AerE 5740: Optimal Control: Spring 2026

Iowa State University

Quiz 19: Discrete Time Infinite Horizon DP

Name:____

For each of the following multiple choice problems, ONLY ONE option is correct. Clearly mark the correct option. Calculators or other electronic devices are NOT allowed.

Recap: A metric space is a pair $(\mathcal{M}, \mathrm{dist})$ consisting of a set $\mathcal{M}$ and a distance function $\mathrm{dist}(\cdot,\cdot)$ that helps to measure distance between any two elements of $\mathcal{M}$.

A complete metric space is a metric space $(\mathcal{M}, \mathrm{dist})$ that has the additional property that every Cauchy sequence in $\mathcal{M}$ has its limit also in $\mathcal{M}$.

A (possibly nonlinear) map $\mathcal{T} : \mathcal{M} \mapsto \mathcal{M}$ over a complete metric space $(\mathcal{M}, \mathrm{dist})$ is a contraction if there exists $0 < \beta < 1$ such that

$$\mathrm{dist}\left(\mathcal{T}(z), \mathcal{T}(\widehat{z})\right) \leq \beta\, \mathrm{dist}\left(z, \widehat{z}\right) \quad \forall z, \widehat{z} \in \mathcal{M}.$$

Problem 1

In the complete metric space $(\mathbb{R}, |\cdot|)$, a linear map $\mathcal{T} : \mathbb{R} \mapsto \mathbb{R}$ given by $\mathcal{T}(z) := az$ is contraction if and only if the scalar a satisfies

(A) $0 < a < 1$.

(B) $-1 < a < 1$. `Reason:` Since $|az - a\widehat{z}| = |a| \cdot |z - \widehat{z}|$, so $\mathcal{T}(z) = az$ is contractive iff $|a| < 1 \Leftrightarrow -1 < a < 1$.

(C) $-\infty < a < 1$.

Problem 2

In the complete metric space $(\mathbb{R}^n, \|\cdot\|_2)$, a linear map $\mathcal{T} : \mathbb{R}^n \mapsto \mathbb{R}^n$ given by $\mathcal{T}(\boldsymbol{z}) := \boldsymbol{A}\boldsymbol{z}$ is contraction if and only if the $n \times n$ matrix $\boldsymbol{A}$ satisfies

(A) $\|\boldsymbol{A}\|_2 < 1$. `Reason:` Since $\|\boldsymbol{A}\boldsymbol{z} - \boldsymbol{A}\widehat{\boldsymbol{z}}\|_2 \leq \|\boldsymbol{A}\|_2 \cdot \|\boldsymbol{z} - \widehat{\boldsymbol{z}}\|_2$, so $\mathcal{T}(\boldsymbol{z}) = \boldsymbol{A}\boldsymbol{z}$ is contractive iff $\|\boldsymbol{A}\|_2 < 1$.

(B) $\|\boldsymbol{A}\|_2 = 1$.

(B) $1 < \|\boldsymbol{A}\|_2$.

Problem 3

The Bellman equation for discrete time infinite horizon DP is a functional fixed point equation $V_\infty = \mathcal{T}V_\infty$, where

$$\mathcal{T}V_\infty(\boldsymbol{x}) := \underset{\boldsymbol{u}\in\mathcal{U}}{\text{minimum}}\; \mathbb{E}\left[c(\boldsymbol{x},\boldsymbol{u},\boldsymbol{w}_k) + \beta V_\infty\left(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{w}_k)\right)\right] \quad \text{for stochastic DP},$$

$$\mathcal{T}V_\infty(i) := \underset{\boldsymbol{u}\in\mathcal{U}}{\text{minimum}}\; \mathbb{E}\left[c(i,\boldsymbol{u}) + \beta\sum_{j=1}^{n} p_{ij}(\boldsymbol{u})V_\infty(j)\right] \;\forall i = 1,\ldots,n, \;\text{for MDP}.$$

(A) $\mathcal{T}$ is a contraction if and only if $0 < \beta < 1$. `Reason:` Lec. 20 p. 16-17.

(B) $\mathcal{T}$ is a contraction if and only if $-1 < \beta < 1$.

(C) $\mathcal{T}$ is a contraction if and only if $1 < \beta$.

Problem 4

Let $\mathcal{X}$ denote the state space. When the Bellman operator $\mathcal{T}$ is contractive, then this contraction guarantee is over the complete metric space

(A) $\left(L_1(\mathcal{X}), \|\cdot\|_{L_1(\mathcal{X})}\right)$.

(B) $\left(L_2(\mathcal{X}), \|\cdot\|_{L_2(\mathcal{X})}\right)$.

(C) $\left(L_\infty(\mathcal{X}), \|\cdot\|_{L_\infty(\mathcal{X})}\right)$. `Reason:` Lec. 20 p. 18-19.

Problem 5

The functional recursion

$$V_\infty(\boldsymbol{x}) \mapsto \left(V_\infty\right)_{\text{next}}(\boldsymbol{x}) = \mathcal{T}V_\infty(\boldsymbol{x})$$

where $\mathcal{T}$ denotes the Bellman operator, is guaranteed to converge to a unique solution if and only if

(A) $\mathcal{T}$ is contractive. `Reason:` Lec. 20 p. 19.

(B) $\mathcal{T}$ is identity.

(C) $\mathcal{T}$ is linear.