Optimal Control Review

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Development

Optimization (OPT)

$$\downarrow$$

Calculus of Variations (CoV)

$$\downarrow$$

Optimal Control Problem (OCP)

Overview

$$\min_{\boldsymbol{x} \in \mathcal{S} \subseteq \mathbb{R}^n} \quad f(\boldsymbol{x})$$

$$\downarrow$$

$$\min_{f \in \mathcal{F}(\mathbb{R}^n) \subseteq C^1(\mathbb{R}^n)} \quad I(f) = \int_{\mathrm{dom}(f)} L\left(\boldsymbol{x}, f, \nabla f\right)\, \mathrm{d}\boldsymbol{x}$$

Overview

$$\min_{\boldsymbol{x}\in\mathcal{S}\subseteq\mathbb{R}^n} \quad f(\boldsymbol{x})$$

$$\downarrow$$

$$\min_{f\in\mathcal{F}(\mathbb{R}^n)\subseteq C^1(\mathbb{R}^n)} \quad I(f) = \int_{\mathrm{dom}(f)} L\left(\boldsymbol{x}, f, \nabla f\right) \, \mathrm{d}\boldsymbol{x}$$

$$\downarrow$$

$$\min_{\boldsymbol{u}(\cdot)\in\mathcal{U}([0,T])\subseteq\mathcal{F}([0,T])} \quad J(\boldsymbol{u})$$

$$\text{subject to} \quad \dot{\boldsymbol{z}}(t) = \boldsymbol{\phi}\left(\boldsymbol{z}(t), \boldsymbol{u}(t), t\right)$$

OPT example: Least squares

OPT template: $$\min_{\boldsymbol{x} \in \mathcal{S} \subseteq \mathbb{R}^n} \quad f(\boldsymbol{x})$$

In this problem: $$\min_{\boldsymbol{x}} \quad \| A\boldsymbol{x} - \boldsymbol{b} \|_2^2$$

OPT example: Least squares

$$\textbf{OPT template:} \quad \min_{\boldsymbol{x} \in \mathcal{S} \subseteq \mathbb{R}^n} \quad f(\boldsymbol{x})$$

$$\textbf{In this problem:} \min_{\boldsymbol{x}} \quad \| A\boldsymbol{x} - \boldsymbol{b} \|_2^2$$

$$\mathcal{S} = \mathbb{R}^n, \qquad f(\boldsymbol{x}) = \| A\boldsymbol{x} - \boldsymbol{b} \|_2^2$$

OPT example: two variable LP

OPT template: $$\min_{x \in \mathcal{S} \subseteq \mathbb{R}^n} \quad f(x)$$

In this problem: $$\max_{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2} \quad 15x_1 + 10x_2$$

$$\text{subject to} \quad \tfrac{1}{4}x_1 + x_2 \leq 65,$$

$$\tfrac{5}{4}x_1 + \tfrac{1}{2}x_2 \leq 90,$$

$$x_1, x_2 \geq 0$$

OPT example: two variable LP

OPT template: $$\min_{\boldsymbol{x}\in\mathcal{S}\subseteq\mathbb{R}^n} \quad f(\boldsymbol{x})$$

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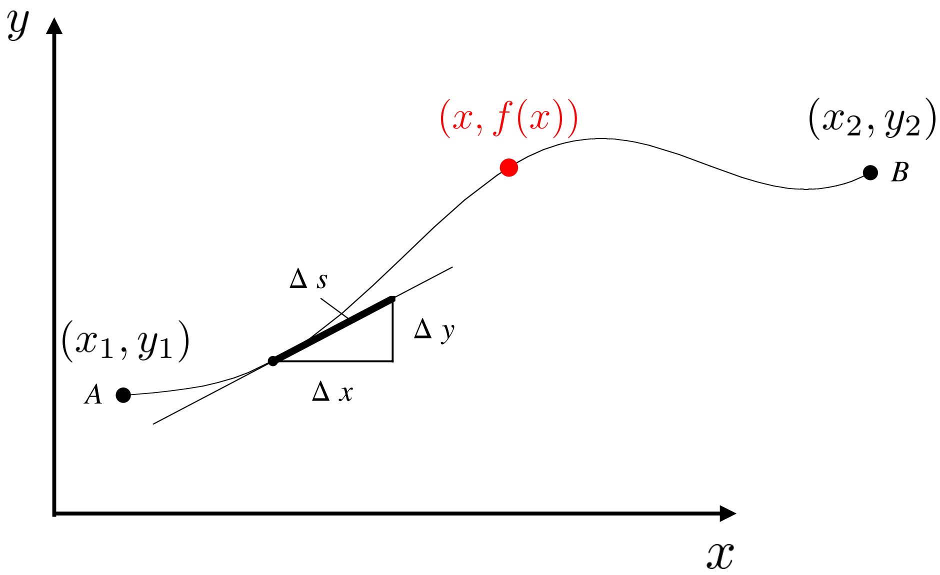
$$\frac{5}{4}x_1 + \frac{1}{2}x_2 \leq 90,$$

$$x_1, x_2 \geq 0$$

$$\mathcal{S} = \{\boldsymbol{x}\in\mathbb{R}^2 \,:\, A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq 0\} \subset \mathbb{R}^2$$

