AerE 5740: Optimal Control: Spring 2025

Iowa State University

Quiz 16: Ruling out singularity in nonlinear OCP

Name:____

For each of the following multiple choice problems, ONLY ONE option is correct. Clearly mark the correct option. Calculators or other electronic devices are NOT allowed.

All problems below are for the minimum time nonlinear OCP (so free final time T):

$$\begin{aligned} \min_{\boldsymbol{u}} \quad & \int_0^T \mathrm{d}t \\ \text{subject to} \quad & \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} x_2 \\ -k\sin x_1 + u\end{pmatrix}, \\ & \boldsymbol{x}(t=0) = \boldsymbol{x}_0 \text{ given}, \quad \boldsymbol{x}(t=T) = \boldsymbol{x}_T \text{ given } \neq \boldsymbol{x}_0, \\ & -1 \leq u \leq 1, \end{aligned} \tag{$\star$}$$

where the constant parameter $k > 0$. We can interpret the ODE above as the equation-of-motion for a controlled pendulum with torque input u.

Problem 1

The ODE in ($\star$) is a control-affine nonlinear system of the form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u$ where the vector fields

(A) $\boldsymbol{f} = \begin{pmatrix} x_2 \\ -k\sin x_1\end{pmatrix}, \boldsymbol{g} = \begin{pmatrix}0\\1\end{pmatrix}$. `Reason:` directly identify from given ODE.

(B) $\boldsymbol{f} = \begin{pmatrix} x_2 \\ -k\sin x_1\end{pmatrix}, \boldsymbol{g} = \begin{pmatrix}1\\0\end{pmatrix}$.

(C) $\boldsymbol{f} = \begin{pmatrix} x_2 \\ -k\sin x_1\end{pmatrix}, \boldsymbol{g} = \begin{pmatrix}1\\1\end{pmatrix}$.

Problem 2

The Lie bracket $[\boldsymbol{f}, \boldsymbol{g}](\boldsymbol{x}) := \dfrac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\boldsymbol{f} - \dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{g}$, is a

(A) scalar field, i.e., scalar that is a function of $\boldsymbol{x}$.

(B) vector field, i.e., vector whose each component is a function of $\boldsymbol{x}$. `Reason:` (matrix $\times$ vector) $-$ (matrix $\times$ vector) $=$ vector.

(C) matrix field, i.e., matrix whose each component is a function of $\boldsymbol{x}$.

Problem 3

In $(\star)$, the Lie bracket $[\boldsymbol{f}, \boldsymbol{g}](\boldsymbol{x})$ equals

(A) $k\cos x_1$.

(B) $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$. `Reason:` $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{f} - \begin{bmatrix} 0 & 1 \\ -k\cos x_1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Alternatively, eliminate the other two options using Problem 2 answer.

(C) $\begin{bmatrix} 0 & 1 \\ k\cos x_1 & 0 \end{bmatrix}$.

Problem 4

In $(\star)$, $\boldsymbol{g}(\boldsymbol{x})$ and $[\boldsymbol{f}, \boldsymbol{g}](\boldsymbol{x})$ are

(A) linearly independent for all $\boldsymbol{x} \in \mathbb{R}^2$. `Reason:` $\det\left(\boldsymbol{g} \mid [\boldsymbol{f}, \boldsymbol{g}]\right) = \det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \neq 0$.

(B) linearly dependent for some $\boldsymbol{x} \in \mathbb{R}^2$.

(C) linearly dependent for all $\boldsymbol{x} \in \mathbb{R}^2$.

Problem 5

In $(\star)$, the optimal control $u^{\text{opt}}(t)$ is

(A) bang-bang. `Reason:` Lec. 17 p. 2.

(B) NOT bang-bang.

(C) bang-singular-bang.