AerE 5740: Optimal Control: Spring 2025

Iowa State University

Quiz 13: Infinite horizon LQR

Name:____

For each of the following multiple choice problems, ONLY ONE option is correct. Clearly mark the correct option. Calculators or other electronic devices are NOT allowed.

Consider the special infinite horizon continuous-time LQR problem:

$$\begin{aligned} &\underset{\boldsymbol{u}}{\text{minimize}} \quad \frac{1}{2}\int_0^\infty \left(\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u}\right) \mathrm{d}t \\ &\text{subject to} \quad \dot{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{u}, \end{aligned} \tag{1}$$

where $\boldsymbol{Q} \succeq \boldsymbol{0}$, $\boldsymbol{R} \succ \boldsymbol{0}$ and $\boldsymbol{B}$ is $n \times n$ nonsingular.

All questions below are for this particular instance of infinite horizon LQR (1) above.

Problem 1

Recall that an LTI pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable if and only if

$$\operatorname{rank}\left(\begin{bmatrix}\boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}^2\boldsymbol{B} & \dots & \boldsymbol{A}^{n-1}\boldsymbol{B}\end{bmatrix}\right) = n.$$

Using this condition, we can guarantee that the CARE for problem (1)

(A) has a unique solution $\boldsymbol{P}_\infty \succeq \boldsymbol{0}$ regardless of the numerical entries in $\boldsymbol{B}$. `Reason:` here $\boldsymbol{A} = \boldsymbol{0}$ and $\operatorname{rank}(\boldsymbol{B}) = n$ since $\boldsymbol{B}$ is nonsingular. So the pair $(\boldsymbol{0}, \boldsymbol{B})$ is controllable, thus stabilizable.

(B) does NOT have a unique solution $\boldsymbol{P}_\infty \succeq \boldsymbol{0}$ regardless of the numerical entries in $\boldsymbol{B}$.

(C) may or may not have unique solution $\boldsymbol{P}_\infty \succeq \boldsymbol{0}$ depending on the numerical entries in $\boldsymbol{B}$.

Problem 2

Consider problem (1) for $n = 2$, with some 2×2 matrices $\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{B}$ as stated, and with 2×1 nonzero initial condition $\boldsymbol{x}_0 = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$. Can the minimum value for the objective be $\alpha^2 - 2\beta^2$?

(A) YES because such finite real numbers α, β could be found.

(B) NO because the corresponding CARE has unique solution $\boldsymbol{P}_\infty \succeq \boldsymbol{0}$. `Reason:` $\alpha^2 - 2\beta^2 = \boldsymbol{x}_0^\top \operatorname{diag}(1, -2)\boldsymbol{x}_0$ which is impossible by Lec. 14 p. 6-7 as $\operatorname{diag}(1, -2)$ is sign-indefinite.

(C) YES because the corresponding CARE does NOT have unique solution $\boldsymbol{P}_\infty \succeq \boldsymbol{0}$.

Problem 3

For any LTI pair $(\boldsymbol{A}, \boldsymbol{B})$, the CARE is

$$\boldsymbol{A}^\top \boldsymbol{P}_\infty + \boldsymbol{P}_\infty \boldsymbol{A} - \boldsymbol{P}_\infty \boldsymbol{B} \boldsymbol{R}^{-1} \boldsymbol{B}^\top \boldsymbol{P}_\infty + \boldsymbol{Q} = \boldsymbol{0}.$$

For problem (1), we have the matrix decomposition $\boldsymbol{Q} = \boldsymbol{C}\boldsymbol{C}^\top$ where

(A) $\boldsymbol{C} = \boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}_\infty$.

(B) $\boldsymbol{C} = \boldsymbol{I}$.

(C) $\boldsymbol{C} = \boldsymbol{P}_\infty \boldsymbol{B} \boldsymbol{R}^{-\frac{1}{2}}$. `Reason:`
$\boldsymbol{A} = \boldsymbol{0} \Rightarrow \boldsymbol{Q} = \boldsymbol{P}_\infty \boldsymbol{B} \boldsymbol{R}^{-1} \boldsymbol{B}^\top \boldsymbol{P}_\infty = \boldsymbol{P}_\infty \boldsymbol{B} \boldsymbol{R}^{-\frac{1}{2}} \left(\boldsymbol{P}_\infty \boldsymbol{B} \boldsymbol{R}^{-\frac{1}{2}}\right)^\top$.

Problem 4

The optimal control for any infinite horizon continuous-time LQR, if exists, is $\boldsymbol{u}^{\text{opt}} = -\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}_\infty \boldsymbol{x}^{\text{opt}}$. For problem (1), we must have

$$\int_0^\infty \left(\boldsymbol{x}^{\text{opt}}\right)^\top \boldsymbol{Q} \boldsymbol{x}^{\text{opt}} \, \mathrm{d}t = \int_0^\infty \left(\boldsymbol{u}^{\text{opt}}\right)^\top \boldsymbol{R} \boldsymbol{u}^{\text{opt}} \, \mathrm{d}t.$$

(A) TRUE. `Reason:` substitute $\boldsymbol{u}^{\text{opt}}$ and use CARE with $\boldsymbol{A} = \boldsymbol{0}$.

(B) FALSE.

Problem 5

To solve the CARE associated with problem (1), which MATLAB command can be used?

(A) `icare` or `lqr`. `Reason:` Lec. 14 p. 9-10.

(B) `idare`.

(C) None of the above.