CoV example: Shortest planar path



CoV example: Shortest planar path

CoV template:

$$\min_{f \in \mathcal{F}(\mathbb{R}^n) \subseteq C^1(\mathbb{R}^n)} \quad I(f) = \int_{\mathrm{dom}(f)} L\left(\boldsymbol{x}, f, \nabla f\right) \, \mathrm{d}\boldsymbol{x}$$

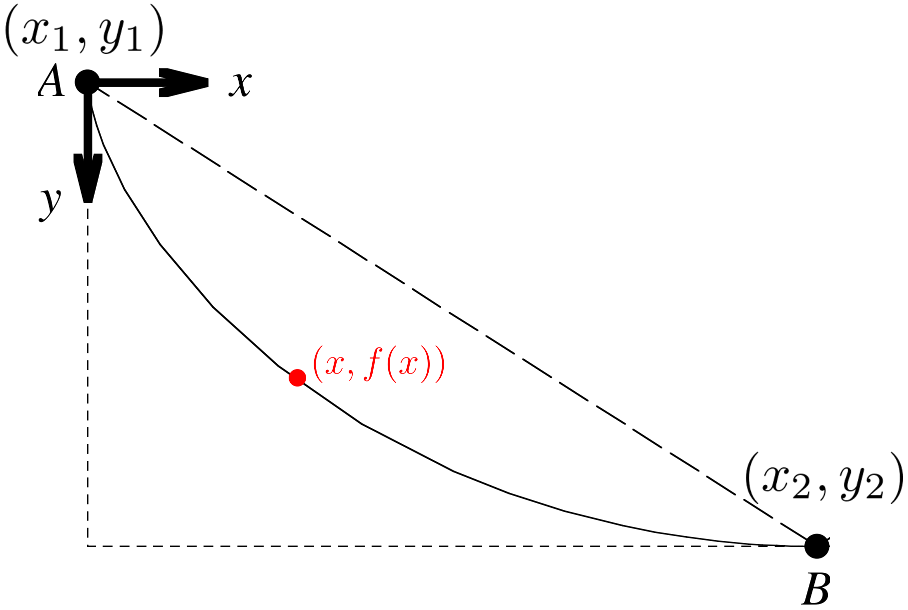
In this problem:

$$I(f) = \int_{x_1}^{x_2} \sqrt{1 + (f')^2} \, \mathrm{d}x$$

$$\mathrm{dom}(f) = [x_1, x_2], \text{ assuming } x_1 \neq x_2$$

$$\mathcal{F}(\mathbb{R}) = \{f \in C^1(\mathbb{R}) \;:\; f(x_1) = y_1, f(x_2) = y_2\}$$

CoV example: Brachistochrone (1696)



CoV example: Brachistochrone (1696)

CoV template:

$$\min_{f \in \mathcal{F}(\mathbb{R}^n) \subseteq C^1(\mathbb{R}^n)} \quad I(f) = \int_{\mathrm{dom}(f)} L\left(\boldsymbol{x}, f, \nabla f\right) \, \mathrm{d}\boldsymbol{x}$$

In this problem:

$$I(f) = \int_{x_1}^{x_2} \sqrt{\frac{1 + (f')^2}{f}} \, \mathrm{d}x$$

$$\mathrm{dom}(f) = [x_1, x_2], \, x_1 \neq x_2, \, y_1 > y_2$$

$$\mathcal{F}(\mathbb{R}) = \{f \in C^1(\mathbb{R}) \, : \, f(x_1) = y_1, f(x_2) = y_2\}$$

CoV theory: EL equation (1740-1760s)

Necessary conditions for $I(f)$ to achieve minimum:

for $f : \mathbb{R} \mapsto \mathbb{R}$: $$\frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\right) = 0,$$

for $f : \mathbb{R}^n \mapsto \mathbb{R}$: $$\frac{\partial L}{\partial f} - \nabla \cdot \frac{\partial L}{\partial \nabla f} = 0,$$

subject to B.C. $f(x_1) = y_1, f(x_2) = y_2$

CoV theory: Beltrami identity

A corollary of EL equation, in the special case $L(x, f, \nabla f)$ has no explicit dependence on x:

for $f : \mathbb{R} \mapsto \mathbb{R}$: $f' \dfrac{\partial L}{\partial f'} - L = \text{constant.}$

