AerE 5740: Optimal Control: Spring 2025

Iowa State University

Quiz 4: Solution of CoV Problem

Name:____

For each of the following multiple choice problems, ONLY ONE option is correct. Clearly mark the correct option. Calculators or other electronic devices are NOT allowed.

All 5 questions below are for the following two Lagrangians (ignoring multiplicative constants) you have seen last class:

$$L_{\text{shortest path}} = \sqrt{1+(u')^2}, \qquad L_{\text{Brachistochrone}} = \sqrt{\frac{1+(u')^2}{u}}.$$

Problem 1

Which of these two Lagrangians is/are convex in u'?

(A) Both of them. `Reason:` the function $\sqrt{1+p^2}$ is 2-norm of the vector $(p,1)$. By triangle inequality,
$\|\theta(1,p)+(1-\theta)(1,q)\|_2 \leq |\theta|\|(1,p)\|_2 + |1-\theta|\|(1,q)\|_2 = \theta\|(1,p)\|_2 + (1-\theta)\|(1,q)\|_2$.
`Alternative reason:` the second derivative of $\sqrt{1+p^2}$ equals $1/(1+p^2)^{3/2} > 0$.

(B) Only $L_{\text{Brachistochrone}}$.

(C) None of them.

Problem 2

Which of these two Lagrangians is/are 1-coercive in u'?

(A) Both of them.

(B) Only $L_{\text{Brachistochrone}}$.

(C) None of them. `Reason:` $\lim\limits_{|p|\to\infty} \frac{\sqrt{1+p^2}}{|p|} = 1 \neq \infty$.

Problem 3

To which of these two Lagrangians, we can apply the Beltrami identity?

(A) Both of them. `Reason:` both Lagrangians have no explicit dependence on x.

(B) Only $L_{\text{Brachistochrone}}$.

(C) None of them.

Problem 4

In one dimension, Hilbert's theorem says: if $\dfrac{\partial^2}{\partial p^2} L(x, u, p) \neq 0$ for all $x \in \Omega$ (bounded domain of x), then $u^{\text{opt}} \in C^2(\Omega)$ provided u^{opt} exists. We then say u^{opt} is nonsingular.

For which of these two Lagrangians, the corresponding u^{opt} (assuming they exist) is/are nonsingular?

(A) Both of them. `Reason:` the second derivative of $\sqrt{1+p^2}$ equals $1/(1+p^2)^{3/2} > 0$.

(B) Only $L_{\text{Brachistochrone}}$.

(C) None of them.

Problem 5

For which of these two Lagrangians, the corresponding u^{opt} (assuming they exist) is/are unique?

Recall **Theorem:** u^{opt} nonsingular and $L(x, u, p) \in C^3(\Omega)$ with respect to $p \Rightarrow$ unique u^{opt}.

(A) Both of them. `Reason:` from the solution of previous question, both u^{opt} are nonsingular. Also, the third derivative of $\sqrt{1+p^2}$ equals $-3p/(1+p^2)^{3/2}$ which is continuous, so $L(x, u, p) \in C^3(\Omega)$ with respect to p. Alternatively, square root of a positive polynomial is in C^∞.

(B) Only $L_{\text{Brachistochrone}}$.

(C) None of them.