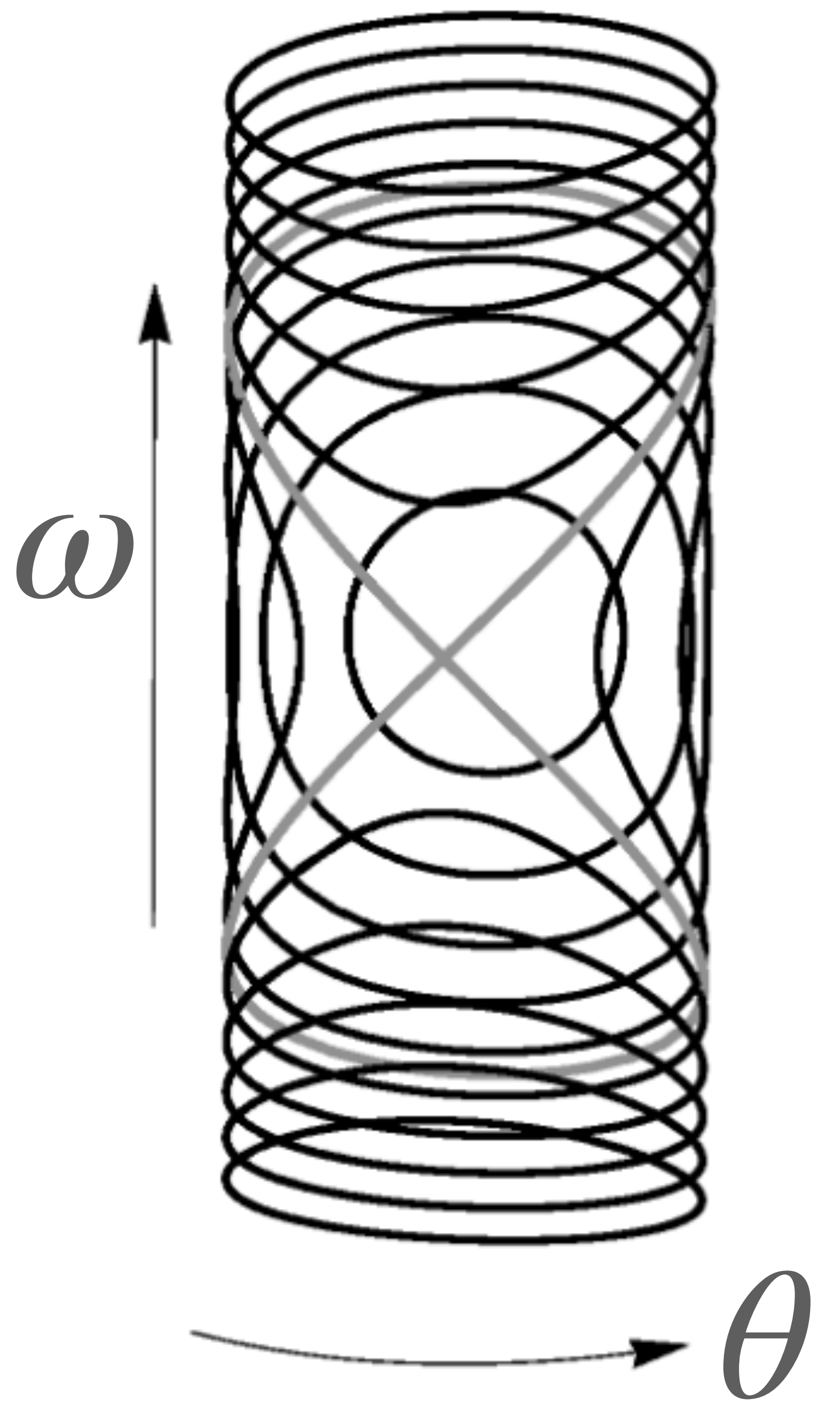
State space: $[-\pi, \pi] \times \mathbb{R}$