EL equation example: f'' need not exist

Problem:

$$\min_{f \in \mathcal{F}([-1,1]) \subseteq C^1([-1,1])} \quad I(f) = \int_{-1}^{+1} f^2 \left(2x - f'\right)^2 \,\mathrm{d}x,$$

$$\mathcal{F}([-1,1]) = \{f \in C^1([-1,1]) \,:\, f(-1) = 0, f(1) = 1\}$$

Solution: From EL equation, minimum $I^* = 0$ is achieved by

$$f^*(x) = \begin{cases} 0 & \text{for} \quad x \in [-1,0] \\ x^2 & \text{for} \quad x \in (0,1] \end{cases}$$

CoV theory: when $f^* \in C^2$

Hilbert's theorem:

If $\dfrac{\partial^2 L}{\partial f'^2} \neq 0$ in the entire dom(f), then the extremal $f^*(\cdot) \in C^2$, and is called *nonsingular*.

Corollary:

If f^* nonsingular and $L \in C^3$, then f^* is the unique extremal.

CoV example: Shortest planar path

Solution:

Set $L = \sqrt{1+(f')^2}$ in EL equation:

$$\frac{\partial}{\partial f}\sqrt{1+(f')^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial}{\partial f'}\left(\sqrt{1+(f')^2}\right)\right] = 0$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{2f'}{2\sqrt{1+(f')^2}}\right] = \frac{f''}{[1+(f')^2]^{\frac{3}{2}}} = 0$$

$$\Rightarrow f(x) = c_1 x + c_2$$

where $c_1 = \dfrac{y_2 - y_1}{x_2 - x_1}$, $c_2 = \dfrac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$

CoV example: Brachistochrone

Solution:

Set $L = \sqrt{\frac{1+(f')^2}{f}}$ in Beltrami identity:

$$\frac{(f')^2}{\sqrt{1+(f')^2}\sqrt{f}} - \frac{\sqrt{1+(f')^2}}{\sqrt{f}} = c$$

$$\Rightarrow \frac{-1}{\sqrt{1+(f')^2}\sqrt{f}} = c$$

$$\Rightarrow f' = \sqrt{\frac{k-f}{f}}, \text{ where } k := \frac{1}{c^2}$$

How to solve this nonlinear ODE?

CoV example: Brachistochrone

Solution (contd.):

Let $f = k\sin^2\phi \Rightarrow \dfrac{\mathrm{d}f}{\mathrm{d}x} = \sqrt{\dfrac{k-f}{f}} = \cot\phi$

By chain rule:

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\mathrm{d}\phi}{\mathrm{d}f}\,\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{2k\sin\phi\cos\phi}\cot\phi = \frac{1}{2k\sin^2\phi}$$

$$\Rightarrow \mathrm{d}x = 2k\sin^2\phi\,\mathrm{d}\phi$$

$$\Rightarrow x = k\int (1-\cos 2\phi)\;\mathrm{d}\phi = k\phi - \frac{k}{2}\sin 2\phi + c_1$$

$$\Rightarrow (x, f(x)) = \left(k\phi - \tfrac{k}{2}\sin 2\phi + c_1,\ \tfrac{k}{2}(1-\cos 2\phi)\right)$$

CoV example: Brachistochrone

Solution (contd.):

Apply B.C. at point A $(0, 0)$: $c_1 = 0$

Introducing $a := \dfrac{k}{2}$ and $\theta := 2\phi$, we get

$x = a\,(\theta - \sin\theta)$, $y \equiv f(x) = a\,(1 - \cos\theta)$

These are parametric equations for a **cycloid**

CoV example: Brachistochrone

Solution (contd.):

Since $\dfrac{\partial^2 L}{\partial f'^2} = \dfrac{1}{\sqrt{f}\,\left[1 + (f')^2\right]^{\frac{3}{2}}} \neq 0$ in $\mathrm{dom}(f)$,

hence $f^* \in C^2$ (by Hilbert's Theorem)

CoV theory: Integral constraints

CoV template:

