S26 AerE 5740: Optimal Control

Practice Set 04: Zermelo's Navigation Problem

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Problem 1. How to Sail Fast.

In a river, a sailor would like to steer a boat through a region of strong currents, from an initial location (x_0, y_0) to the final location $(0, 0)$. Knowing that a storm is coming, the sailor would like to reach $(0, 0)$ as fast as possible, i.e., in minimum time. The boat moves with constant speed V but its heading $\theta(t)$ can be adjusted, i.e., $\theta(t)$ is a control variable. The current c in the river varies as

$$c(y) := \frac{Vy}{h}, \quad \text{for some fixed } h.$$

Letting (x, y) to be the position coordinate of the boat, its dynamics is given by

$$\dot{x} = V\cos\theta + c(y), \qquad \dot{y} = V\sin\theta.$$

(a) Linear tangent law for optimal heading.

Prove that the tangent of the optimal heading at time t, must be a linear function of t.

Solution for part (a):

The Hamiltonian $H = 1 + \lambda_1 (V\cos\theta + Vy/h) + \lambda_2 V \sin\theta$ gives the costate equations:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = 0, \quad \dot{\lambda}_2 = -\frac{\partial H}{\partial y} = -\frac{\lambda_1 V}{h},$$

which implies $\lambda_1 =$ constant, and $\lambda_2(t) = \lambda_2(T) + \lambda_1 V(T-t)/h$.

From PMP, we have $0 = \dfrac{\partial H}{\partial \theta} = -\lambda_1 V \sin\theta + \lambda_2 V\cos\theta$, or equivalently, $\tan\theta(t) = \lambda_2(t)/\lambda_1$, wherein substituting $\lambda_2(t)$ gives

$$\tan\theta(t) = \frac{\lambda_2(T)}{\lambda_1} + \frac{V(T-t)}{h},$$

which is the desired result.

(b) Optimal heading dynamics.

Show that the optimal heading angle θ has dynamics: $\dot{\theta} = -\frac{V}{h}\cos^2\theta$.

Solution for part (b):

The final state being fixed at $(0,0)$, makes terminal constraint as $\psi(T) = \begin{pmatrix} x(T) \\ y(T) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore, the transversality condition gives $H(T) = 0$. Since the Hamiltonian is not an explicit function of t, this implies $H(t) = 1 + \lambda_1 (V\cos\theta + Vy/h) + \lambda_2 V \sin\theta = 0$ for all $0 \leq t \leq T$. Combining this together with the PMP, gives us two equations in two unknowns λ_1, λ_2 as

$$\begin{pmatrix} V\cos\theta + \frac{Vy}{h} & V\sin\theta \\ -V\sin\theta & V\cos\theta \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

which yields

$$\lambda_1 = \frac{-\cos\theta}{V + V(y/h)\cos\theta}, \tag{1}$$

$$\lambda_2 = \frac{-\sin\theta}{V + V(y/h)\cos\theta}. \tag{2}$$

Using constancy of λ_1, the above expression for λ_1, and the state equation for $\dot{y}$, we get

$$0 = \dot{\lambda}_1 = \frac{\partial \lambda_1}{\partial \theta}\dot{\theta} + \frac{\partial \lambda_1}{\partial y}\dot{y} = \frac{V\sin\theta}{(V + V(y/h)\cos\theta)^2}\dot{\theta} + \frac{(V/h)\cos^2\theta}{(V + V(y/h)\cos\theta)^2} V\sin\theta \ \Rightarrow \ \dot{\theta} = -\frac{V}{h}\cos^2\theta,$$

as desired.

(c) Optimal state feedback.

Let T be the (free) final time. Prove that the optimal state feedback θ as function of (x, y)

(available from real-time GPS measurement, for example), is given implicitly by the following two coupled nonlinear equations, to be solved for $(\theta(t), \theta(T))$ as function of $(x(t), y(t))$:

$$\frac{x(t)}{h} = \frac{1}{2}\left[\sec\theta(T)\left(\tan\theta(T) - \tan\theta(t)\right) - \tan\theta(t)\left(\sec\theta(T) - \sec\theta(t)\right) + \log\frac{\sec\theta(T) + \tan\theta(T)}{\sec\theta(t) + \tan\theta(t)}\right],$$

$$\frac{y(t)}{h} = \sec\theta(T) - \sec\theta(t).$$

Solution for part (c):

