S26 AerE 5740: Optimal Control

Practice Set 03: Brachistochrone as OCP

Instructor: Abhishek Halder

Problem 1

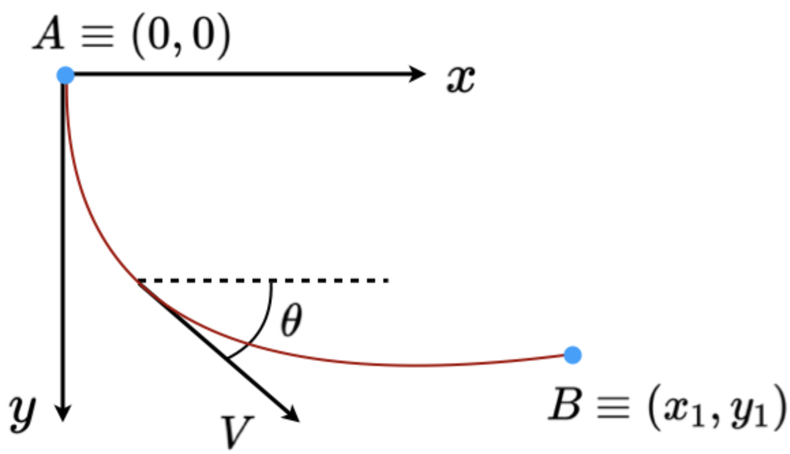

Figure 1: Brachistochrone problem.

In this exercise, we will solve the Brachistochrone problem shown above using OCP techniques.

(a) OCP reformulation.

Let the control variable be the angle θ that the velocity vector V (tangent to the curve) makes with the horizontal axis, as shown in Fig. 1. Reformulate the Brachistochrone problem of going from point $A \equiv (0, 0)$ to point $B \equiv (x_1, y_1)$ as a minimum time optimal control problem with terminal time T free, and terminal state $(x(T), y(T))$ fixed. In other words, write the corresponding problem in standard OCP template.

Solution for part (a):

The optimal control problem is to

$$\underset{\theta(\cdot)}{\text{minimize}} \quad \int_0^T 1 \, \mathrm{d}t$$

subject to $\dot{x} = V \cos\theta, \dot{y} = V \sin\theta$, where $V = \sqrt{2gy}$. The initial condition $(x(0), y(0)) \equiv (0, 0)$, and the terminal condition $(x(T), y(T)) \equiv (x_1, y_1)$ are given. There is no terminal cost ($\phi \equiv 0$);

the (vectorial) terminal constraint is

$$\boldsymbol{\psi}(x(T),T) := \begin{pmatrix} x(T)-x_1 \\ y(T)-y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) Inverse of the optimal state feedback.

Prove that the optimal position tuple $(x(t), y(t))$ at any time $t \in [0,T]$, is given by

$$x(t) = x_1 + \frac{y_1}{2\cos^2\theta(T)}\left\{2\left(\theta(T)-\theta(t)\right) + \sin\left(2\theta(T)\right) - \sin\left(2\theta(t)\right)\right\}, \quad y(t) = y_1 \frac{\cos^2\theta(t)}{\cos^2\theta(T)}.$$

The above two identities allow (numerically) solving for $(\theta(t), \theta(T))$ as function of $(x(t), y(t))$. This helps define optimal state feedback: $\theta(t)$ as function of $(x(t), y(t))$.

Solution for part (b):

The Hamiltonian $H = 1 + \sqrt{2gy}\,(\lambda_1\cos\theta + \lambda_2\sin\theta)$ gives the necessary conditions:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = 0, \tag{1}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial y} = -\frac{g}{V}\left(\lambda_1\cos\theta + \lambda_2\sin\theta\right), \tag{2}$$

$$0 = \frac{\partial H}{\partial \theta} = -\lambda_1 V\sin\theta + \lambda_2 V\cos\theta. \tag{3}$$

