

# S26 AerE 5740: Optimal Control

## Practice Set 07: Singular OCP

Instructor: Abhishek Halder

All rights reserved

### Problem 1. Launching Rocket on A New Planet

Suppose we have a time invariant OCP in Mayer form with free final time  $T$ , given by

$$\underset{u(\cdot)}{\text{minimize}} \quad \phi(x(T))$$

subject to

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^3, \quad u \in \mathbb{R}, \quad x(0) \text{ given, terminal constraint: } \psi(x(T)) = 0, \quad u_{\min} \leq u \leq u_{\max}.$$

The corresponding Hamiltonian  $H = \langle \lambda, f(x) + g(x)u \rangle$  is linear in the control  $u$ .

#### (a) Mathematical foundation.

(i) Write down the costate ODE, the PMP, the transversality conditions, and the singular arc or switching curve involving costate for the optimal control, if any.

(ii) A singular surface is a 2D surface in our 3D state space such that the optimal control is singular exactly when the state vector is on that surface. Derive the equation of the singular surface only in terms of the components of the vector fields  $f, g$  and  $h := [f, g]$  (Lie bracket).

#### Solution for part (a):

(i) The Hamiltonian is  $H = \langle \lambda, f(x) + g(x)u \rangle$ . This gives the costate ODE

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\left(\frac{\partial f}{\partial x}\right)' \lambda - \left(\frac{\partial g}{\partial x}\right)' \lambda u.$$

The PMP gives

$$u^{\text{opt}} = \begin{cases} u_{\min} & \text{if } \langle \lambda, g \rangle > 0, \\ u_{\max} & \text{if } \langle \lambda, g \rangle < 0, \\ \text{singular} & \text{if } \langle \lambda, g \rangle = 0. \end{cases}$$

For the transversality conditions, notice that since  $dx(T) \neq 0$ ,  $dT \neq 0$ ,  $\psi(x(T)) = 0$ , we get

$$\lambda(T) = \frac{\partial \phi(x(T))}{\partial x(T)}, \quad H(T) = 0.$$

The singular arc/switching curve is  $\langle \lambda, g \rangle = 0$ .

(ii) Following Lec. 16 end and Lec. 17 beginning, at the singular surface, we have the two conditions

$$\langle \lambda, g \rangle = 0, \quad \langle \lambda, \underbrace{[f, g]}_{=:h} \rangle = 0.$$

On the other hand, this being a time-invariant OCP, we must have  $H(t) = \text{constant}$  for all  $t \in [0, T]$ . However, transversality condition tells us that  $H(T) = 0$ , which implies  $H(t) = 0$  for all  $t \in [0, T]$ . Thus, at the singular surface, we get a third condition

$$H(t) \Big|_{\text{singular surface}} = \langle \lambda, f \rangle = 0.$$

We observe that the three conditions we derived are three linear homogeneous equations in  $(\lambda_1, \lambda_2, \lambda_3)$  of the form:

$$\begin{pmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To have nontrivial solution for  $(\lambda_1, \lambda_2, \lambda_3)$ , the determinant of the corresponding coefficient matrix must vanish, i.e.,

$$\det \begin{pmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{pmatrix} = 0,$$

which is the equation of singular surface.

### (b) Application.

Suppose that a space robotics mission has successfully landed a rover on a planet whose atmospheric properties are not precisely known. To take direct atmospheric measurements, the rover launches a small rocket with initial mass  $m_0$  that vertically ascends up to some height within that atmosphere and collects data during its flight. For this robotic experiment to be scientifically meaningful, it is desired to maximize the final altitude of the rocket. The idea is that during its ascent, the rocket will record and transmit data to the rover, and then simply drop off on the surface.

Define the state vector of the rocket as  $x := (v, h, m)^\top$  where  $v$  is its vertical velocity,  $h$  is its altitude, and  $m$  is its mass. Because the rocket needs to burn propellant, its mass is time-varying. At the free final time  $T$  when the maximum altitude is reached, the rocket has final mass  $m_T$  (given).

A known drag force  $D(v, h)$ , that depends on  $v$  and  $h$ , acts during the rocket's ascent. The only control variable is the magnitude of the thrust  $F$ . The OCP becomes

$$\underset{F(\cdot)}{\text{maximize}} \quad h(T)$$

subject to

$$\begin{pmatrix} \dot{v} \\ \dot{h} \\ \dot{m} \end{pmatrix} = \begin{pmatrix} \frac{1}{m}(F - D) - g \\ v \\ -\frac{1}{c}F \end{pmatrix}, \quad \begin{pmatrix} v(0) \\ h(0) \\ m(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ m_0 \end{pmatrix} \text{ given, } \quad m(T) = m_T \text{ given, } \quad 0 \leq F \leq F_{\max},$$

where  $g > 0$  is the planet's acceleration due to gravity (known constant), and  $c > 0$  is the specific impulse of the rocket engine (known constant).