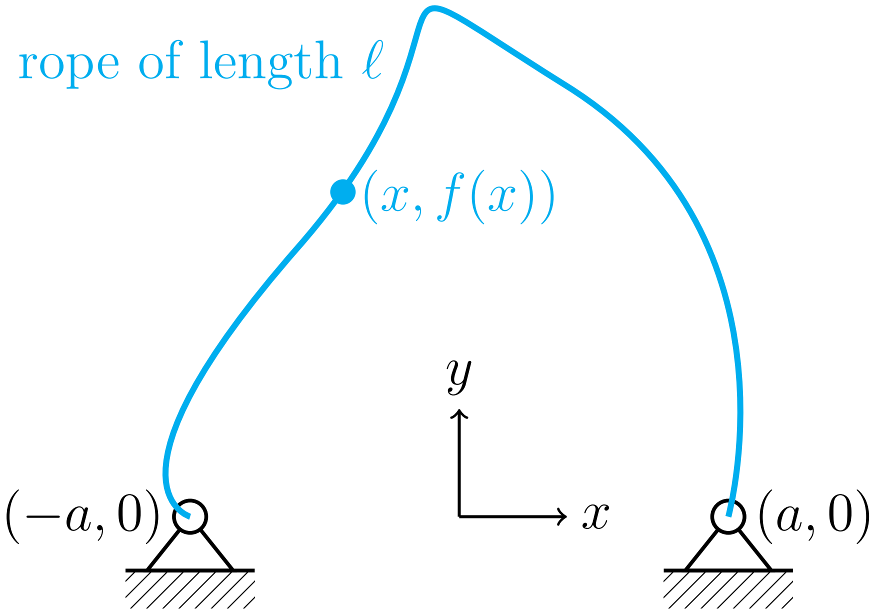
$$\min_{f \in \mathcal{F}(\mathbb{R}^n) \subseteq C^1(\mathbb{R}^n)} \quad I(f) = \int_{\mathrm{dom}(f)} L\left(\boldsymbol{x}, f, \nabla f\right)\, \mathrm{d}\boldsymbol{x}$$

$$\text{subject to } \int_{\mathrm{dom}(f)} \boldsymbol{M}\left(\boldsymbol{x}, f, \nabla f\right)\, \mathrm{d}\boldsymbol{x} = \boldsymbol{k}$$

Euler-Lagrange equation:

$$\frac{\partial}{\partial f}\left(L + \boldsymbol{\lambda}^\top \boldsymbol{M}\right) - \nabla \cdot \frac{\partial}{\partial \nabla f}\left(L + \boldsymbol{\lambda}^\top \boldsymbol{M}\right) = 0$$

CoV example: Isoperimetric problem



CoV example: Isoperimetric problem

CoV template:

$$\min_{f \in \mathcal{F}(\mathbb{R}^n) \subseteq C^1(\mathbb{R}^n)} \quad I(f) = \int_{\mathrm{dom}(f)} L\left(\boldsymbol{x}, f, \nabla f\right)\, \mathrm{d}\boldsymbol{x}$$

$$\text{subject to } \int_{\mathrm{dom}(f)} \boldsymbol{M}\left(\boldsymbol{x}, f, \nabla f\right)\, \mathrm{d}\boldsymbol{x} = \boldsymbol{k}$$

In this problem:

minimize $I(f) = \int_{-a}^{+a} f(x)\, \mathrm{d}x$, $0 < 2a < \ell$, subject to

$$\int_{-a}^{+a} \sqrt{1 + (f')^2}\, \mathrm{d}x = \ell \text{ (given)}, f(-a) = f(a) = 0$$

CoV example: Isoperimetric problem

Solution:

$$\text{EL equation: } \frac{\partial}{\partial f}\left(f + \lambda\sqrt{1+(f')^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial}{\partial f'}\left(f + \lambda\sqrt{1+(f')^2}\right)\right] = 0$$

$$\Rightarrow 1 + \frac{\lambda f''}{\left[1+(f')^2\right]^{\frac{3}{2}}} = 0$$

Set $f' = \tan\theta \Rightarrow f'' = \sec^2\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}x}$ (by chain rule)

EL equation becomes: $1 + \lambda\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}x} = 0$

$$\Rightarrow \mathrm{dx} = -\lambda\cos\theta\mathrm{d}\theta \Rightarrow x = -\lambda\sin\theta + c_1$$

CoV example: Isoperimetric problem

Solution (contd.):

On the other hand: $\mathrm{d}f = \tan\theta\,\mathrm{d}x = -\lambda\sin\theta\,\mathrm{d}\theta$

$$\Rightarrow y \equiv f(x) = \lambda\cos\theta + c_2$$

$$\Rightarrow (c_1 - x)^2 + (y - c_2)^2 = \lambda^2 \text{ (circular arc)}$$

To determine c_1, c_2 and λ, first use endpoint BCs:

$$f(-a) = 0 \Rightarrow (c_1 + a)^2 + c_2^2 = \lambda^2$$

$$f(+a) = 0 \Rightarrow (c_1 - a)^2 + c_2^2 = \lambda^2$$

These yield: $c_1 = 0, c_2 = \sqrt{\lambda^2 - a^2}$

CoV example: Isoperimetric problem

Solution (contd.):

Now use the integral constraint:

$$\ell = \int_{x=-a}^{x=+a} \sqrt{1+(f')^2}\, \mathrm{d}x$$

$$= \int_{\theta=\arcsin\left(\frac{a}{\lambda}\right)}^{\theta=-\arcsin\left(\frac{a}{\lambda}\right)} \sec\theta\ \left(-\lambda\cos\theta\right)\ \mathrm{d}\theta$$

$$= 2\lambda \arcsin\left(\frac{a}{\lambda}\right)$$

λ solves transcendental equation: $\sin\left(\frac{\ell}{2\lambda}\right) = \frac{a}{\lambda}$

Think: Our solution makes sense for

$$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Leftrightarrow \ell < \pi a$$

OCP (in continuous time)

OCP template for $\boldsymbol{x} : [0, T] \mapsto \mathbb{R}^n$, $\boldsymbol{u} : [0, T] \mapsto \mathbb{R}^m$

$$\min_{\boldsymbol{u}(\cdot)\in\mathcal{U}([0,T])} J(\boldsymbol{u}) := \underbrace{\phi\left(\boldsymbol{x}(T), T\right)}_{\text{terminal cost}} + \underbrace{\int_0^T \mathcal{L}\left(\boldsymbol{x}(t), \boldsymbol{u}(t), t\right) \mathrm{d}t}_{\text{cost-to-go}}$$

subject to

$$(1) \qquad \underbrace{\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\left(\boldsymbol{x}(t), \boldsymbol{u}(t), t\right)}_{\text{dynamics}}, \quad \underbrace{\boldsymbol{x}(0) = \boldsymbol{x}_0}_{\text{initial condition}} \quad \text{given}$$

$$(2) \qquad \underbrace{\boldsymbol{\psi}\left(\boldsymbol{x}(T), T\right) = 0}_{\text{terminal constraint}}$$

OCP theory: Necessary conditions

Hamiltonian $\mathcal{H}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\lambda}(t), t)$

$$:= \mathcal{L}(\boldsymbol{x}(t), \boldsymbol{u}(t), t) + \boldsymbol{\lambda}^\top(t)\boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t)$$

State equation: $\dot{x}(t) = \nabla_\lambda \mathcal{H} = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t)$

Costate equation: $\dot{\boldsymbol{\lambda}}(t) = -\nabla_x \mathcal{H}$

Pontryagin's Maximum Principle (PMP): $\mathbf{0} = \nabla_u \mathcal{H}$

Transversality conditon:

$$\left(\nabla_x \phi + (\nabla_x \boldsymbol{\psi})^\top \boldsymbol{\nu} - \boldsymbol{\lambda}\right)^\top \Big|_{t=T} \mathrm{d}\boldsymbol{x}(T) +$$

$$\left(\frac{\partial \phi}{\partial t} + \left(\frac{\partial \boldsymbol{\psi}}{\partial t}\right)^\top \boldsymbol{\nu} + \mathcal{H}\right)\Big|_{t=T} \mathrm{d}T = 0$$

OCP theory: Optimized Hamiltonian $\mathcal{H}^*$

By chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}\left(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\lambda},t\right)$$

$$=\frac{\partial\mathcal{H}}{\partial t}+\left(\nabla_{\boldsymbol{x}}\mathcal{H}\right)^{\top}\dot{\boldsymbol{x}}+\left(\nabla_{\boldsymbol{u}}\mathcal{H}\right)^{\top}\dot{\boldsymbol{u}}+\left(\dot{\boldsymbol{\lambda}}\right)^{\top}\boldsymbol{f}$$

$$=\frac{\partial\mathcal{H}}{\partial t}+\underbrace{\left(\nabla_{\boldsymbol{u}}\mathcal{H}\right)^{\top}}_{=0}\dot{\boldsymbol{u}}+\underbrace{\left(\nabla_{\boldsymbol{x}}\mathcal{H}+\dot{\boldsymbol{\lambda}}\right)^{\top}}_{=0}\boldsymbol{f}$$

$$=\frac{\partial\mathcal{H}}{\partial t}\Rightarrow\mathcal{H}^*\text{ is constant for time invariant OCP}$$

OCP example: Shortest planar path redux

In this problem: $\dot{x}(t) = u(t), \dot{y}(t) = v(t)$,

$$\mathcal{L} = \sqrt{1 + \frac{v^2}{u^2}}, \phi = 0$$

Hamiltonian $\mathcal{H} = \sqrt{1 + \frac{v^2}{u^2}} + \lambda_1 u + \lambda_2 v$

$\dot{\lambda}_1 = -\frac{\partial \mathcal{H}}{\partial x} = 0 \Rightarrow \lambda_1 = c_1$, $\dot{\lambda}_2 = -\frac{\partial \mathcal{H}}{\partial y} = 0 \Rightarrow \lambda_2 = c_2$

$0 = \frac{\partial \mathcal{H}}{\partial u} = \frac{\frac{v^2}{u^2}}{\sqrt{u^2+v^2}} + \lambda_1$, $0 = \frac{\partial \mathcal{H}}{\partial v} = \frac{\frac{v}{u}}{\sqrt{u^2+v^2}} + \lambda_2$

$\Rightarrow (u^*, v^*) = (k_1, k_2) \Rightarrow (x^*, y^*) = (k_1 t + \widetilde{k}_1, k_2 t + \widetilde{k}_2)$

$\Rightarrow y^* = \kappa_1 x^* + \kappa_2$, $\kappa_1 := \frac{k_2}{k_1}$, $\kappa_2 := \widetilde{k}_2 - \frac{k_2}{k_1}\widetilde{k}_1$

Use B.C. $y(x_1) = y_1, y(x_2) = y_2$ to find κ_1, κ_2

OCP example: Shortest planar path redux

$$\mathcal{H}^* = \sqrt{1 + \frac{k_2^2}{k_1^2}} + c_1 k_1 + c_2 k_2 = \text{constant}$$

OCP example: LQR with terminal cost

In this problem: $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$,

$$\mathcal{L} = \frac{1}{2}\left(\boldsymbol{x}^\top(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^\top(t)R\boldsymbol{u}(t)\right),$$

$\phi = \frac{1}{2}\boldsymbol{x}^\top(T)M\boldsymbol{x}(T)$, $\boldsymbol{\psi} \equiv \mathbf{0}$, where T is fixed

Here: $M, Q \in \mathbb{S}^n_+$, $R \in \mathbb{S}^m_{++}$, $(A, B) \in \left(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times m}\right)$

$$\mathcal{H} = \frac{1}{2}\left(\boldsymbol{x}^\top Q\boldsymbol{x} + \boldsymbol{u}^\top R\boldsymbol{u}\right) + \boldsymbol{\lambda}^\top\left(A\boldsymbol{x} + B\boldsymbol{u}\right)$$

$$\dot{\boldsymbol{\lambda}} = -\nabla_x\mathcal{H} = Q\boldsymbol{x} + A^\top\boldsymbol{\lambda}$$

$$\mathbf{0} = \nabla_u\mathcal{H} = R\boldsymbol{u} + B^\top\boldsymbol{\lambda} \Rightarrow \boldsymbol{u}(t) = -R^{-1}B^\top\boldsymbol{\lambda}(t)$$

Transversality: $\mathrm{d}T = 0$, $\mathrm{d}\boldsymbol{x}(T) \neq 0 \Rightarrow \boldsymbol{\lambda}(T) = M\boldsymbol{x}(T)$

OCP example: LQR with terminal cost

Two point boundary value problem (TPBVP):

$$\begin{pmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{\lambda}} \end{pmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{bmatrix}}_{\text{Hamiltonian matrix } H} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{pmatrix}$$

$$\boldsymbol{x}(0) = \boldsymbol{x}_{\mathbf{0}}, \ \boldsymbol{\lambda}(T) = M\boldsymbol{x}(T)$$

To solve TPBVP, consider ansatz: $\boldsymbol{\lambda}(t) = P(t)\boldsymbol{x}(t)$

We find: $\dot{\boldsymbol{\lambda}} = \dot{P}\boldsymbol{x} + P\dot{\boldsymbol{x}} = \dot{P}\boldsymbol{x} + P\left(A\boldsymbol{x} - BR^{-1}B^\top P\boldsymbol{x}\right)$

But LHS $= -Q\boldsymbol{x} - A^\top\boldsymbol{\lambda} = -Q\boldsymbol{x} - A^\top P\boldsymbol{x}$

This gives: $-\dot{P}\boldsymbol{x} = \left(A^\top P + PA - PBR^{-1}B^\top P + Q\right)\boldsymbol{x}$

OCP example: LQR with terminal cost

For this to hold for all $\boldsymbol{x}_0$, and hence for all $\boldsymbol{x}(t)$ where $t \in [0, T]$, we must have:

$$\underbrace{-\dot{P} = A^\top P(t) + P(t)A - P(t)BR^{-1}B^\top P(t) + Q}_{\text{Riccati matrix differential equation in unknown } P(t)}$$

B.C.: $\boldsymbol{\lambda}(T) = P(T)\boldsymbol{x}(T) = M\boldsymbol{x}(T) \Rightarrow P(T) = M$

Back integrate Riccati $\rightarrow P(t) \rightarrow \boldsymbol{u}^*(t) = -K(t)\boldsymbol{x}(t)$

where $\underbrace{K(t) = R^{-1}B^\top P(t)}_{\text{Kalman gain}}$

Forward integrate $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}^*(t), \boldsymbol{x}(0) = \boldsymbol{x}_0$ to get $\boldsymbol{x}^*(t)$

OCP example: LQR with terminal cost

Optimal costate trajectories: $\boldsymbol{\lambda}^*(t) = P(t)\boldsymbol{x}^*(t)$

Closed-loop system: $\dot{\boldsymbol{x}}(t) = (A - BK(t))\,\boldsymbol{x}(t)$

Sufficiency: $\nabla_{\boldsymbol{u}} \circ \nabla_{\boldsymbol{u}} J_{\text{LQR}} = R \succ 0$

Same derivation goes through for LTV dynamics $(A(t), B(t))$

OCP example: LQR with terminal cost

Solving *quadratic* Riccati matrix ODE via *linear* Hamiltonian matrix ODE (a.k.a. Bernoulli substituition):

Intuition: $\boldsymbol{\lambda}(t) = P(t)\boldsymbol{x}(t)$ suggests that $P(t) = \boldsymbol{\lambda}(t)\left(\boldsymbol{x}(t)\right)^{-1}$ (nonsense unless $n = 1$)

Now consider linear Hamiltonian ODE in matrix (not vector) variables $X(t), \Lambda(t) \in \mathbb{R}^{n\times n}$

$$\begin{pmatrix} \dot{X} \\ \dot{\Lambda} \end{pmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{bmatrix}}_{\text{Hamiltonian matrix } H} \begin{pmatrix} X \\ \Lambda \end{pmatrix}$$

with final conditions $X(T) = I_n$, $\Lambda(T) = M$

OCP example: LQR with terminal cost

Theorem: $P(t) = \Lambda(t)\,(X(t))^{-1}$

Proof: Let $\Psi(t) := \Lambda(t)\,(X(t))^{-1}$. We will show that $\Psi(t) \equiv P(t)$.

$$\dot{\Psi} = \dot{\Lambda}X^{-1} - \Lambda X^{-1}\dot{X}X^{-1}$$

$$= \left(-QX - A^\top\Lambda\right)X^{-1} - \Lambda X^{-1}\left(AX - BR^{-1}B^\top\right)X^{-1}$$

$$= -Q - A^\top\Psi - \Psi A + \Psi BR^{-1}B^\top\Psi$$

with $\Psi(T) = \Lambda(T)(X(T))^{-1} = M\,I_n^{-1} = M$

This is the Riccati ODE we derived for $P(t)$ ■

OCP example: LQR with terminal cost

Think: From the Hamiltonian matrix ODE, $X(t)$ is nonsingular (invertible)

For **LTI case**, solution of Hamiltonian matrix ODE:

$$\begin{pmatrix} X(t) \\ \Lambda(t) \end{pmatrix} = \underbrace{e^{H(t-T)}}_{=:\Theta(t)} \begin{pmatrix} I_n \\ M \end{pmatrix} = \underbrace{\begin{bmatrix} \Theta_{11}(t) & \Theta_{12}(t) \\ \Theta_{21}(t) & \Theta_{22}(t) \end{bmatrix}}_{\text{four } n \times n \text{ blocks}} \begin{pmatrix} I_n \\ M \end{pmatrix}$$

$$\therefore P(t) = (\underbrace{\Theta_{21}(t) + \Theta_{22}(t)M}_{\Lambda(t)})(\underbrace{\Theta_{11}(t) + \Theta_{12}(t)M}_{X(t)})^{-1}$$

OCP example: LQR with terminal cost

Davison-Maki Algorithm *[Davison and Maki, TAC 1973]*

Motivation: Direct Runge-Kutta on Riccati matrix ODE may be slow and numerically unstable, depending on the problem data

Idea: Avoid direct numerical integration by taking advantage of the linear Hamiltonian matrix ODE solution. For LTI, matrix exponential evaluation can be fast

OCP example: LQR with terminal cost

Davison-Maki Algorithm *[Davison and Maki, TAC 1973]*

Let $\Theta^{(1)} := \Theta(1\Delta t) = e^{H(\Delta t - T)}$, where Δt is step-size

Then recursively $\Theta^{(k+1)} = \Theta^{(k)}\Theta^{(1)}$

Computational cost = startup cost to evaluate $2n \times 2n$ matrix exponential $\Theta^{(1)}$ + cost of multiplying two $n \times n$ matrices to form $\Theta^{(k+1)}$ + cost for evaluating
$P^{(k+1)} = (\Theta_{21}^{(k+1)} + \Theta_{22}^{(k+1)} M)(\Theta_{11}^{(k+1)} + \Theta_{12}^{(k+1)} M)^{-1}$

Issue: may still be numerically unstable for large t since inversion may cause ill-conditioning

OCP example: LQR with terminal cost

Modified Davison-Maki Algorithm

[Kenney and Leipnik, TAC 1985]

To keep t small, use Bernoulli substitution in each interval $[k\Delta t, (k+1)\Delta t]$ resetting B.C.:

i.e., solve $\begin{pmatrix} \dot{X} \\ \dot{\Lambda} \end{pmatrix} = H \begin{pmatrix} X \\ \Lambda \end{pmatrix}$, $\begin{pmatrix} X(k\Delta t) \\ \Lambda(k\Delta t) \end{pmatrix} = \begin{pmatrix} X_k \\ \Lambda_k \end{pmatrix}$

This yields recursion

$$P^{(k+1)} = \left(\Theta_{21}(\Delta t) + \Theta_{22}(\Delta t)P^{(k)}\right)\left(\Theta_{11}(\Delta t) + \Theta_{12}(\Delta t)P^{(k)}\right)^{-1}$$

Also works for LTV by taking Θ as STM

OCP example: LQR with cross-weights

More general Lagrangian:

$$\mathcal{L} = \frac{1}{2} \begin{pmatrix} \boldsymbol{x}(t) & \boldsymbol{u}(t) \end{pmatrix}^\top \Pi \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{u}(t) \end{pmatrix}$$

Popov matrix: $\Pi := \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \in \mathbb{S}_+^{(m+n)}$,

where cross-weight matrix $S \in \mathbb{R}^{n \times m}$

Then $K(t) = R^{-1}B^\top P(t) + S^\top$, Riccati ODE:

$$-\dot{P} = A^\top P(t) + P(t)A - (P(t)B + S)R^{-1}(P(t)B + S)^\top + Q$$

and $H = \begin{bmatrix} A - BR^{-1}S^\top & -BR^{-1}B^\top \\ -Q + SR^{-1}S^\top & -A^\top + SR^{-1}B^\top \end{bmatrix}$

OCP example: Finite Horizon LQR with Terminal Cost for Tracking

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),\ \ \boldsymbol{y}(t) = C\boldsymbol{x}(t),\ \ t \in [0, T]$$

Reference/desired trajectory to track: $\boldsymbol{y}_d(t)$

$$J = \frac{1}{2}(\underbrace{\boldsymbol{y}(T)}_{C\boldsymbol{x}(T)} - \boldsymbol{y}_d(T))^\top M(\underbrace{\boldsymbol{y}(T)}_{C\boldsymbol{x}(T)} - \boldsymbol{y}_d(T)) + \int_0^T [(\underbrace{\boldsymbol{y}(t)}_{C\boldsymbol{x}(t)} - \boldsymbol{y}_d(t))^\top Q(\underbrace{\boldsymbol{y}(t)}_{C\boldsymbol{x}(t)} - \boldsymbol{y}_d(t)) + \boldsymbol{u}(t)^\top R\boldsymbol{u}(t)]\mathrm{d}t$$

Optimal control:

$$\boldsymbol{u}^*(\boldsymbol{x}(t), t) = \boldsymbol{u}^*_{\text{feedback}}(\boldsymbol{x}(t)) + \boldsymbol{u}^*_{\text{feedforward}}(t)$$

OCP example: Finite Horizon LQR with Terminal Cost for Tracking (contd.)

$$\boldsymbol{u}^*_{\text{feedback}}(\boldsymbol{x}(t)) = -K(t)\boldsymbol{x}(t),\ \ K(t) = R^{-1}B^\top P(t)$$

Riccati ODE:

$$-\dot{P}(t) = A^\top P(t) + P(t)A - P(t)BR^{-1}B^\top P(t) + C^\top QC$$

terminal condition: $P(T) = C^\top MC$

$$\boldsymbol{u}^*_{\text{feedforward}}(t) = R^{-1}B^\top \boldsymbol{v}(t)$$

Feedforward ODE:

$$-\dot{\boldsymbol{v}}(t) = (A - BK(t))^\top \boldsymbol{v}(t) + C^\top Q\,\boldsymbol{y}_d(t)$$

terminal condition: $\boldsymbol{v}(T) = C^\top M\,\boldsymbol{y}_d(T)$