AerE 5740: Optimal Control: Spring 2025

Iowa State University

Quiz 8: OCP formulation, necessary conditions for optimality

Name:____

For each of the following multiple choice problems, ONLY ONE option is correct. Clearly mark the correct option. Calculators or other electronic devices are NOT allowed.

Consider the scalar state $x(t)$ of national economy governed by the second order ODE

$$\ddot{x} = -\alpha^2 x + u, \quad \alpha \text{ nonzero real}, \quad t \geq 0, \quad x(0) = \dot{x}(0) = 0,$$

where $u(t)$ is the effort a Government puts at time t for economic reform.

The Government would like to maximize its chance of getting re-elected at the fixed terminal time T by bringing the national economy at a healthy state at the time of re-election, while not spending too much effort in economic reform during its tenure, i.e.,

$$\underset{u(\cdot)}{\text{maximize}} \quad x(T) - \int_0^T u^2 \, \mathrm{d}t.$$

All 5 questions below are for this problem.

Problem 1

Letting the state $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \in \mathbb{R}^2$, the corresponding OCP in standard form is

(A)

$$\begin{aligned} &\underset{u(\cdot)}{\text{minimize}} \quad -x_1(T) + \int_0^T u^2 \, \mathrm{d}t \\ &\text{subject to} \quad \dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad \boldsymbol{x}(0) = \boldsymbol{0}. \end{aligned}$$

`Reason:` rewrite given second order ODE in state space form.

(B)

$$\begin{aligned} &\underset{u(\cdot)}{\text{minimize}} \quad -x_1(T) + \int_0^T u^2 \, \mathrm{d}t \\ &\text{subject to} \quad \dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad \boldsymbol{x}(0) = \boldsymbol{0}. \end{aligned}$$

(C)

$$\begin{aligned} &\underset{u(\cdot)}{\text{minimize}} \quad -x_1(T) + \int_0^T u^2 \, \mathrm{d}t \\ &\text{subject to} \quad \dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \quad \boldsymbol{x}(0) = \boldsymbol{0}. \end{aligned}$$

Problem 2

In this problem, which of the following is the Hamiltonian $H := L + \langle \boldsymbol{\lambda}, \boldsymbol{f} \rangle$?

(A) $H = u^2$.

(B) $H = \lambda_1 x_2 + \lambda_2 \left(-\alpha^2 x_1 + u\right)$.

(C) $H = u^2 + \lambda_1 x_2 + \lambda_2 \left(-\alpha^2 x_1 + u\right)$. `Reason:` substitute $L = u^2$ and $\boldsymbol{f} = (x_2, -\alpha^2 x_1 + u)$ in $H := L + \langle \boldsymbol{\lambda}, \boldsymbol{f} \rangle$.

Problem 3

In this problem, the costate ODEs (recall: $\dot{\boldsymbol{\lambda}} = -\nabla_{\boldsymbol{x}} H$) are

(A) $\dot{\lambda}_1 = \alpha^2 \lambda_2, \quad \dot{\lambda}_2 = -\lambda_1$. `Reason:` take the given gradient of H from Problem 2.

(B) $\dot{\lambda}_1 = -\alpha^2 \lambda_2, \quad \dot{\lambda}_2 = -\lambda_1$.

(C) $\dot{\lambda}_1 = \alpha^2 \lambda_2, \quad \dot{\lambda}_2 = \lambda_1$.

Problem 4

In this problem, Pontryagin Minimum Principle (PMP) $0 = \left.\dfrac{\partial H}{\partial u}\right|_{\text{opt}}$ gives optimal control

(A) $u^{\text{opt}} = \lambda_2^{\text{opt}}/2$

(B) $u^{\text{opt}} = -\lambda_2^{\text{opt}}/2$. `Reason:` take the given gradient of H from Problem 2.

(C) $u^{\text{opt}} = -\lambda_2^{\text{opt}}$.

Problem 5

In this problem, the terminal cost $\phi(T, \boldsymbol{x}(T)) = -x_1(T)$. So the transversality condition

$$\left\langle \left. \left(\nabla_{\boldsymbol{x}} \phi + (\nabla_{\boldsymbol{x}} \boldsymbol{\psi})^{\top} \boldsymbol{\nu} - \boldsymbol{\lambda} \right) \right|_{t=T}, \mathrm{d}\boldsymbol{x}(T) \right\rangle + \left. \left(\frac{\partial \phi}{\partial t} + \left\langle \frac{\partial \boldsymbol{\psi}}{\partial t}, \boldsymbol{\nu} \right\rangle + H \right) \right|_{t=T} \mathrm{d}T = 0$$

gives (Hint: in this problem, T is fixed, so $\mathrm{d}T = 0$. Here $\boldsymbol{x}(T)$ is not fixed, so $\mathrm{d}\boldsymbol{x}(T) \neq \mathbf{0}$.)

(A) $\begin{pmatrix} \lambda_1(T) \\ \lambda_2(T) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. `Reason:` use the given gradient of terminal cost ϕ.

(B) $\begin{pmatrix} \lambda_1(T) \\ \lambda_2(T) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(C) $\begin{pmatrix} \lambda_1(T) \\ \lambda_2(T) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.