AerE 5740: Optimal Control: Spring 2025

Iowa State University

Quiz 11: Continuous-time finite horizon LQR solution

Name:____

For each of the following multiple choice problems, ONLY ONE option is correct. Clearly mark the correct option. Calculators or other electronic devices are NOT allowed.

For fixed terminal time T, consider the scalar (i.e., 1 state and 1 input) LQR problem:

$$\begin{aligned} \min_{u} \quad & \int_0^T \frac{1}{2}\left(x^2 + u^2\right) \mathrm{d}t \\ \text{subject to} \quad & \dot{x} = u, \\ & x(0) = x_0 \text{ given.} \end{aligned}$$

All 5 questions are for this problem. All variables below are understood at optimality: we drop the superscript $^{\text{opt}}$ to reduce notational clutter.

Problem 1

This LQR problem has terminal cost $\frac{1}{2}M(x(T))^2$ with

(A) $M = 0$. `Reason:` no terminal cost in this LQR objective.

(B) $M = 1$.

(C) $M = 2$.

Problem 2

In this problem, which of the following is the Hamiltonian $H := L + \langle \boldsymbol{\lambda}, \boldsymbol{f} \rangle$?

(A) λu.

(B) $\dfrac{1}{2}\left(x^2 + u^2\right)$.

(C) $\dfrac{1}{2}\left(x^2 + u^2\right) + \lambda u$. `Reason:` here $L = \frac{1}{2}(x^2 + u^2)$, $f = u$.

Problem 3

In this problem, the transversality condition gives

(A) $\lambda(T) = 0$. `Reason:` $\lambda(T) = Mx(T) = 0 \cdot x(T) = 0$.

(B) $\lambda(T) = 1$.

(C) $\lambda(T) = -1$.

Problem 4

For this problem, using the costate ODE $\dot{\lambda} = -\dfrac{\partial H}{\partial x}$ and the PMP $0 = \dfrac{\partial H}{\partial u}$, we get the two point boundary value problem (TPBVP)

(A) $\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}, \quad x(0) = x_0, \quad \lambda(T) = 0.$

(B) $\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}, \quad x(0) = x_0, \quad \lambda(T) = 0.$ `Reason:` $\dot{\lambda} = -x$, $\dot{x} = u = -\lambda$.

(C) $\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}, \quad x(0) = x_0, \quad \lambda(T) = 0.$

Problem 5

To solve the TPBVP in Problem 4, we make the guess $\lambda(t) = p(t)x(t)$ for some to-be-determined function $p(t)$. The ODE IVP for $p(t)$ is

Hint: by product rule of differentiation, $\dot{\lambda} = p\dot{x} + x\dot{p}$.

(A) $\dot{p} = p^2 - 1, \quad p(T) = 0$. `Reason:` Using hint:
$-x = p \cdot (-\lambda) + x\dot{p} = -p^2 x + x\dot{p} \;\Rightarrow\; \dot{p} = p^2 - 1$. To generate terminal condition, use transversality (Problem 3): $0 = \lambda(T) = p(T)x(T) \;\Rightarrow\; p(T) = 0$. See Lec. 12 p. 5-6.

(B) $\dot{p} = p^2 + 1, \quad p(T) = 0$.

(C) $\dot{p} = p^2, \quad p(T) = 1$.

Complete solution of this LQR problem:
To solve the ODE IVP in Problem 5, we notice that

$$\frac{\mathrm{d}p}{(p+1)(p-1)} = \mathrm{d}t \Rightarrow \frac{1}{2}\frac{(p+1)-(p-1)}{(p+1)(p-1)}\mathrm{d}p = \mathrm{d}t \Rightarrow \frac{\mathrm{d}p}{p-1} - \frac{\mathrm{d}p}{p+1} = 2\ \mathrm{d}t \Rightarrow \ln\left|\frac{p-1}{p+1}\right| = 2t + c,$$

and using $p(T) = 0$, the constant of integration $c = -2T$. This gives

$$\ln\left|\frac{p-1}{p+1}\right| = 2(t-T) \Rightarrow \frac{p-1}{p+1} = \exp(2(t-T)),$$

where we dropped the absolute value sign since the exponential is nonnegative. Letting $\tau := t - T$, we then have

$$\frac{p-1+p+1}{p-1-p-1} = \frac{\exp(2\tau)+1}{\exp(2\tau)-1} \Rightarrow -p = \frac{e^{\tau}+e^{-\tau}}{e^{\tau}-e^{-\tau}} = \tanh(\tau) \Rightarrow p(t) = \tanh(-\tau) = \tanh(T-t).$$

Hence $\lambda = px = x\tanh(T-t)$, so the optimal feedback control is

$$u^{\mathrm{opt}}(t,x) = -\lambda = -x\tanh(T-t).$$

The optimal state solves

$$\dot{x}^{\mathrm{opt}} = -x^{\mathrm{opt}}\tanh(T-t) \ \Rightarrow\ \log x^{\mathrm{opt}} = \int \tanh(t-T)\,\mathrm{d}t = \log\cosh(t-T) + \log k,$$

where k is the constant of integration. This gives $x^{\mathrm{opt}} = k\cosh(t-T)$, and using the given initial condition $x(0) = x_0$, we find $k = x_0/\cosh(T)$. Therefore, the optimal state is

$$x^{\mathrm{opt}} = \frac{x_0\cosh(t-T)}{\cosh(T)}.$$

The optimal cost:

$$\begin{aligned}
J^{\mathrm{opt}} = \frac{1}{2}\int_0^T \left(\left(x^{\mathrm{opt}}\right)^2 + \left(u^{\mathrm{opt}}\right)^2\right)\mathrm{d}t &= \frac{1}{2}\int_0^T \left(x^{\mathrm{opt}}\right)^2\left(1+\tanh^2(T-t)\right)\mathrm{d}t \\
&= \frac{x_0^2}{2\cosh^2(T)}\int_0^T \cosh(2\tau)\mathrm{d}\tau \\
&= \frac{x_0^2}{2\cosh^2(T)}\left[\frac{\sinh(2\tau)}{2}\right]_{\tau=0}^{\tau=T} \\
&= \frac{x_0^2}{2}\tanh(T),
\end{aligned}$$

where the third equality used $\cosh^2\xi + \sinh^2\xi = \cosh 2\xi$, and the last equality used $\sinh(2T) = 2\sinh(T)\cosh(T)$.