From (1), $\lambda_1 =$ constant. Since $\mathrm{d}\boldsymbol{x}(T) = 0$ and $\mathrm{d}T \neq 0$, the transversality condition gives $H(T) = 0$. But this being a time-invariant OCP, $H(t) =$ constant, which implies $H(t) = 0$ $\forall t \in [0,T]$. Therefore,

$$H = 1 + \lambda_1 V\cos\theta + \lambda_2 V\sin\theta = 0. \tag{4}$$

Eliminating λ_2 from (3) and (4), we obtain

$$\lambda_1 = -\frac{\cos\theta}{V}. \tag{5}$$

Likewise, eliminating λ_1 from (3) and (4), we get

$$\lambda_2 = -\frac{\sin\theta}{V}. \tag{6}$$

Combining (5) with $\dot{\lambda}_1 = 0$, we get

$$0 = \dot{\lambda}_1 = \frac{\partial\lambda_1}{\partial\theta}\dot{\theta} + \frac{\partial\lambda_1}{\partial y}\dot{y} = \frac{\sin\theta}{V}\dot{\theta} + \frac{g\sin\theta\cos\theta}{V^2} \Rightarrow 0 = \dot{\theta} + \frac{g}{V}\cos\theta \Rightarrow \dot{\theta} = -\frac{g}{V}\cos\theta. \tag{7}$$

Now, evaluating (5) at $t = t$ and $t = T$, and using the fact that λ_1 is constant, we obtain

$$\frac{\cos\theta(t)}{\sqrt{y(t)}} = \frac{\cos\theta(T)}{\sqrt{y_1}} \Rightarrow y(t) = y_1\frac{\cos^2\theta(t)}{\cos^2\theta(T)}. \tag{8}$$

Next, we use $\dot{x} = V\cos\theta$, together with (7), to get

$$V\cos\theta = \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}\theta}\dot{\theta} = \frac{\mathrm{d}x}{\mathrm{d}\theta} \times \left(-\frac{g}{V}\cos\theta\right) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = -2y = -\frac{2y_1}{\cos^2\theta(T)}\cos^2\theta, \tag{9}$$

wherein the last equality follows from (8). Integrating (9), we have

$$\begin{aligned} &\int_{x=x}^{x=x_1} \mathrm{d}x = -\frac{2y_1}{\cos^2\theta(T)}\int_{\theta=\theta}^{\theta=\theta(T)} \cos^2\theta \; \mathrm{d}\theta \\ \Rightarrow & x(t) = x_1 + \frac{y_1}{2\cos^2\theta(T)}\{2(\theta(T)-\theta(t)) + \sin(2\theta(T)) - \sin(2\theta(t))\}. \end{aligned} \tag{10}$$

We are done because (8) and (10) are the desired identities.

(c) Properties of optimal solution.

(c.1) Prove that the optimal control $\theta(t)$ satisfies $\dot{\theta}$ = constant.

(c.2) Prove that $\theta(0) = \dfrac{\pi}{2}$. Give brief physical interpretation of this result.

(c.3) Letting $\beta := \pi - 2\theta$, use your answer in part (b) to show the optimal curve is a cycloid.

Solution for part (c):

(c.1) Combining (5) and (7), we get $\dot{\theta} = g\lambda_1$ = constant, assuming g is constant as in the original Brachistochrone formulation.

(c.2) Evaluating (8) at $t = 0$ yields $\theta(0) = \pi/2$. Vertical initial heading is natural since the terminal point B is below the initial point A. In other words, to minimize the time to go, it is optimal to increase the vertical coordinate as fast as possible.

(c.3) In equations (8) and (10), we substitute θ as function of ϕ, and define a constant

$$a := \frac{y_1}{2\cos^2\beta(T)} = \frac{y_1}{1-\cos\beta(T)}.$$

Then (8) gives

$$y = a\left[1 + \cos(\pi - \beta)\right] = a\left(1 - \cos\beta\right).$$

Likewsie, (10) gives

$$x - x_1 + a\left[\beta(T) - \sin\beta(T)\right] = a(\beta - \sin\beta).$$

The last two equations above give the parametric form of a cycloid passing through (x_1, y_1). Here, $\theta(T)$ (and thus $\beta(T)$) is such that the cycloid passes though point A with coordinate $(0, 0)$.