(i) Use your answer in part (a) to clearly write down the costate ODEs, the PMP, the transversality conditions, the singular arc or switching curve, and the equation of singular surface for this particular problem.

(ii) Clearly derive the optimal singular feedback control, that is, the optimal state feedback  $F$  when the state is on the singular surface.

(iii) What type of policy is the optimal state feedback? Is it bang-bang or bang-off-bang or something else? Explain.

### **Solution for part (b):**

(i) In this case,  $(x_1, x_2, x_3) = (v, h, m)$ ,  $D \equiv D(x_1, x_2)$ ,  $\phi(x(T)) \equiv -x_2(T)$ ,  $u \equiv F$ ,  $u_{\min} = 0$ ,  $u_{\max} = F_{\max}$ . Specializing the answer of part (a) for this case, we get

$$\begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{x_3} \frac{\partial D}{\partial x_1} \lambda_1 - \lambda_2 \\ \frac{1}{x_3} \frac{\partial D}{\partial x_2} \lambda_1 \\ \frac{F - D}{x_3^2} \lambda_1 \end{pmatrix}.$$

The PMP yields

$$F^{textopt} = \begin{cases} 0 & \text{if } \frac{\lambda_1}{x_3} - \frac{\lambda_3}{c} > 0, \\ F_{\max} & \text{if } \frac{\lambda_1}{x_3} - \frac{\lambda_3}{c} < 0, \\ \text{singular} & \text{if } \frac{\lambda_1}{x_3} - \frac{\lambda_3}{c} = 0. \end{cases}$$

The transversality condition gives

$$\begin{pmatrix} \lambda_1(T) \\ \lambda_2(T) \\ \lambda_3(T) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad H(T) = 0.$$

The singular arc/switching curve is

$$\sigma(t) := \frac{\lambda_1}{x_3} - \frac{\lambda_3}{c} = 0.$$

The Lie bracket

$$\begin{aligned} h := [f, g] &= \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = \begin{pmatrix} 0 & 0 & -\frac{1}{x_3^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{D}{x_3} - g \\ x_1 \\ 0 \end{pmatrix} - \begin{pmatrix} -\frac{1}{x_3} \frac{\partial D}{\partial x_1} & -\frac{1}{x_3} \frac{\partial D}{\partial x_2} & \frac{D}{x_3^2} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{x_3} \\ 0 \\ -\frac{1}{c} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{x_3^2} \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) \\ -\frac{1}{x_3} \\ 0 \end{pmatrix}. \end{aligned}$$

Thus, the singular surface is

$$\det \begin{pmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{pmatrix} = \det \begin{pmatrix} -\frac{D}{x_3} - g & x_1 & 0 \\ \frac{1}{x_3} & 0 & -\frac{1}{c} \\ \frac{1}{x_3^2} \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) & -\frac{1}{x_3} & 0 \end{pmatrix} = \frac{D}{cx_3^2} + \frac{g}{cx_3} - \frac{x_1}{cx_3^2} \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) = 0.$$

Since  $m := x_3 \neq 0$ , multiplying both sides of the above equation by  $cx_3^2$ , we obtain  $D + x_3g - x_1 \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) = 0$ , or equivalently, in original state symbols:

$$D + mg - v \left( \frac{\partial D}{\partial v} + \frac{D}{c} \right) = 0.$$

(ii) To derive the optimal state feedback when the state is on the singular surface derived in part (b)(i), we take the second derivative of the singular arc and set it equal to zero, i.e.,  $\ddot{\sigma} = 0$ . Recall that  $\dot{\sigma} = 0$  is equivalent to

$$\langle \lambda, h \rangle = \lambda_1 \frac{1}{x_3^2} \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) - \lambda_2 \frac{1}{x_3} = 0 \quad \Leftrightarrow \quad \lambda_1 \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) - \lambda_2 x_3 = 0.$$

