AerE 5740: Optimal Control: Spring 2025

Iowa State University

Quiz 0: Assessment for Mathematical Preparation

For each of the following multiple choice problems, ONLY ONE option is correct. Clearly mark the correct option. Calculators or other electronic devices are NOT allowed.

Problem 1

Consider vectors

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Then $\boldsymbol{a}\boldsymbol{b}^\top$ equals

(A) 11.

(B) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(C) $\begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$. `Reason:` $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 & 1 \cdot 4 \\ 2 \cdot 3 & 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$.

Problem 2

Which of the following is a valid state space representation for the input-output ODE $\dddot{y} + 3\ddot{y} + 2\dot{y} + y = u$?

(A) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

`Reason:` Rewrite using $y = x_1, \dot{y} = x_2, \ddot{y} = x_3$.

(B) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

(C) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

Problem 3

A function $f \in \mathcal{C}^1(\mathbb{R}^n)$. This is equivalent to which of the following?

(A) f is continuous with respect to any vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

(B) $\forall i = 1, \ldots, n$, $\dfrac{\partial f}{\partial x_i}$ exists at any vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

(C) $\forall i = 1, \ldots, n$, $\dfrac{\partial f}{\partial x_i}$ is continuous with respect to any vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$.
`Reason`: Differentiability is not enough; each derivative must be continuous too. See 1D counter-example in Lecture 1 notes.

Problem 4

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be skew-symmetric, i.e., $\boldsymbol{A}^\top = -\boldsymbol{A}$. Then the state transition matrix for the linear time-invariant (LTI) system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{R}^n$, must be

(A) symmetric.

(B) skew-symmetric.

(C) orthogonal. `Reason`: State transition matrix is $\exp(\tau\boldsymbol{A})$ for some $\tau > 0$. Now, $(\exp(\tau\boldsymbol{A}))^\top = \exp(\tau\boldsymbol{A}^\top) = \exp(-\tau\boldsymbol{A})$. So we have $(\exp(\tau\boldsymbol{A}))^\top \exp(\tau\boldsymbol{A}) = \exp(-\tau\boldsymbol{A} + \tau\boldsymbol{A}) = \exp(\boldsymbol{0}) = \boldsymbol{I}$. `Reason for more advanced students`: by definition, the exponential map $\exp : \mathfrak{so}(n) = T_{\boldsymbol{I}}\mathrm{SO}(n) \to \mathrm{SO}(n)$.

Problem 5

Let the continuous function $f : [0,1] \to [0,1]$, $y := \min\limits_{x\in[0,1]} f(x)$, and $Y := \max\limits_{x\in[0,1]} -f(x)$, where := denotes "defined as".

(A) $y = Y$.

(B) $y = -Y$. `Reason`: Draw a picture.

(C) There is no relation between y and Y.

Problem 6

Let the continuous function $f : [0,1] \to [0,1]$, $y := \min\limits_{x\in[0,1]} f(x)$, and $z := \min\limits_{x\in\{0,1\}} f(x)$, where := denotes "defined as".

(A) $y \le z$. `Reason`: $\{0,1\} \subset [0,1]$.

(B) $y \ge z$.

(C) $y = z$.

Problem 7

If matrix $\boldsymbol{S}$ is symmetric, then which of the following is IMPOSSIBLE?

(A) $\boldsymbol{S}$ has a zero eigenvalue.

(B) $\boldsymbol{S}$ has repeated real eigenvalues.

(C) $\boldsymbol{S}$ has a pair of complex conjugate eigenvalues. `Reason`: spectral theorem.

Problem 8

Consider a real invertible matrix $\boldsymbol{X}(t)$ that is differentiable with respect to the parameter $t \in \mathbb{R}$. If $\boldsymbol{Y} = \boldsymbol{X}^{-1}$, then

(A) $\dot{\boldsymbol{Y}} = \boldsymbol{Y}\dot{\boldsymbol{X}}\boldsymbol{Y}$.

(B) $\dot{\boldsymbol{Y}} = -\boldsymbol{Y}\dot{\boldsymbol{X}}\boldsymbol{Y}$. `Reason`: differentiate both sides of $\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{I}$ or $\boldsymbol{Y}\boldsymbol{X} = \boldsymbol{I}$ with respect to the scalar parameter t. Then re-arrange.

(C) $\dot{\boldsymbol{Y}} = -\dot{\boldsymbol{X}}$.

Problem 9

Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{M}\boldsymbol{x}$ for some constant $\boldsymbol{M} \in \mathbb{R}^{n\times n}$. Then ∇f, that is, the gradient of f with respect to $\boldsymbol{x}$, is

(A) $2\boldsymbol{M}\boldsymbol{x}$.

(B) $2\boldsymbol{M}^\top\boldsymbol{x}$.

(C) $\left(\boldsymbol{M} + \boldsymbol{M}^\top\right)\boldsymbol{x}$. `Reason:` use the failsafe way to extract ∇f from the definition of directional derivative $D_{\boldsymbol{z}}f(\boldsymbol{x})$, see Lecture 1 notes.

Problem 10

Consider the two dimensional linear dynamical system in discrete time:

$$\begin{aligned} x_1(t+1) &= \alpha x_1(t) + 2026 x_2(t), \\ x_2(t+1) &= \frac{1}{\alpha} x_2(t), \end{aligned}$$

where the time index $t = 0, 1, \ldots$, and the real parameter $\alpha \neq 0$. This system is

(A) stable for ALL nonzero $\alpha \in \mathbb{R}$.

(B) NOT stable for ANY nonzero $\alpha \in \mathbb{R}$. `Reason`: matrix $\begin{bmatrix} \alpha & 2026 \\ 0 & 1/\alpha \end{bmatrix}$ has eigenvalues $\alpha, 1/\alpha$. For Schur-Cohn stability, need $|\alpha| < 1$ and $1/|\alpha| < 1$, mutual contradiction.

(C) stable for SOME nonzero $\alpha \in \mathbb{R}$.