S26 AerE 5740: Optimal Control

Practice Set 05: Hamiltonian Matrix

Instructor: Abhishek Halder

Problem 1.

In both CoV and Pontryagin's necessary conditions of optimality for continuous-time OCP, we have encountered the Hamiltonian dynamical system, which is a special-structured ODE for some $2n \times 1$ state vector $\boldsymbol{z}$ given by

$$\dot{\boldsymbol{z}} = \boldsymbol{\Omega} \nabla_{\boldsymbol{z}} H, \quad \text{where} \quad \boldsymbol{\Omega} := \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix}}_{2n \times 2n}.$$

The scalar-valued mapping H is called the Hamiltonian. In this exercise, we investigate the closely related concept of Hamiltonian matrix encountered in LQ OCP.

(a) Properties of matrix $\boldsymbol{\Omega}$.

Prove that the matrix $\boldsymbol{\Omega}$ is orthogonal and skew symmetric, i.e., $\boldsymbol{\Omega}^{-1} = \boldsymbol{\Omega}^{\top} = -\boldsymbol{\Omega}$.

We say a $2n \times 2n$ matrix $\boldsymbol{M}$ is symplectic if $\boldsymbol{M}^{\top} \boldsymbol{\Omega} \boldsymbol{M} = \boldsymbol{\Omega}$. Prove that $\boldsymbol{\Omega}$ itself is symplectic.

Solution for part (a):

By direct block matrix multiplication, we verify that

$$\boldsymbol{\Omega} \boldsymbol{\Omega}^{\top} = \boldsymbol{I}.$$

Taking determinant to both sides of above, we find $\det \boldsymbol{\Omega} = \pm 1$ implying $\boldsymbol{\Omega}$ is nonsingular, i.e., $\boldsymbol{\Omega}^{-1}$ exists. Pre-multiplying both sides of the above equation by $\boldsymbol{\Omega}^{-1}$ gives $\boldsymbol{\Omega}^{-1} = \boldsymbol{\Omega}^{\top}$. That $\boldsymbol{\Omega}^{\top} = -\boldsymbol{\Omega}$ follows by taking the transpose of the block matrix. Therefore, $\boldsymbol{\Omega}^{-1} = \boldsymbol{\Omega}^{\top} = -\boldsymbol{\Omega}$. Notice that a consequence of this result is that $-\boldsymbol{\Omega}^2 = \boldsymbol{I}$.

To show $\boldsymbol{\Omega}$ is symplectic, we compute $\boldsymbol{\Omega}^{\top} \boldsymbol{\Omega} \boldsymbol{\Omega} = -\boldsymbol{\Omega} \boldsymbol{\Omega} \boldsymbol{\Omega} = \boldsymbol{\Omega}$, where the first equality used $\boldsymbol{\Omega}^{\top} = -\boldsymbol{\Omega}$, and the second equality used $-\boldsymbol{\Omega}^2 = \boldsymbol{I}$, from the earlier part of our proof.

(b) Hamiltonian matrix $\boldsymbol{H}$.

We say $\boldsymbol{H} \in \mathbb{R}^{2n\times 2n}$ is a Hamiltonian matrix if $\boldsymbol{\Omega H}$ is symmetric, i.e., $(\boldsymbol{\Omega H})^{\top} = \boldsymbol{\Omega H}$. Prove that this definition is equivalent to saying $\boldsymbol{H}$ satisfies the linear equation

$$\boldsymbol{\Omega H} + \boldsymbol{H}^{\top}\boldsymbol{\Omega} = \boldsymbol{0}.$$

Solution for part (b):

We have

$$\begin{aligned} &(\boldsymbol{\Omega H})^{\top} = \boldsymbol{\Omega H} \\ \Rightarrow \quad &\boldsymbol{H}^{\top}\boldsymbol{\Omega}^{\top} = \boldsymbol{\Omega H}. \end{aligned}$$

Substituting $\boldsymbol{\Omega}^{\top} = -\boldsymbol{\Omega}$ (from part (a)) in the left-hand-side above, and bringing all terms to one side, the desired equality follows.

(c) Spectrum of Hamiltonian matrix.

Prove that the spectrum of a Hamiltonian matrix is symmetric about the imaginary axis, i.e., if $\lambda \in \mathbb{C}$ is an eigenvalue then so is $-\lambda \in \mathbb{C}$.

Solution for part (c):

Let $\boldsymbol{H}$ be a Hamiltonian matrix, and let $\lambda \in \mathbb{C}$. We have

$$\det(\boldsymbol{H}-\lambda\boldsymbol{I}) = \det(\boldsymbol{H}^{\top}-\lambda\boldsymbol{I}) = \det\left(\boldsymbol{\Omega H \Omega} + \lambda\boldsymbol{\Omega}^2\right) = \det\left(\boldsymbol{\Omega}\left(H + \lambda\boldsymbol{I}\right)\boldsymbol{\Omega}\right) = \left(\det\boldsymbol{\Omega}\right)^2 \det(\boldsymbol{H}+\lambda\boldsymbol{I}),$$

where the first equality used that determinant is invariant under transposition; the second equality used from part (b): $\boldsymbol{H}^{\top}\boldsymbol{\Omega} = -\boldsymbol{\Omega H} \;\Rightarrow\; \boldsymbol{H}^{\top} = -\boldsymbol{\Omega H \Omega}^{-1} = \boldsymbol{\Omega H \Omega}$, and from part (a) that $\boldsymbol{I} = -\boldsymbol{\Omega}^2$. Since $\boldsymbol{\Omega}$ is orthogonal (from part (a)), $\left(\det\boldsymbol{\Omega}\right)^2 = 1$, and the previous equality simplifies to

$$\det(\boldsymbol{H}-\lambda\boldsymbol{I}) = \det(\boldsymbol{H}+\lambda\boldsymbol{I}).$$

Therefore, if $\lambda \in \mathbb{C}$ is a root of the characteristic polynomial $\det(\boldsymbol{H}-\lambda\boldsymbol{I}) = 0$, so is $-\lambda \in \mathbb{C}$.

(d) When is a block matrix Hamiltonian?

Prove that the $2n \times 2n$ block matrix $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$ with all four blocks of size $n \times n$, is Hamiltonian if and only if $\boldsymbol{B}, \boldsymbol{C}$ are symmetric and $\boldsymbol{D} = -\boldsymbol{A}^{\top}$.

Solution for part (d):

From part (b), for the block matrix to be Hamiltonian, it must satisfy

$$\left(\begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}\right)^{\top} = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \boldsymbol{A}^{\top} & \boldsymbol{C}^{\top} \\ \boldsymbol{B}^{\top} & \boldsymbol{D}^{\top} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\boldsymbol{I} \\ \boldsymbol{I} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ -\boldsymbol{A} & -\boldsymbol{B} \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \boldsymbol{C}^{\top} & -\boldsymbol{A}^{\top} \\ \boldsymbol{D}^{\top} & -\boldsymbol{B}^{\top} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ -\boldsymbol{A} & -\boldsymbol{B} \end{bmatrix}.$$

Equating respective blocks in both sides, the claim follows.