Cylinder



Example: state space for angular dynamics

Dynamics:

$$\theta_1(t+1) = \theta_1(t) - \frac{1}{3}\left[\sin(\theta_1(t) - \theta_2(t)) + \sin\left(2\theta_1(t) + \theta_2(t)\right)\right]$$

$$\theta_2(t+1) = \theta_2(t) - \frac{1}{3}\left[\sin(\theta_2(t) - \theta_1(t)) + \sin\left(2\theta_2(t) + \theta_1(t)\right)\right]$$

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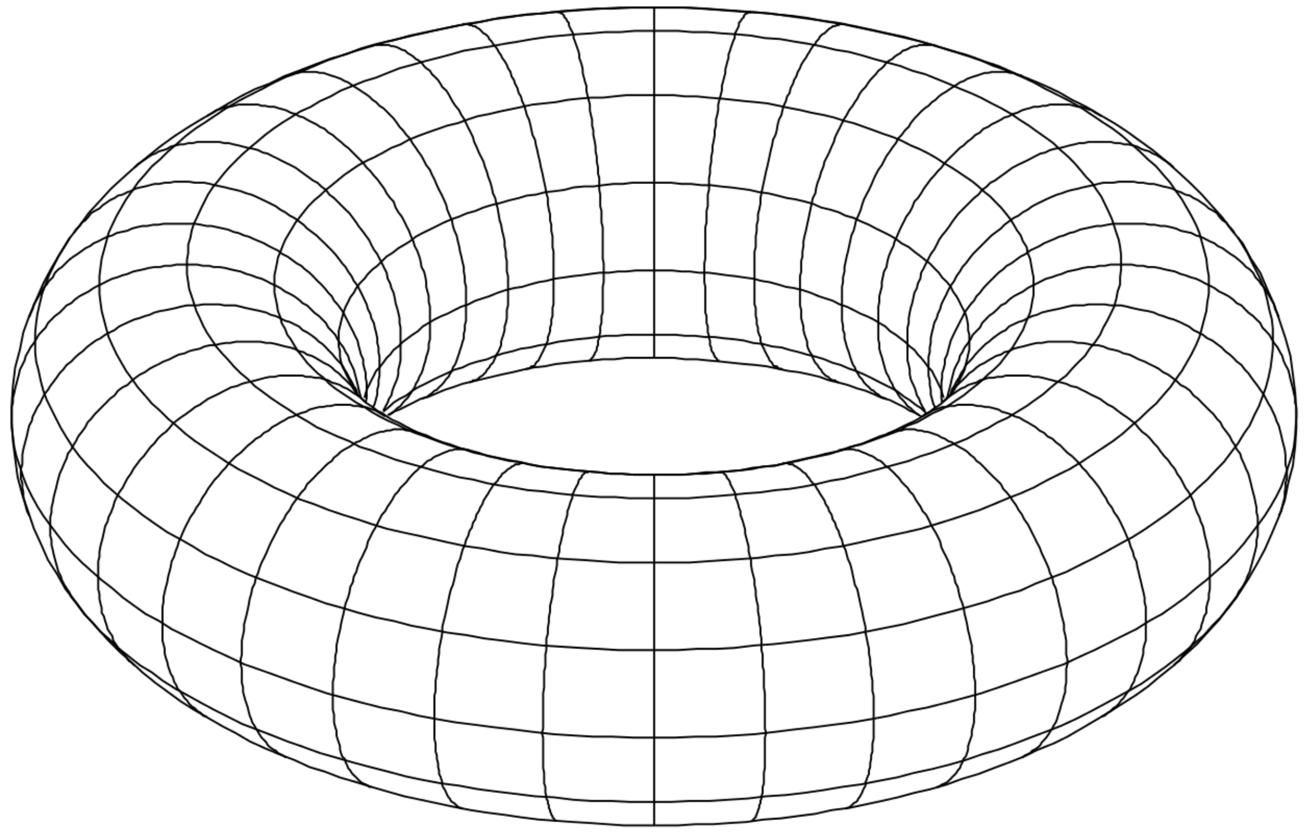
State space: $[-\pi, \pi] \times [-\pi, \pi]$