To enforce  $\ddot{\sigma} = 0$ , we take  $\frac{d}{dt}$  to both sides of the last expression above to get

$$\begin{aligned} & \frac{d}{dt} \left[ \lambda_1 \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) - \lambda_2 x_3 \right] = 0 \\ \Rightarrow & \dot{\lambda}_1 \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) + \lambda_1 \left( \frac{d}{dt} \frac{\partial D}{\partial x_1} + \frac{1}{c} \frac{dD}{dt} \right) - \dot{\lambda}_2 x_3 - \lambda_2 \dot{x}_3 = 0 \\ \Rightarrow & \left( \frac{1}{x_3} \frac{\partial D}{\partial x_1} \lambda_1 - \lambda_2 \right) \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) + \lambda_1 \left( \frac{\partial^2 D}{\partial x_1^2} \dot{x}_1 + \frac{\partial^2 D}{\partial x_1 \partial x_2} \dot{x}_2 + \frac{1}{c} \frac{\partial D}{\partial x_1} \dot{x}_1 + \frac{1}{c} \frac{\partial D}{\partial x_2} \dot{x}_2 \right) - \frac{\partial D}{\partial x_2} \lambda_1 - \lambda_2 \dot{x}_3 = 0 \\ \Rightarrow & \left( \frac{1}{x_3} \frac{\partial D}{\partial x_1} \lambda_1 - \lambda_2 \right) \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) + \lambda_1 \left( \left( \frac{\partial^2 D}{\partial x_1^2} + \frac{1}{c} \frac{\partial D}{\partial x_1} \right) \left( \frac{1}{x_3} (F^{\text{opt}} - D) - g \right) + \left( \frac{\partial^2 D}{\partial x_1 \partial x_2} + \frac{1}{c} \frac{\partial D}{\partial x_2} \right) x_1 \right) \\ & \quad - \frac{\partial D}{\partial x_2} \lambda_1 + \frac{1}{c} \lambda_2 F^{\text{opt}} = 0, \quad (*) \end{aligned}$$

wherein the last but one line used chain rule and costate ODEs. The last line used the state ODEs. From  $\dot{\sigma} = 0$ , we know that on the singular surface,  $\lambda_2 = \frac{1}{x_3} \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) \lambda_1$ , which upon substitution in (\*) above, allows us to write

$$\begin{aligned} & \left[ -\frac{1}{x_3} \frac{D}{c} \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) + \left( \frac{\partial^2 D}{\partial x_1^2} + \frac{1}{c} \frac{\partial D}{\partial x_1} \right) \left( \frac{1}{x_3} (F^{\text{opt}} - D) - g \right) + \left( \frac{\partial^2 D}{\partial x_1 \partial x_2} + \frac{1}{c} \frac{\partial D}{\partial x_2} \right) x_1 - \frac{\partial D}{\partial x_2} \right. \\ & \quad \left. + \frac{1}{cx_3} \left( \frac{\partial D}{\partial x_1} + \frac{D}{c} \right) F^{\text{opt}} \right] \lambda_1 = 0. \end{aligned}$$

For  $\lambda_1$  to be nontrivial, the term inside the square bracket must vanish, wherein further simplification results in after substituting  $\frac{\partial D}{\partial x_1} + \frac{D}{c} = \frac{1}{x_1} (D + x_3 g)$  (from the singular surface equation). Upon solving for  $F^{\text{opt}}$ , we obtain the optimal feedback control on singular surface:

$$F^{\text{opt}} = \frac{(D + x_3 g) \left( \frac{D}{c} + x_1 \left( \frac{\partial^2 D}{\partial x_1^2} + \frac{1}{c} \frac{\partial D}{\partial x_1} \right) \right) + x_3 x_1 \left( \frac{\partial D}{\partial x_2} - x_1 \left( \frac{\partial^2 D}{\partial x_1 \partial x_2} + \frac{1}{c} \frac{\partial D}{\partial x_2} \right) \right)}{x_1 \left( \frac{\partial^2 D}{\partial x_1^2} + \frac{1}{c} \frac{\partial D}{\partial x_1} \right) + \frac{1}{c} (D + x_3 g)}.$$

(iii) The optimal state feedback policy is bang-singular-bang. In other words, the singular control derived in part (b)(ii) is active only for the singular surface derived in part (b)(i). If the state is NOT on the singular surface, the optimal control remains bang-bang.

(c) Numerically compute and plot the singular surface.

Let the drag force

$$D(v, h) := \frac{1}{2} \rho_0 S C_D v^2 \exp(-\beta h),$$

where  $\rho_0 S C_D = 620$ ,  $\beta = 500$ . Set  $g = 1$ ,  $c = 0.5$ . For this case, write a simple MATLAB/Python code to make a plot of the singular surface.

