AerE 5740: Optimal Control: Spring 2026

Iowa State University

Quiz 22: Solving HJB PDE

Name:____

For each of the following multiple choice problems, ONLY ONE option is correct. Clearly mark the correct option. Calculators or other electronic devices are NOT allowed.

Consider the following minimum time deterministic OCP in continuous time:

$$\min_{\gamma\in\Gamma} \quad \int_0^T 1 \,\mathrm{d}t$$
$$\text{subject to} \quad \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{u}, \quad \boldsymbol{x}\in\mathbb{R}^n, \quad \|\boldsymbol{u}\|_2 \leq 1, \quad \boldsymbol{A} = -\boldsymbol{A}^\top, \quad \boldsymbol{x}(T)\in\mathcal{X}_{\text{target}}\subset\mathbb{R}^n,$$

where Γ denotes the set of all history-dependent randomized policies. So the objective is to design controller to hit a closed bounded target set $\mathcal{X}_{\text{target}}$ (which could be a single point) as fast as possible.

All five questions below are about the DP equation for this specific OCP. That the constant matrix $\boldsymbol{A}$ is skew-symmetric will only be needed for Problem 4.

The following observations are helpful:

- Since cost only depends on time-to-go, so the value function V must NOT be an explicit function of time, i.e., instead of $V(t,\boldsymbol{x})$, we must have $V(\boldsymbol{x})$.
- From the definition of value function, if $\boldsymbol{x}\in\mathcal{X}_{\text{target}}$ then $V(\boldsymbol{x}) = 0$.
- Using the above two, the DP equation for $V(\boldsymbol{x})$ becomes the following the HJB PDE boudary value problem:

$$\underbrace{\frac{\partial V}{\partial t}}_{=0} + \underset{\|\boldsymbol{u}\|_2\leq 1}{\text{minimum}}\left\{\underbrace{L(t,\boldsymbol{x},\boldsymbol{u})}_{=1} + \left\langle\frac{\partial V}{\partial \boldsymbol{x}}, \underbrace{\boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{u})}_{=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{u}}\right\rangle\right\} = 0, \quad \text{if} \quad \boldsymbol{x}\notin\mathcal{X}_{\text{target}}$$
$$V(\boldsymbol{x}) = 0, \quad \text{if} \quad \boldsymbol{x}\in\mathcal{X}_{\text{target}}.$$

Problem 1

This particular HJB PDE cannot be solved by Hopf-Lax formula because the minimum (i.e., optimized Hamiltonian) has

(A) explicit state dependence. `Reason:` the $\boldsymbol{x}$ appears explicitly in the inner product.

(B) explicit time dependence.

(C) explicit control dependence.

Problem 2

For any two vectors $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^n$, the Cauchy-Schwarz inequality gives tight bound on the magnitude of inner product:

$$|\langle \boldsymbol{p}, \boldsymbol{q} \rangle| \leq \|\boldsymbol{p}\|_2 \|\boldsymbol{q}\|_2.$$

Using this, our specific HJB PDE: $\underset{\|\boldsymbol{u}\|_2 \leq 1}{\text{minimum}} \left\{ 1 + \left\langle \frac{\partial V}{\partial \boldsymbol{x}}, \boldsymbol{A}\boldsymbol{x} + \boldsymbol{u} \right\rangle \right\} = 0$ simplifies to

(A) $1 + \left\langle \frac{\partial V}{\partial \boldsymbol{x}}, \boldsymbol{A}\boldsymbol{x} \right\rangle - \left\| \frac{\partial V}{\partial \boldsymbol{x}} \right\|_2 = 0$. `Reason:` $|\langle \frac{\partial V}{\partial \boldsymbol{x}}, \boldsymbol{u} \rangle| \leq \|\frac{\partial V}{\partial \boldsymbol{x}}\|_2 \|\boldsymbol{u}\|_2 \leq \|\frac{\partial V}{\partial \boldsymbol{x}}\|_2$, so $\underset{\|\boldsymbol{u}\|_2 \leq 1}{\text{minimum}} \langle \frac{\partial V}{\partial \boldsymbol{x}}, \boldsymbol{u} \rangle = -\|\frac{\partial V}{\partial \boldsymbol{x}}\|_2$.

(B) $1 + \left\langle \frac{\partial V}{\partial \boldsymbol{x}}, \boldsymbol{A}\boldsymbol{x} \right\rangle + \left\| \frac{\partial V}{\partial \boldsymbol{x}} \right\|_2 = 0$.

(C) $1 + \left\langle \frac{\partial V}{\partial \boldsymbol{x}}, \boldsymbol{A}\boldsymbol{x} \right\rangle = 0$.

Problem 3

For our specific OCP, the optimal control $\boldsymbol{u}^{\text{opt}}$ written in terms of value function V is

(A) $\boldsymbol{u}^{\text{opt}} = -\frac{\partial V}{\partial \boldsymbol{x}}$.

(B) $\boldsymbol{u}^{\text{opt}} = -\frac{\frac{\partial V}{\partial \boldsymbol{x}}}{\|\frac{\partial V}{\partial \boldsymbol{x}}\|_2}$. `Reason:` To achieve $\underset{\|\boldsymbol{u}\|_2 \leq 1}{\text{minimum}} \langle \frac{\partial V}{\partial \boldsymbol{x}}, \boldsymbol{u} \rangle = -\|\frac{\partial V}{\partial \boldsymbol{x}}\|_2$, this is the only possibility.

Problem 4

For our specific OCP, the solution of the HJB PDE is
[**Hint:** $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} = 0$ for skew-symmetric $\boldsymbol{A}$.]

(A) $V(\boldsymbol{x}) = \sqrt{\boldsymbol{x}^\top \boldsymbol{x}} \quad \forall \boldsymbol{x} \notin \mathcal{X}_{\text{target}}$. `Reason:` We get $\frac{\partial V}{\partial \boldsymbol{x}} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}$, which being a unit vector, has magnitude one. So the HJB PDE left-hand-side becomes $1 + \frac{\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}}{\|\boldsymbol{x}\|_2} - 1$, which is indeed zero $\forall \boldsymbol{x} \notin \mathcal{X}_{\text{target}}$. Hence this $V(\boldsymbol{x})$ satisfies the HJB PDE.

(B) $V(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{x} \quad \forall \boldsymbol{x} \notin \mathcal{X}_{\text{target}}$.

Problem 5

For our specific OCP, the optimal control $\boldsymbol{u}^{\text{opt}}$ in state feedback form is

(A) $\boldsymbol{u}^{\text{opt}} = -2\boldsymbol{x}$.

(B) $\boldsymbol{u}^{\text{opt}} = -\boldsymbol{x}$.

(C) $\boldsymbol{u}^{\text{opt}} = -\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}$. `Reason:` combine answers for Prob. 3 and Prob. 4.