AerE 5740: Optimal Control: Spring 2025

Iowa State University

Quiz 6: Newtonian mechanics as special case of Euler-Lagrange equation

Name:____

For each of the following multiple choice problems, ONLY ONE option is correct. Clearly mark the correct option. Calculators or other electronic devices are NOT allowed.

Problem 1

In 1D position coordinate x, the standard spring-mass equation of motion is $m\ddot{x} = -kx$ for constant mass $m > 0$ and constant spring coefficient $k > 0$. This equation of motion is simply the Euler-Lagrange equation

$$\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) \quad \text{where the Lagrangian } L = T(\dot{x}) - V(x),$$

and

(A) $T = \frac{1}{2}m\dot{x}^2$, $V = \frac{1}{2}kx^2$. `Reason:` directly verify using EL equation. `Alternatively`, identify these T, V as the kinetic and spring potential energies, respectively.

(B) $T = m\dot{x}$, $V = -kx$.

(C) $T = m\dot{x}$, $V = kx$.

Problem 2

In a resonant circuit, an inductor with constant inductance $\Lambda > 0$ is connected in series with a capacitor with constant capacitance $C > 0$. If $q(t)$ is the electric charge at time t, then the Euler-Lagrange equation $\frac{\partial L}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)$ with

$$\text{Lagrangian } L = \text{inductor kinetic energy} - \text{capacitor potential energy} = \frac{1}{2}\Lambda\dot{q}^2 - \frac{q^2}{2C}$$

is exactly the Kirchoff's voltage law, given by

(A) $\frac{q}{C} = \Lambda\ddot{q}$.

(B) $-\frac{q}{C} = \Lambda\ddot{q}$. `Reason:` substitute given L in EL equation.

(C) $-\frac{q}{C} = \Lambda\dot{q}$.

Problem 3

For a Hamiltonian H, Hamilton's canonical equations is of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{q}\\ \boldsymbol{p}\end{pmatrix} = \boldsymbol{\Omega}\nabla_{\begin{pmatrix}\boldsymbol{q}\\ \boldsymbol{p}\end{pmatrix}} H, \quad \text{where } \boldsymbol{q}, \boldsymbol{p} \in \mathbb{R}^n.$$

Which of the following statements is TRUE?

(A) The matrix $\boldsymbol{\Omega}$ is of size $n \times n$.

(B) The matrix $\boldsymbol{\Omega}$ is of size $n \times 2n$.

(C) The matrix $\boldsymbol{\Omega}$ is of size $2n \times 2n$. `Reason:` left-hand-side is a vector of size $2n \times 1$.

Problem 4

For any $f : \mathbb{R}^n \mapsto \mathbb{R}$, its Legendre-Fenchel conjugate f^* is defined as

$$f^*(\boldsymbol{y}) := \sup_{\boldsymbol{x}\in\mathbb{R}^n} \left\{\langle \boldsymbol{y}, \boldsymbol{x}\rangle - f(\boldsymbol{x})\right\}.$$

If $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{M}\boldsymbol{x}$ for given $n \times n$ matrix $\boldsymbol{M} \succ \boldsymbol{0}$, then $f^*(\boldsymbol{y})$ equals

(A) the linear function $\boldsymbol{M}\boldsymbol{y}$.

(B) the same quadratic function $\frac{1}{2}\boldsymbol{y}^\top \boldsymbol{M}\boldsymbol{y}$.

(C) the quadratic function $\frac{1}{2}\boldsymbol{y}^\top \boldsymbol{M}^{-1}\boldsymbol{y}$. `Reason:` Since this involves unconstrained maximization, so taking the gradient of $\left\{\langle \boldsymbol{y}, \boldsymbol{x}\rangle - f(\boldsymbol{x})\right\}$ with respect to $\boldsymbol{x}$ to zero, we find the maximizer $\boldsymbol{x}^{\max} = \boldsymbol{M}^{-1}\boldsymbol{y}$. So the maximum value $f^*(\boldsymbol{y}) = \boldsymbol{y}^\top \boldsymbol{M}^{-1}\boldsymbol{y} - \frac{1}{2}\boldsymbol{y}^\top \boldsymbol{M}^{-1}\boldsymbol{y}$ $= \frac{1}{2}\boldsymbol{y}^\top \boldsymbol{M}^{-1}\boldsymbol{y}$.

Problem 5

If the Lagrangian $L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ has no explicit dependence on time t, then the corresponding Hamiltonian H

(A) must be a numerical constant independent of time t. `Reason:` see Lec. 7 p. 14.

(B) must be a function of time t.

(C) may or may not be a function of time t.