Since λ_1 is a constant, from (1) we have $\lambda_1(t) = \lambda_1(T)$, which together with $y(T) = 0$ gives

$$\frac{y(t)}{h} = \sec\theta(T) - \sec\theta(t). \tag{3}$$

Next, we use chain rule in the state equation for $\dot{x}$, to obtain

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}\dot{\theta} = V\cos\theta + \frac{Vy}{h}. \tag{4}$$

In (4), we substitute for $\dot{\theta}$ from part (b), and for $y(t)/h$ from (3), to get

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{V\cos\theta + V\sec\theta(T) - V\sec\theta}{-(V/h)\cos^2\theta} = -h\left(\sec\theta + \sec\theta(T)\sec^2\theta - \sec^3\theta\right). \tag{5}$$

We integrate (5) as the independent variable goes from $\theta = \theta(t)$ to $\theta = \theta(T)$, and the dependent variable goes from $x = x(t)$ to $x = x(T) = 0$, to arrive at

$$\frac{x(t)}{h} = \frac{1}{2}\left[\sec\theta(T)\left(\tan\theta(T) - \tan\theta(t)\right) - \tan\theta(t)\left(\sec\theta(T) - \sec\theta(t)\right) + \log\frac{\sec\theta(T) + \tan\theta(T)}{\sec\theta(t) + \tan\theta(t)}\right].$$

In performing the above integration, we only needed the following three integrals:

$$\int \sec\theta\,\mathrm{d}\theta = \int \sec\theta\,\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\,\mathrm{d}\theta = \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\,\mathrm{d}\theta = \int \frac{\mathrm{d}\eta}{\eta} = \log\left(\sec\theta + \tan\theta\right), \tag{6}$$

$$\int \sec^2\theta\,\mathrm{d}\theta = \int \mathrm{d}\left(\tan\theta\right) = \tan\theta, \tag{7}$$

$$\begin{aligned}\int \sec^3\theta\,\mathrm{d}\theta &= \int \underbrace{\sec\theta}_{\text{1st function}}\;\underbrace{\sec^2\theta}_{\text{2nd function}}\,\mathrm{d}\theta = \sec\theta\tan\theta - \int \left(\sec\theta\tan\theta\right)\tan\theta\,\mathrm{d}\theta\\ &= \sec\theta\tan\theta - \int \sec\theta\left(\sec^2\theta - 1\right)\mathrm{d}\theta\end{aligned}$$

$$\Rightarrow \int \sec^3\theta\,\mathrm{d}\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\log\left(\sec\theta + \tan\theta\right). \tag{8}$$

(d) Plot optimal state and optimal control trajectories.

Let $V = 1$, $h = 1$, $(x_0, y_0) = (4.9, 1.66)$. Use parts (b) and (c) above to numerically compute and then plot the optimal state trajectory in the xy plane. In the same figure, plot arrows to show how the optimal heading varies along that optimal state trajectory.

Solution for part (d):

See the following figure. The MATLAB implementation to generate this figure is in CANVAS: `Problem Sets/PS04Code.zip`. The executable is `runZermelo.m`.

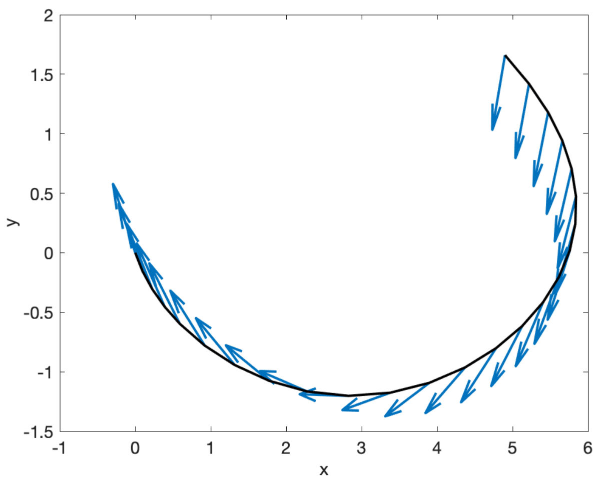

Figure 1: The optimal state (black) and control (blue) trajectories in the xy plane.