AerE 5740: Optimal Control: Spring 2025

Iowa State University

Quiz 5: Solving Euler-Lagrange equation

Name:____

For each of the following multiple choice problems, ONLY ONE option is correct. Clearly mark the correct option. Calculators or other electronic devices are NOT allowed.

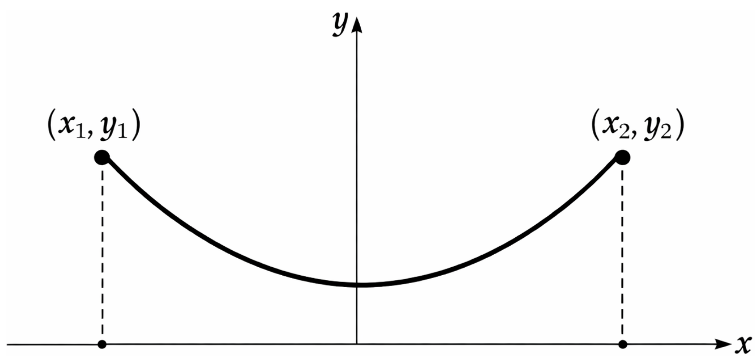

Figure 1: A chain or cable of fixed length ℓ with fixed ends at (x_1, y_1) and (x_2, y_2).

Let us revisit the setting in earlier Quiz 3 shown in Fig. 1. We have hanging chain or cable of given length $\ell > 0$ whose ends have given co-ordinates (x_1, y_1) and (x_2, y_2). We want to determine the "natural" shape $y = u(x)$ that minimizes the gravitational potential energy. Assume that $x_1 \neq x_2$ and $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \leq \ell$.

All 5 questions below are for the corresponding CoV formulation:

$$\min_{u \in \mathcal{F}(\mathbb{R})} \int_{x_1}^{x_2} u\sqrt{1 + (u')^2}\,\mathrm{d}x$$

where
$\mathcal{F}(\mathbb{R}) := \{u \in \mathcal{C}^1([x_1, x_2]) \mid u(x_1) = y_1, u(x_2) = y_2, \int_{x_1}^{x_2} \sqrt{1 + (u')^2}\,\mathrm{d}x = \ell\}$.

Problem 1

Let λ be a Lagrange multiplier associated with the integral equality constraint. Here λ

(A) must be a numerical constant. `Reason:` We explained in Lec. 5 p. 1 that for integral equality constraint, λ is constant; and for pointwise equality constraint beyond natural BC, λ is a function of x.

(B) must be a function of x.

(C) must be a function of u'.

Problem 2

In this problem, the augmented Lagrangian $L_{\text{augmented}}$ is equal to

(A) $\lambda\sqrt{1+(u')^2}$.

(B) $\lambda u\sqrt{1+(u')^2}$.

(C) $(u+\lambda)\sqrt{1+(u')^2}$. `Reason:` The augmented Lagrangian is $L+\lambda M$. Here $L = u\sqrt{1+(u')^2}$, $M = \sqrt{1+(u')^2}$.

Problem 3

Here we can apply the Beltrami identity to $L_{\text{augmented}}$ because

(A) $L_{\text{augmented}}$ has NO explicit dependence on x. `Reason:` follows from the previous answer.

(B) $L_{\text{augmented}}$ has NO explicit dependence on u.

(C) $L_{\text{augmented}}$ has NO explicit dependence on u'.

Problem 4

Letting $v(x) := u(x)+\lambda$, the Beltrami identity: $L_{\text{augmented}} - u'\dfrac{\partial}{\partial u'}L_{\text{augmented}} = c$ for this problem, where c is some constant, simplifies to

(A) $v' = \sqrt{\dfrac{v^2}{c^2}-1}$. `Reason:` Substituting $L_{\text{augmented}}$ in the Beltrami identity, we get

$$\frac{u+\lambda}{\sqrt{1+(u')^2}} = c \;\Rightarrow\; \frac{v}{\sqrt{1+(v')^2}} = c \;\Rightarrow\; v' = \sqrt{\frac{v^2}{c^2}-1}.$$

(B) $v' = \sqrt{v^2-c^2}$.

(C) $v' = \sqrt{v^2-vc}$.

Problem 5

By visual inspection, one possible solution of the Beltrami identity for this problem is $v(x) = c$, and hence $u(x) = \text{constant}$. This solution is valid only if

(A) $y_1 = y_2$.

(B) $x_2 - x_1 = \ell$.

(C) $y_1 = y_2$ and $x_2 - x_1 = \ell$. `Reason:` Follows from the constraint set $\mathcal{F}(\mathbb{R})$. Specifically, if u is constant then the natural BC gives $y_1 = y_2$, and the integral equality constraint gives $x_2 - x_1 = \ell$.

Additional material NOT needed for this quiz:
Here is the detailed solution of the hanging chain/cable problem. To solve the nonlinear ODE derived in Problem 4 option A:

$$v' = \sqrt{\frac{v^2}{c^2} - 1},$$

we make the cosine hyperbolic substitution: $v := c\cosh w \;\Rightarrow\; v' = c(\sinh w)w'$. Then our nonlinear ODE becomes

$$c(\sinh w)w' = \sinh w.$$

Now there are two possibilities: either $\sinh w = 0$ or $\sinh w \neq 0$. For the first possibility, we must have $w = 0 \;\Rightarrow\; v = c \;\Rightarrow\; u^{\text{opt}} = \text{constant}$, which is admissible iff we satisfy the conditions in Problem 5 option C. For the second possibility, we obtain

$$cw' = 1 \;\Rightarrow\; w(x) = \frac{x}{c} + k \;\Rightarrow\; v(x) = c\cosh\left(\frac{x}{c} + k\right).$$

Then, $\sqrt{1+(u')^2} = \sqrt{1+(v')^2} = \cosh\left(\frac{x}{c} + k\right)$, and the integral equality constraint gives

$$\begin{aligned} \ell = \int_{x_1}^{x_2} \sqrt{1+(u')^2}\mathrm{d}x &= \int_{x_1}^{x_2} \cosh\left(\frac{x}{c} + k\right)\mathrm{d}x \\ &= c\int_{\frac{x_1}{c}+k}^{\frac{x_2}{c}+k} \cosh z\,\mathrm{d}z \\ &= c\left(\sinh\left(\frac{x_2}{c} + k\right) - \sinh\left(\frac{x_1}{c} + k\right)\right). \qquad (1) \end{aligned}$$

We also have the natural BC:

$$y_1 + \lambda = c\cosh\left(\frac{x_1}{c} + k\right), \qquad (2)$$

$$y_2 + \lambda = c\cosh\left(\frac{x_2}{c} + k\right). \qquad (3)$$

From here we can determine the three constants (c, k, λ) by numerically solving the three coupled equations (1), (2) and (3). This non-constant solution: $u^{\text{opt}}(x) = c\cosh\left(\frac{x}{c} + k\right) - \lambda$ is called a catenary. See Fig. 2.

Figure 2: Catenary shape of the hanging chain. Source.