**Solution for part (c):**

Substituting the specific form of  $D(v, h)$  in the expression of the singular surface derived in part (b), we get

$$\frac{1}{2} \rho_0 S C_D v^2 \exp(-\beta h) + \frac{1}{2c} \rho_0 S C_D v^3 \exp(-\beta h) = mg \quad \Leftrightarrow \quad \left(1 + \frac{v}{c}\right) D = mg.$$

The following Fig. 1 plots the above singular surface.

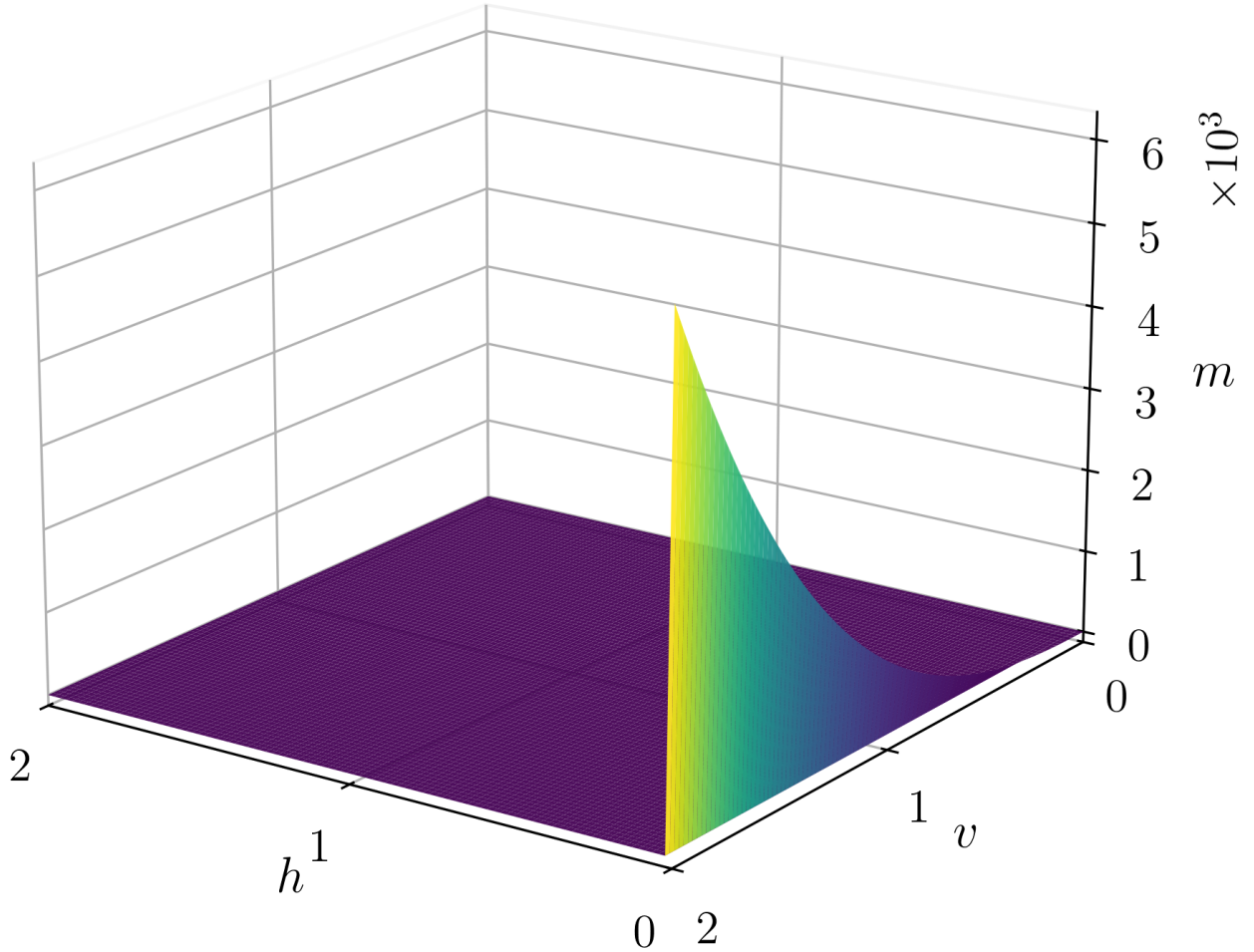


Figure 1: Singular surface.