Torus



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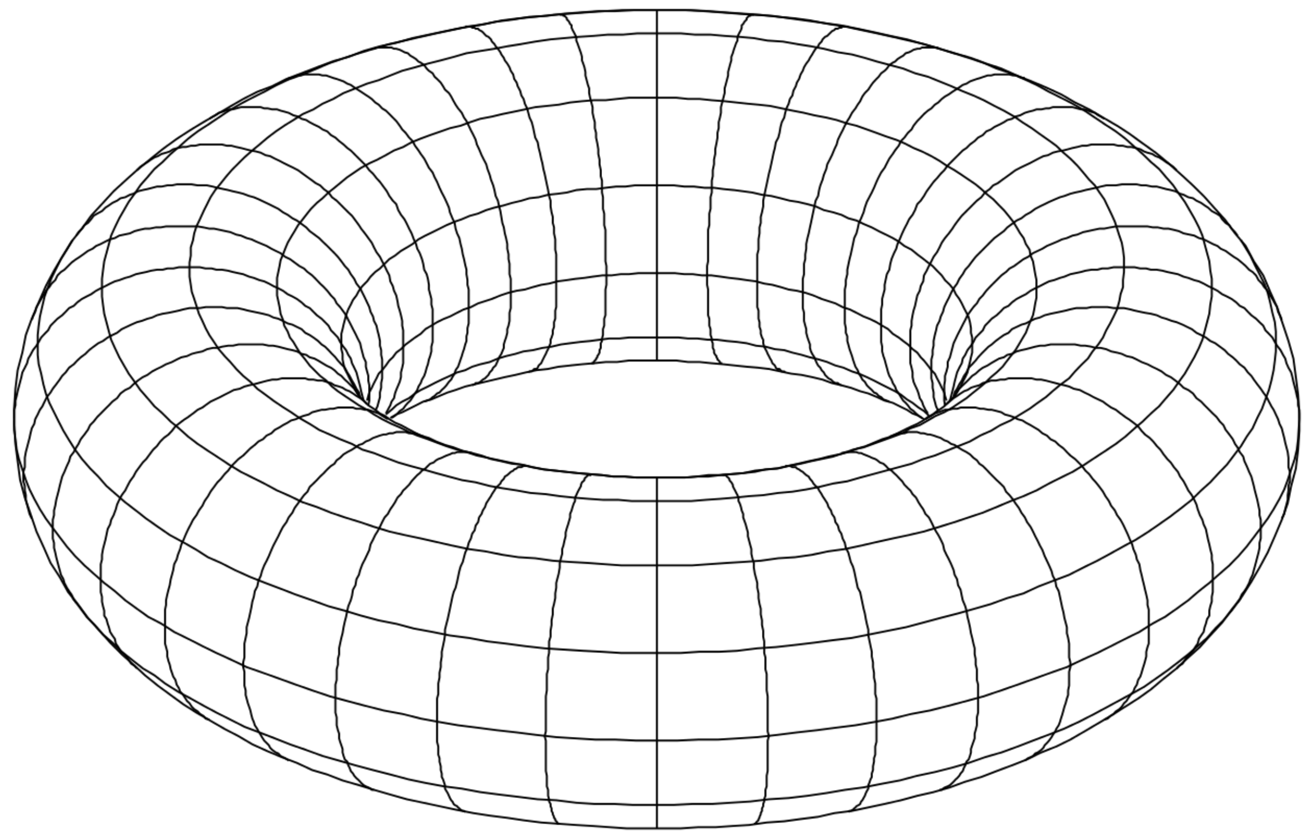

Fixed points: