S26 AerE 5740: Optimal Control

Practice Set 02: Equation-of-motion for Pendulum with Moving Pivot

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Problem 1

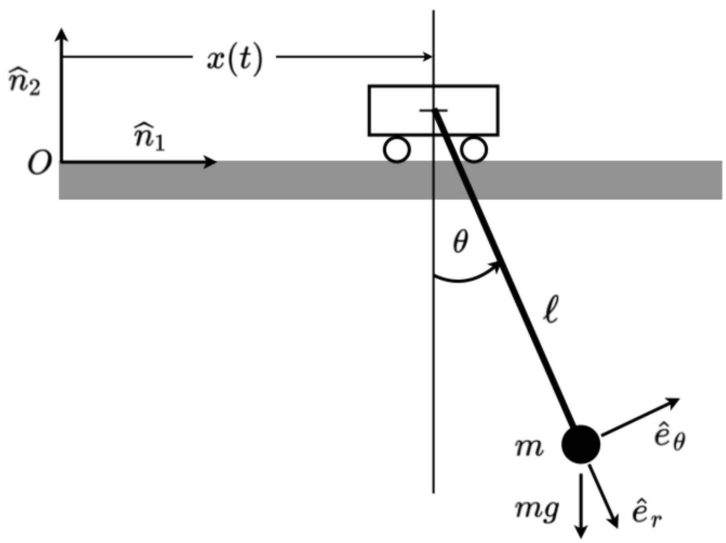

Figure 1: Pendulum with moving pivot.

Consider a pendulum with moving pivot as shown in Fig. 1 where the pivot motion is a sinusoidal function of time, i.e.,

$$x(t) = A \sin \Omega t,$$

for some given constants A, Ω. The mass of the bob is m, and the massless rod is of length ℓ, as shown. Neglect friction.

(a) Equation of motion.

Use Euler-Lagrange equation to derive the equation of motion for the pendulum.

Solution for part (a):

Since the position vector for the pendulum is $r(t) = (A \sin \Omega t)\widehat{n}_1 + \ell\widehat{e}_r$, we have

$$\dot{r}(t) = (A\Omega \cos \Omega t)\widehat{n}_1 + \frac{\mathrm{d}}{\mathrm{d}t}(\ell\widehat{e}_r) = (A\Omega \cos \Omega t)\widehat{n}_1 + \frac{\mathrm{d}\ell}{\mathrm{d}t}\widehat{e}_r + \dot{\theta}\widehat{e}_z \times \ell\widehat{e}_r = (A\Omega \cos \Omega t)\widehat{n}_1 + 0 + \ell\dot{\theta}\widehat{e}_\theta.$$

Therefore, the kinetic energy T for the pendulum is

$$T = \frac{1}{2}m\langle \dot{r}, \dot{r}\rangle = \frac{1}{2}m\left[(A\Omega\cos\Omega t)^2 + \left(\ell\dot{\theta}\right)^2 + 2\langle A\Omega\cos(\Omega t)\widehat{n}_1, \ell\dot{\theta}\widehat{e}_\theta\rangle\right]$$
$$= \frac{1}{2}m\left[(A\Omega\cos\Omega t)^2 + \left(\ell\dot{\theta}\right)^2 + 2A\ell\dot{\theta}\Omega\cos(\Omega t)\cos\theta\right].$$

The potential energy for the pendulum is $V = mg\ell\,(1-\cos\theta)$. So, the Lagrangian L is

$$L = T - V = \frac{1}{2}m\left[(A\Omega\cos\Omega t)^2 + \left(\ell\dot{\theta}\right)^2 + 2A\ell\dot{\theta}\Omega\cos(\Omega t)\cos\theta\right] - mg\ell\,(1-\cos\theta)\,.$$

Thus, the EL equation

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = m\ell^2\ddot{\theta} + mg\ell\sin\theta - mA\Omega^2\ell\cos\theta\sin(\Omega t)\,,$$

gives the equation of motion for the pendulum as

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = \left(\frac{A\Omega^2}{\ell}\right)\cos\theta\sin\Omega t.$$

(b) Conserved quantities.

(b.1) Compute the total energy E, and the Hamiltonian H. Is $H = E$ in this case?

(b.2) Use your answer in part (b.1) to conclude if any of the two quantities among H and E is a constant of motion.

Solution for part (b):

(b.1) The total energy is

$$E = T + V = \frac{1}{2}m\left[(A\Omega\cos\Omega t)^2 + \left(\ell\dot{\theta}\right)^2 + 2A\ell\dot{\theta}\Omega\cos(\Omega t)\cos\theta\right] + mg\ell\,(1-\cos\theta)\,.$$

The Hamiltonian is

$$H = \frac{\partial L}{\partial\dot{\theta}}\dot{\theta} - L = -\frac{1}{2}mA^2\Omega^2\cos^2(\Omega t) + mg\ell(1-\cos\theta) + \frac{1}{2}m\ell^2\dot{\theta}^2.$$

<u>Comparing the above two expressions, $H \neq E$ in this case.</u>

(b.2) By direct differentiation, we get

$$\frac{\mathrm{d}E}{\mathrm{dt}} = -mA\Omega\cos(\Omega t)\left(A\Omega\sin(\Omega t) + \ell\dot{\theta}\sin\theta\right),$$

where we have used the equation of motion in part (a) to substitute for $\ddot{\theta}$. From above, we conclude that $\frac{\mathrm{d}E}{\mathrm{d}t} \neq 0$, i.e., E is not constant. In retrospect, this is intuitive since the constraint force needed to ensure $x(t) = A\sin\Omega t$ does work on the system.

Similarly, we have

$$\frac{\mathrm{d}H}{\mathrm{d}t} = mA\Omega^2 \sin(\Omega t) \left(A\Omega \cos(\Omega t) + \ell\dot{\theta} \cos\theta \right),$$

where again, we have used the equation of motion in part (a) to substitute for $\ddot{\theta}$. From above, we conclude that $\frac{\mathrm{d}H}{\mathrm{d}t} \neq 0$, i.e., H is not constant. This is expected since the Lagrangian L was an explicit function of time t.